Asymmetric Models of Sales

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May 2025. Final version for *American Economic Journal: Microeconomics*.[1]

Abstract. We generalize the captive-and-shopper model of sales to allow asymmetries in production costs and captive audiences in an oligopoly. Both kinds of asymmetry determine the firms that compete (via randomized sales) to serve the price-comparing shoppers, while other firms exploit their captive audiences. In contrast to a model with symmetric costs (but asymmetric captive audiences) there are natural situations in which more than two firms use sales by engaging in pairwise battles across different price intervals. We then study the choice of production technologies via costly process innovations. A distinctive asymmetry emerges endogenously: one firm innovates more and becomes the dominant supplier of shoppers. The pattern of innovations connects to the size of firms' captive bases and the shape of technological opportunity. We also provide a trio of extensions to consider costly acquisitions of captives and shoppers, and captives' choice of captor.

The dispersion of prices for similar products is a well-established regularity. Some researchers have observed some stability of dispersion, with different firms maintaining distinct price positions for some period of time, while others have emphasized inter-temporal price movements.[2] In any case, the existence of different prices for the same product means that buyers benefit by considering

[1]We thank Julian Wright for encouraging this agenda, and many colleagues for comments on this paper and our companion work. These include Simon Anderson, Mark Armstrong, Atabek Atayev, Dan Bernhardt, Yongmin Chen, Alex de Cornière, Andrea Galeotti, Zheng Gong, Paul Heidhues, Roman Inderst, Maarten Janssen, Justin Johnson, Johannes Johnen, Jeanine Miklós-Thal, José Moraga-González, Volker Nocke, Martin Obradovits, Martin Peitz, Martin Pesendorfer, Régis Renault, Michael Riordan, Nicolas Schutz, Robert Somogyi, Martin Spann, Greg Taylor, Juuso Välimäki, Hal Varian, and Chris Wilson. Ronayne is grateful for support from Deutsche Forschungsgemeinschaft through CRC TRR 190 (project number 280092119). We also thank Alexander Wolitzky and several referees for excellent revision suggestions.

[2]Established studies have reported a standard deviation (relative to the mean) at brick-and-mortar stores ranging from 19% to 36% (Kaplan and Menzio, 2015) with most variation attributed to persistent differences (Nakamura and Steinsson, 2008; Wulfsberg, 2016; Kaplan et al., 2019; Moen, Wulfsberg and Aas, 2020) but with more intertemporal changes in online prices (Gorodnichenko, Sheremirov and Talavera, 2018). Several industry studies also support this summary (Sorensen, 2000; Moraga-González and Wildenbeest, 2008; Hong and Shum, 2010; Galenianos, Pacula and Persico, 2012). Nevertheless, other have emphasized movement within the price distribution (Lach, 2002) and substantial dynamic movements in specific industries (Chandra and Tappata, 2011; Pennerstorfer et al., 2020).

multiple offers. Of course, not all buyers may be able to shop around. Sales, which are reductions from established or regular prices, are a long-standing and common practice and can offer firms a resolution to the trade-off presented by such a heterogeneous population of buyers: either price high to exploit those customers who cannot access other offers, or price low to capture those who compare many prices. A central task for theorists, therefore, is to predict the pattern of sale prices and to relate that pattern to the awareness of firms' offers and other firm characteristics.

The canonical "model of sales" (Varian, 1980) parsimoniously captures the key pricing trade-off by specifying buyers who are either "shoppers" (who consider every price) or "captive" customers (who are locked in to a single firm). This succinctly represents situations in which, for example, firms experience demand from local clientele as well as from a central clearinghouse for those who see all offers. The classic model-of-sales pricing game features firms with symmetric costs and captive bases and has many Nash equilibria, all involving the play of mixed strategies. The natural focus for Varian (1980) was on the symmetric equilibrium in which all firms compete for shoppers (or "use sales") by mixing continuously over the same interval of prices stretching from the monopoly price (that fully exploits captive customers) down to a price at which a firm earns only its captive-exploiting monopoly profit even if it wins the business of all shoppers. This provided a rationale for dispersed and unpredictable price offers by all members of an oligopoly.

However, even small deviations from exact symmetry are important. If firms have differently sized captive-audiences (but are otherwise identical) then a unique Nash equilibrium offers a narrower prediction (Baye, Kovenock and De Vries, 1992): the two firms with the smallest captive audiences compete for the business of shoppers via randomized sales. Those authors memorably called this a "tango" danced by de facto duopolists. Other firms "stay off the dance floor" (we call them "wallflowers") and charge the monopoly price to their captive customers. The firm with the smallest captive audience sets a lower price more often and so is the dominant supplier of shoppers.

The literature omits a full treatment of the captive-and-shopper pricing game with asymmetric marginal costs and more than two firms. Our contribution does this, pushes further by allowing those costs to be influenced endogenously by firms' technology choices, and offers several extensions. This fills a notable and long-standing gap in the literature, and provides a springboard for future work and extensions. There are also important economic reasons to consider cost asymmetries.

Firstly, we already noted that the classic model is sensitive to asymmetries in the sizes of captive audiences. Natural questions arise. Is there sensitivity to asymmetries in costs? Do more than two firms engage in sales? Which firms do so? What is the pattern of sale-price offers? Efficiency is also relevant when costs are asymmetric: are shoppers served at the lowest cost?

Secondly, results in asymmetric settings allow the model of sales to be embedded in deeper frameworks. For example, in a recent paper by Hagiu and Wright (2024) a platform specifies benefits and per-sales fees to firms. The equilibrium analysis requires the platform to anticipate profits in a captive-and-shopper pricing subgame with asymmetric marginal costs.[3]

Thirdly, and relatedly, firms face a choice of technologies: choosing a higher fixed cost (for example, via an innovative investment) can yield a lower marginal cost (corresponding to a process innovation). An asymmetric-cost model of sales allows us to investigate such choices. One possibility is that firms (even if they begin symmetrically) may choose very different technologies, which (if true) reinforces the requirement for an asymmetric-marginal-cost model.

Bearing in mind these motivations, we preview here the results that follow in the rest of the paper.

Our first contribution (in Section 1) is to characterize a unique equilibrium of the captive-and-shopper model with asymmetric costs. If firms with smaller captive audiences also have lower costs then there is a de facto duopoly: the two most "aggressive" firms (with the smallest captive audiences and so the lowest costs) dance the familiar "tango" by using randomized sales while all other (less aggressive) "wallflower" firms charge the monopoly price to their captives.

The competition for shoppers moves beyond a de facto duopoly whenever a firm with a higher marginal cost has relatively few captive customers. As an illustrative example, consider a triopoly with two incumbent firms and a new entrant, where that new entrant has an inferior production technology (higher costs) and has yet to develop a locked-in customer base (fewer captives). The (lower cost) incumbents compete for sales across prices that lie below the marginal cost of the entrant. At elevated prices the entrant (with a hunger to attract shoppers) "steps on to the dance floor" to replace one of the incumbents in a "thrango" pattern of mixed-strategy pricing.

[3]Hagiu and Wright (2024) used results from an earlier version of our paper (for example, see their Footnotes 9 and 25 and their Online Appendix A.5). The payoff characterization of Siegel (2009) would also have served their needs, with an appropriate parameter mapping between contest and model-of-sales settings.

The equilibrium exhibits an interesting comparative-static result: a reduction in the marginal cost of the most aggressive firm pushes up the prices charged by the second-most aggressive firm. This result is also found in the duopoly analysis of Golding and Slutsky (2000).[4] This positive strategic effect gives that firm a distinct enhanced incentive to reduce its marginal cost.

This effect is central to our second contribution (in Section 2) in which we consider endogenous technology choice. We add a pre-pricing stage in which firms engage in process innovations: a firm can pay (via a higher fixed cost) to lower its marginal cost of production.

Asymmetric capabilities and a uniquely most aggressive firm emerge in equilibrium. For example, if firms are symmetric ex ante then exactly one firm chooses a distinct technology with more innovation, operates with a lower marginal cost, uses sales, and most often supplies the shoppers.

If firms have differently sized captive audiences then we can identify the firms that use sales. This depends on the strength of technological opportunity embodied in an innovation production function. A firm that expects to serve more customers innovates more, achieves a lower marginal cost, and so generates a greater surplus from each unit. If the relationship between this (endogenously chosen) per-unit surplus and expected output is elastic (so that "technological opportunity is strong") then firms with the most captive customers innovate sufficiently (and so lower their marginal costs sufficiently) to become those that compete for shoppers. Furthermore, the inverse relationship of marginal cost and captive-audience size is precisely the feature needed for more than to two firms to engage in randomized sales.

As well as considering welfare and efficiency (throughout Sections 1 and 2) we offer further contributions (in Section 3) via several extensions: we study the incentives to invest in acquiring captive customers, we evaluate how such captive buyers might wish to move between the firms, and we study a clearinghouse that charges fees for firms to reach price-comparing shoppers.

Related Literature. The original model of sales (Varian, 1980) specified complete symmetry of firms, while others (Narasimhan, 1988; Baye, Kovenock and De Vries, 1992; Kocas and Kiyak, 2006) studied firms with different captive-audience sizes. A central "tango" finding is that the two firms with fewest captives compete (via mixed strategies) to attract shoppers.

[4]This also appears as a related result but in a narrower setting within Inderst (2002).

A full treatment of asymmetric marginal costs has (to our knowledge) not moved beyond the case of duopoly. The earliest duopoly analysis (which we know of) by Golding and Slutsky (2000) is not well-known in the literature but offered comprehensive welfare and comparative-static results.

More recently, Shelegia and Wilson (2021) considered a model with asymmetric costs and where costly advertising is required for shoppers to see prices. A model of sales is the limiting case when advertising costs are removed.[5] They required a condition that is not satisfied in a model of sales (or for their model when advertising costs are small) when more than two firms use sales, and so their paper is primarily relevant only for the de facto duopoly case. Specifically, Shelegia and Wilson (2021) restricted to equilibria (if and when such an equilibrium exists) in which any firm offering randomized sales mixes continuously over the same interval up to the monopoly price.[6] For our asymmetric model of sales, we show (Proposition 3 in Section 1) that this condition is only satisfied in a duopoly, or in a de facto duopoly by which we mean that only two firms use randomized sales.[7] We identify (Proposition 2 in Section 1) conditions for such a de facto duopoly, for which the equilibrium characterization corresponds to that of Golding and Slutsky (2000).

Models of sales are closely related to all-pay auctions and contests: the contest effort corresponds to captive profit sacrificed by charging a lower price, and the prize is to sell to shoppers. Helpfully, the broad framework of Siegel (2009, 2010) can incorporate an asymmetric model of sales. Our Proposition 1 (concerning equilibrium profits) can be derived from Siegel (2009), but our other results (notably the derivation of equilibria) cannot.[8] Methodologically, our equilibrium-construction algorithm identifies "partner swapping" points at which the identities of actively mixing firms change. The construction is reminiscent of algorithms used elsewhere in contests and contest-like settings (Bulow and Levin, 2006; Siegel, 2010, 2014; Xiao, 2016, 2018).[9]

[5]In their full model description firms make utility offers à la Armstrong and Vickers (2001). However, their "Assumption U" restricts to unit demand when costs are asymmetric. They also specify a cost of advertising which is equivalent to a clearinghouse fee. We also study costly advertising (for an oligopoly) as an extension (in Section 3).

[6]Shelegia and Wilson (2021) did not state conditions for the existence of such an equilibrium. Their results (for example, their Proposition 2, p. 209) either condition on "when a sales equilibrium exists under [their] restrictions" or implicitly do so, for example, their Corollary 1 (p. 210) which says more compactly "when a sales equilibrium exists." The key restriction is not a labeled assumption, but instead is documented and explained in their main text (on their p. 204) where they focus "only on sales equilibria where all advertising firms use the full convex support [of prices]."

[7]Our result here also holds if costly advertising is required to make sales to shoppers (see our Section 3).

[8]Siegel (2010) characterized equilibria with m prizes and $m + 1$ players, and so covers models of sales ($m = 1$) only for duopoly. See Appendix C of our working paper and Shelegia and Wilson (2023) for a broader discussion.

[9]There are differences too, of course. For example, Xiao (2016) considered an all-pay auction with asymmetric costs and heterogeneous prizes. There can be intervals in which more than two players mix (see, for example, Xiao, 2016,

Our buyers are either captives or shoppers. Armstrong and Vickers (2022) have made welcome progress with more general consideration sets, so revealing "patterns of competitive interaction" that emerge.[10] They focused on the demand side (with symmetry for $n > 3$) and assumed cost symmetry. Our simpler demand-side specification allows us to consider more general costs.

Beyond the incorporation of asymmetric marginal costs, our second contribution is our study of endogenous innovative activity in the spirit of Dasgupta and Stiglitz (1980). The key property that we exploit is that one firm faces a distinctly different innovation incentive to others. This property is present in work by Chioveanu (2008) which identified (in a model of sales with symmetric costs) a distinctly lower incentive for one (most aggressive) firm to retain captive customers. Relatedly, models with independent awareness of each firm (Ireland, 1993; McAfee, 1994) feature a stronger incentive for the firm with the greatest awareness to increase that awareness.

One of our extensions (in Section 3) specifies a "clearinghouse" for access to shoppers. Pioneering work on clearinghouses (Baye and Morgan, 2001) has been extended to include brand advertising (Baye and Morgan, 2009); asymmetric distributions of customers (Arnold et al., 2011); differentiation and discrimination (Moraga-González and Wildenbeest, 2012); per-sale fees (Baye, Gao and Morgan, 2011; Ronayne, 2021); and auto-switching services (Garrod, Li and Wilson, 2023).[11] See Baye, Morgan and Scholten (2004*a*, 2006) for empirical tests and a review of the earlier work.

1. A Model of Sales with Fully Asymmetric Firms

Model. There are $n \geq 2$ firms who simultaneously choose their prices, where $p_i \in [0, v]$ is the price chosen by firm $i \in \{1, \ldots, n\}$ and $v > 0$ is customers' (common) maximal willingness to pay. Firm i faces a constant marginal cost $c_i \in [0, v)$ to serve any customer.[12]

Fig. 1, p. 184) whereas here battles are always pairwise. As a second example, Bulow and Levin (2006) considered a model in which employers bid for workers. The partner swapping can be such that no firm mixes across the whole price interval (illustrated in Bulow and Levin, 2006, Fig. 2, p. 658) whereas here the most aggressive firm does so.

[10]In other work (Myatt and Ronayne, 2025*a*,*b*) we study richer consideration sets together with our stable prices solution concept, but make progress by joining Armstrong and Vickers (2022) in specifying symmetric marginal costs.

[11]Arnold and Zhang (2014) showed that the introduction of a fixed access fee to the Varian (1980) model leaves the symmetric equilibrium (like the one studied by Baye and Morgan, 2001) as the unique equilibrium.

[12]Equilibria are unaffected if the cost of serving captive customers is different from serving shoppers.

A mass of $\lambda_i > 0$ customers are "captive" to firm i. A mass of $\lambda_S > 0$ customers are "shoppers" who buy from the cheapest firm, or from one of the cheapest (in the event of a tie).[13]

Firm i earns $\lambda_i(p_i - c_i)$ from its captive customers and $\lambda_S(p_i - c_i)$ from the shoppers if it sells to them. These components sum to form a (risk neutral) firm's payoff. This specification nests that of Varian (1980) if $\lambda_i = \lambda$ and $c_i = c$ for every i, so that firms are symmetric.[14]

Equilibrium Play. Central to a characterization of equilibrium play is to identify the firms that compete for sales to shoppers. This depends on a notion of a firm's potential "aggression", which is associated with a firm's willingness to set a low price in order to win over those shoppers.

Firm i guarantees a profit of at least $\lambda_i(v - c_i)$ by setting $p_i = v$ and selling only to captive customers. The lowest price it would be willing to set in order to win the business of shoppers is $p_i^\dagger$, satisfying $\lambda_i(v - c_i) = (\lambda_i + \lambda_S)(p_i^\dagger - c_i)$, or explicitly

$$p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i}{\lambda_i + \lambda_S}. \tag{1}$$

This lowest undominated price is a measure of how aggressive (in terms of pricing) a firm is willing to be. It is higher when a firm has more captive customers (it is more costly to lose revenue from them by lowering price) or when the marginal cost of serving shoppers is higher (making it less tempting to serve those shoppers). Using this measure, firm j is strictly more aggressive than firm i and if and only if $p_j^\dagger < p_i^\dagger$ which holds if and only if

$$\underbrace{(c_i - c_j)}_{\text{cost adv. } j \text{ vs. } i} > \frac{v - (c_i + c_j)/2}{\lambda_S + (\lambda_i + \lambda_j)/2} \underbrace{(\lambda_j - \lambda_i)}_{\text{captive adv. } j \text{ vs. } i}. \tag{2}$$

If firm j has an advantage over firm i on the supply side (lower costs) and a disadvantage on the demand side (fewer captives) then this holds; otherwise, this inequality can break either way.

We choose labels for the firms (without loss of generality) so that the three highest-indexed firms n, $n-1$, and $n-2$ are the most aggressive: $p_n^\dagger \le p_{n-1}^\dagger \le p_{n-2}^\dagger \le \min_{i \in \{1,\dots,n-3\}} p_i^\dagger$.

[13]For technical convenience we break any ties in favor of a lowest-cost firm. This allows us to apply an off-the-shelf equilibrium-existence result (Dasgupta and Maskin, 1986, Theorem 5).

[14]It also nests Baye, Kovenock and De Vries (1992) and Kocas and Kiyak (2006), which allowed for asymmetry in "captives", $\lambda_i \neq \lambda_j$ for $i \neq j$, but retained common (and so without further loss of generality, zero) marginal costs.

Our notion of aggression, and so the ordering of the firms, focuses on the lowest price that a firm would charge in order to be sure of capturing shoppers. A more general notion asks how confident a firm would need to be of such capture in order to charge each possible price $p \leq v$.

To explore briefly a more general notion of pricing aggression, we write $\underline{w}_i(p)$ for the "minimum win probability" (of winning the business of shoppers) for firm i contemplating charging a price p instead of simply exploiting captive customers. Explicitly, this is the probability that satisfies

$$\lambda_i(v - c_i) = (\lambda_i + \lambda_S \underline{w}_i(p))(p - c_i) \quad \text{and so} \quad \underline{w}_i(p) = \frac{\lambda_i(v - p)}{\lambda_S(p - c_i)}. \tag{3}$$

In a contest setting (as in, for example, Siegel, 2009, 2010, 2012, 2014) the numerator (this is the lost profit from sales to captive customers from a discount of size $v - p$) is the cost of effort, while the denominator (the profit from sales to shoppers made at price p) is the value of the prize. Using this notation, the lowest undominated price for firm i satisfies $\underline{w}_i(p_i^\dagger) = 1$.

An unambiguous "aggression" ranking of firms i and j is obtained if $\underline{w}_i(p) < \underline{w}_j(p)$ for all relevant p: firm i is more willing to drop to price p to win the shoppers' business. From eq. (3) this is true if i has both a supply-side advantage ($c_i < c_j$) and demand-side disadvantage ($\lambda_i < \lambda_j$). However, if one inequality is switched then $\underline{w}_i(p)$ and $\underline{w}_j(p)$ can readily cross. This possibility arises naturally when one firm has an advantage in both dimensions. In our Supplemental Appendix (Appendix S) we build a full characterization of equilibria using minimum win probabilities. Here, we simply note that such functions that cross are required for equilibria in which more than two firms actively use sales.

Baye, Kovenock and De Vries (1992, Section V) found a unique Nash equilibrium when firms with the same marginal cost ($c_i = c$ for all i) have differently sized captive audiences: ordering firms so that $\lambda_n < \lambda_{n-1} < \lambda_{n-2}$, the equilibrium involves mixing (the "tango" of their paper) by firms $n - 1$ and n while firms $i \in \{1, \ldots, n - 2\}$ set $p_i = v$. In other situations (including symmetry) there can be many equilibria, all of which generate the same expected profits.[15]

In our more general setting we find (just as with cost symmetry) that a firm earns its "captive-only" profit, $\lambda_i(v - c_i)$, for $i < n$. The exception is firm n. By setting its price equal to $p_n^\dagger$ it would

[15] Under symmetry there are infinitely many Nash equilibria (Baye, Kovenock and De Vries, 1992, Theorem 1). Two firms mix over a common interval, while others may mix over any low subinterval of that interval. Johnen and Ronayne (2021, Proposition 1) showed that multiplicity depends on the absence of customers who compare exactly two prices.

guarantee to sell to all shoppers (recall that we order firms so that $p_n^\dagger$ is the lowest) and so would earn $p_n^\dagger(\lambda_n + \lambda_S)$ which (by construction of $p_n^\dagger$) is equal to its captive-only profit. However, it has the option to raise its price to match the lowest undominated price of the next-most aggressive competitor by setting $p_n = p_{n-1}^\dagger$. Following this price rise of $p_{n-1}^\dagger - p_n^\dagger$ it continues to sell to all shoppers and its captives, and so earns $(p_{n-1}^\dagger - p_n^\dagger)(\lambda_n + \lambda_S)$ on top of its captive-only profit.

Proposition 1 (Nash Equilibrium and Profits). *For any parameter values, there exists a Nash equilibrium of the single-stage game in which firm i's expected profit is given by*

$$\pi_i = \underbrace{\lambda_i(v - c_i)}_{\textit{captive-only profit}} + \begin{cases} (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger) & \textit{if } i = n\textit{, and} \\ 0 & \textit{otherwise,} \end{cases} \tag{4}$$

so that only a most aggressive firm earns (weakly) more than its captive-only (expected) profit.

For generic parameter values satisfying $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$, eq. (4) *holds in any Nash equilibrium.*

Our main proofs are contained within Appendix A; some details are relegated to our Supplemental Appendix.

We noted (in our introduction) that a model of sales is a contest: the effort cost is the captive profit lost from a lower price, while the shoppers are the prize. A statement equivalent to Proposition 1 using a contest setting can be derived from Theorem 1 and Corollary 2 of Siegel (2009).[16]

Proposition 1 says that the profits given by eq. (4) arise in the non-generic case of $p_n^\dagger = p_{n-1}^\dagger$ or $p_{n-1}^\dagger = p_{n-2}^\dagger$, but leaves scope for other profit levels too. However, we deem any other equilibria to be "pathological" in the sense that a small perturbation to parameters away from such a case would cause a discontinuous change to equilibrium profits—back to those given by eq. (4).

Definition. *An equilibrium is* pathological *if it gives payoffs that differ from those of eq.* (4).

For the remainder of the main paper, we do not consider pathological equilibria further.[17]

[16] In our Supplemental Appendix we provide an explicit mapping between Siegel's contest setting and models of sales.
[17] We give an example of one in our Supplemental Appendix and show that in any, profits differ from eq. (4) for exactly one firm.

Note that (at most) one firm strictly benefits from its access to shoppers beyond its captive-only profit. With symmetric captive bases ($\lambda_i = \lambda$ for all i), profits are the same as if firms offered a discriminatory price to shoppers: firm n earns $\lambda(v - c_n) + \lambda_S(c_{n-1} - c_n)$, which is what it would earn if shoppers are served by it (the lowest-cost firm) at a price equal to the second-lowest cost.[18]

More generally, by pricing below $p^\dagger_{n-1}$, the most aggressive firm n is sure to sell to all shoppers. Any higher price invites an "undercut" from the next-most aggressive firm, $n-1$, and for prices ranging upward from $p^\dagger_{n-1}$ we see the (familiar) mixing from (at least) two firms. In the classic setting (with asymmetric captive shares) only two firms are involved in such randomized sales.[19]

The "two to tango" result does not extend fully. Such a tango is danced by the two most aggressive firms over some interval ranging upward from $p^\dagger_{n-1}$. However, there is a possibility that (for higher prices) other firms step onto the dance floor. For strictly asymmetric firms, there are situations in which more than two tango. We will identify the exact condition so that only two tango. A sufficient condition is that the most aggressive firms have the two smallest captive audiences.

Our Supplemental Appendix describes the (unique, generically) construction of an equilibrium for all parameters.[20] Here, however, to ease exposition we focus on (generic) cases satisfying $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$. We use F_i to refer to the cdf of firm i's prices, which may be in mixed strategies in equilibrium.

Proposition 2 (Nash Prices I: When Two Tango). *Suppose that $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.*

(i) Firms $i < n$ place an atom at $p_i = v$, while n mixes continuously over all $p \in [p^\dagger_{n-1}, v]$.

(ii) There is $p^\ddagger \in (p^\dagger_{n-2}, v]$ such that for $p \in [p^\dagger_{n-1}, p^\ddagger)$, $F_i(p) = 0$ for $i \le n-2$, while

$$F_n(p) = \frac{(p - p^\dagger_{n-1})(\lambda_{n-1} + \lambda_S)}{\lambda_S(p - c_{n-1})} \quad \textit{and} \quad F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda_n + \lambda_S)}{\lambda_S(p - c_n)}. \tag{5}$$

(iii) If $c_n \le c_{n-1}$ then $F_{n-1}(p)$ first order stochastically dominates $F_n(p)$.

[18] As usual, a Bertrand construction requires a careful treatment of tie-break rules; for example by breaking a tie in favor of a lowest-cost firm. Our reference to discriminatory pricing refers to unit-demand customers. A more general analysis of captive-vs-shoppers discrimination was reported by Armstrong and Vickers (2019).

[19] If $c_n < \cdots < c_1$, then for sufficiently symmetric captive audience sizes, firms n and $n-1$ are the most aggressive and the only two that mix (we approach the "two to tango" of Baye, Kovenock and De Vries, 1992, Section V).

[20] Siegel (2010) described an algorithm to construct an equilibrium in a related class of contest games. His approach applies to a duopoly model of sales, but not for the broader oligopoly environment studied here; see our Footnote 8.

(iv) If $\lambda_{n-1} \leq \min_{i \in \{1,\ldots,n-2\}}\{\lambda_i\}$, then $p^{\ddagger} = v$ and so all firms $i \in \{1, \ldots, n-2\}$ choose $p_i = v$ and serve only captives, while firms n and $n-1$ mix via eq. (5) *over prices $p \in [p^{\dagger}_{n-1}, v)$.*

Equation (5) characterizes a "tango" by firms $n-1$ and n below $p^{\dagger}_{n-2}$. Firm $n-1$ can earn its captive-only profit of $\lambda_{n-1}(v - c_{n-1})$ by charging v. It is indifferent to charging $p < v$ if

$$\underbrace{(v-p)\lambda_{n-1}}_{\text{loss on captives}} = \underbrace{(p - c_{n-1})\lambda_S(1 - F_n(p))}_{\text{gain from shopper sales}}, \tag{6}$$

which solves for $F_n(p)$. The desire to compete for shoppers is lessened if a firm has more captives, and sales to shoppers are less valuable if its marginal cost is higher. Any lower-indexed (and less aggressive) firm $i \in \{1, \ldots, n-2\}$ that has both more captives ($\lambda_i > \lambda_{n-1}$) and higher costs ($c_i > c_{n-1}$) has a strictly weaker incentive to set a price $p < v$. Such a firm does not wish to "step onto the dance floor" and so (if this is true, and in fact under weaker conditions) we can construct a unique equilibrium in which firms $n-1$ and n "tango" with the distributions of eq. (5) all of the way up to v. In this case the oligopoly reduces to a de facto duopoly for which the analysis of Golding and Slutsky (2000) applies and where a characterization based on contests (Siegel, 2010) can also be used. Other (lower indexed) firms are simply wallflowers.[21]

However, a less aggressive firm $i \leq n-2$ might have both higher costs but fewer captives. This combination can result in a higher $p^{\dagger}_i$ but a greater temptation to charge some intermediate price. To demonstrate this explicitly, we (attempt to) construct a "two to tango" scenario in which firms n and $n-1$ mix over $[p^{\dagger}_{n-1}, v)$ according to the distributions reported in eq. (5), while other firms charge v. For this to be an equilibrium, we must be sure that this condition holds:

$$(v-p)\lambda_i \geq (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)) \quad \text{for all} \quad p \in [p^{\dagger}_{n-1}, v) \quad \text{and} \quad i \leq n-2, \tag{7}$$

where $F_n(p)$ and $F_{n-1}(p)$ are taken from the explicit solutions of eq. (5).

Suppose, however, $c_i \in (p^{\dagger}_{n-1}, v)$. This guarantees that $p^{\dagger}_i > p^{\dagger}_{n-1}$, and so firm i is not one of the two most aggressive firms. We can now choose λ_i sufficiently small such that eq. (7) fails. This means that there is a price at which firm i wishes to join the dance floor.

[21]In the study of contests, "participation" in the contest is akin to the "dancing" we and others refer to in the study of models of sales. Siegel (2012) identifies when any number of players in a contest can participate in equilibrium. A difference to models of sales is that there the valuation of the prize is independent of the player's choice variable.

This argument adds a third firm to disrupt the tango danced by two existing firms. By choosing this firm's marginal cost to be high, we guarantee that the two existing firms compete in a lower price range. However, if this third firm has a sufficiently small captive base then it wishes to join the action at some point, to sell to shoppers at least sometimes. This implies that any equilibrium must involve mixing from that third firm. This logic extends such that if $k \geq 2$ firms play mixed strategies in equilibrium there are parameters such that an additional firm wants to join the dance.

Proposition 3 (Nash Prices II: When Three or More Tango).

(i) For any $k \geq 2$ there is an open set of parameters such that k firms uses sales in equilibrium.

(ii) The support for any firm that uses sales is a single interval plus a possible atom at v. The intervals for firms $i < n$ do not overlap. They partition $[p^{\dagger}_{n-1}, v]$, which is the support of firm n.

(iii) If $p^{\dagger}_{n} < p^{\dagger}_{n-1} < p^{\dagger}_{n-2}$, then at least three firms uses sales if and only if eq. (7) fails.

A firm that joins a dance "late" (rather than at the "start", i.e., at $p^{\dagger}_{n-1}$) may be a new entrant with higher costs and a smaller captive base. Indeed, if its captive base is small then it has an incentive to use a production technology with a relatively high marginal cost if that results in a lower fixed cost. (This corresponds to technology choice studied in Section 2.) This suggests that the presence of such a firm (and so randomized sales by more than two firms) is not only a theoretical curiosity.

We earlier summarized how the preceding literature covers symmetric parameters or asymmetries on one side of the market. In the first instance many firms can mix over a common interval in equilibrium. In the latter case, no more than two firms can mix over a common interval. Our Proposition 3 shows (as we state below as a corollary) that this result is maintained, despite the fact that any number of firms can mix in equilibrium.[22] Notably, Shelegia and Wilson (2021) studied an asymmetric-cost model in which costly advertising is required for shoppers to see prices and sought equilibria in which advertising firms must mix over a common interval up to the monopoly price

[22]As the equilibrium derivation in our Supplemental Appendix shows, were three firms to dance simultaneously, the density of one firm's strategy would have to turn negative in order to keep the others indifferent.

(see their p. 204).[23] Our results show this is not a possible equilibrium feature when advertising costs are removed (as in a traditional model of sales); firms only "dance in pairs."

Corollary 1 (Dancing in Pairs). *For generic parameter values, there does not exist an equilibrium in which three or more firms randomize continuously over any given interval of prices.*

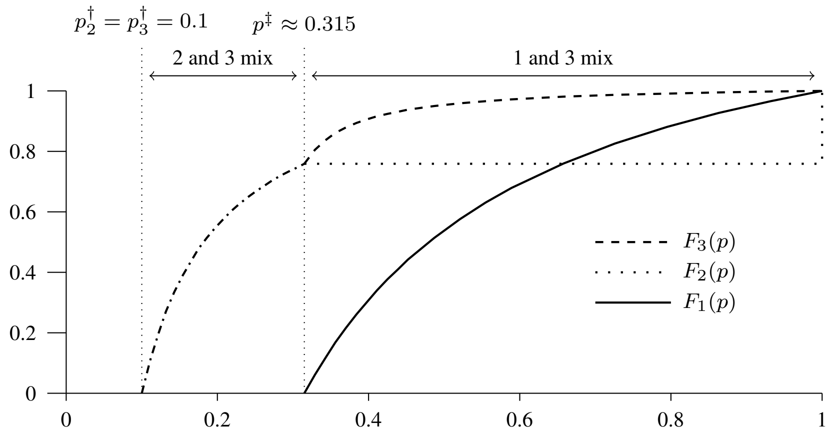
When parameters are "asymmetric enough on both sides" so that three or more firms engage in randomized sales, the pattern of sale prices they use takes a particular form, which we now discuss.

Randomized Sales from Multiple Firms. In our Supplemental Appendix we offer an equilibrium characterization that is unique except for knife-edge cases (such as exact symmetry). If k firms play mixed strategies then in equilibrium the interval $[p^{\dagger}_{n-1}, v)$ is partitioned into $k-1$ subintervals. The most aggressive firm n mixes over the entire range of prices. However, each of the other $k-1$ firms mix only over a single subinterval. Moving upward through prices, the "dance partner" of firm n switches. More than two firms mix; but only two firms mix within any particular interval of prices.

Propositions 2 and 3 confirm existence and properties of equilibria with $k > 2$. Here we build an illustrative "thrango" example with $k = n = 3$. For simplicity and to simplify exposition, two firms are symmetric, but this is easily modified so that the three firms are (generically) different.

Specifically, assume that firms 2 and 3 have the same characteristics, comprising low costs and larger captive audiences: $c_2 = c_3 = 0$ and $\lambda_2 = \lambda_3 = \lambda_H$. Firm 1 has higher costs, but a smaller captive audience: $c_1 = c > 0$ and $\lambda_1 = \lambda_L$ where $\lambda_L < \lambda_H$. By setting $c > (\lambda_H - \lambda_L)v/(\lambda_H + \lambda_S)$ we guarantee that firm 1 is less aggressive than the others. An equilibrium involves the two jointly

[23] Specifically, Shelegia and Wilson (2021, Corollary 1 point ii) showed that when such advertising costs are symmetric and close enough to a specific positive value, then an equilibrium with common-interval mixing (should one exist) involves all firms. In addition, if symmetric advertising costs are sufficiently small, then in such an equilibrium (again, should one exist) common-interval mixing involves only two firms (Corollary 1, point i). Their stated claim was that "only two firms use sales when advertising costs are sufficiently small," which contrasts with our Corollary 1. What they showed is that if more than two firms dance, then they cannot do so over a common interval. This leaves open the possibility that an equilibrium can involve pairwise mixing over different intervals; a property they ruled out by assumption. Their equilibrium construction also used several other features. They required buyers to treat as equivalent non-advertising firms and those that advertise the monopoly price; this essentially prevents the existence of a best reply to undercut the set of non-advertising competitors. In the case that no firm advertises, they required specific tie-break probabilities.



This illustrates the mixed-strategy distribution functions for a triopoly in which all three firms use randomized sales. The specification is from the text, where $c_2 = c_3 = 0$, $\lambda_2 = \lambda_3 = \lambda_H = 0.1$, $\lambda_S = 0.9$, $v = 1$ and so $p_2^\dagger = p_3^\dagger = 0.1$. For the less aggressive firm, $\lambda_1 = \lambda_L = 0.005$ and $c_1 = c = 0.25$. There are two distinct "dance floor" segments, with firm 3 "partner swapping" from firm 2 to firm 1 at $p^\ddagger \approx 0.315$.

FIGURE 1. Mixing CDFs in a Three-Firm "Thrango" Example

most aggressive firms engaged in a "tango" over the lower interval of prices, with

$$1 - F_3(p) = 1 - F_2(p) = \frac{\lambda_H(v-p)}{\lambda_S p}. \tag{8}$$

The inequality in eq. (7) holds for all relevant p if and only if $\lambda_L \geq [\lambda_H^2(v-c)^2]/[4\lambda_S vc]$. Our "thrango" situation arises when this fails, as it does, at a price $p^\ddagger$, for $\lambda_L > 0$ sufficiently small. If so, then the "tango" between firms 2 and 3 ends at the price $p^\ddagger$, which satisfies eq. (7), and here is:

$$p^\ddagger = \frac{v + c - \sqrt{(v+c)^2 - 4\left(1 + [\lambda_L\lambda_S/\lambda_H^2]\right)vc}}{2\left(1 + [\lambda_L\lambda_S/\lambda_H^2]\right)}. \tag{9}$$

At this point, $p^\ddagger$, there is a partner swap: firm 2 shifts all remaining weight to price at v, while firm 1 then begins mixing.[24] Over the interval $[p^\ddagger, v)$ the relevant mixing distributions are

$$1 - F_3(p) = \frac{\lambda_L}{\lambda_H}\frac{v-p}{v-p^\ddagger}\frac{p^\ddagger}{p-c} \quad \text{and} \quad 1 - F_1(p) = \frac{v-p}{v-p^\ddagger}\frac{p^\ddagger}{p}. \tag{10}$$

The equilibrium CDFs of this example are illustrated in Figure 1 for suitable parameter choices.

[24]Because firms 2 and 3 are completely symmetric in this example, there is a second equilibrium in which firm 3 (instead of 2) leaves the dance floor and firms 1 and 2 mix over $[p^\ddagger, v)$. Departures from symmetry leave only one.

Changing Costs. The asymmetric solution allows us to vary costs . To ease exposition, we consider local changes that maintain the ranking $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. We also assume (a sufficient condition is $\lambda_{n-1} \leq \min_{i\in\{1,\ldots,n-2\}}\{\lambda_i\}$) that the equilibrium involves mixing by only two firms. Each firm $i \in \{1, \ldots, n-2\}$ sets $p_i = v$, which is unaffected by any local cost changes, and such firms' costs of do not influence prices. The interesting exercises concern firms n and $n-1$.

Inspecting eq. (5), c_n enters only into the solution for firm $n-1$. The distribution $F_{n-1}(p)$ is increasing in c_n: an increase in the marginal cost of firm n lowers the prices charged by firm $n-1$. This is because firm $n-1$ prices more aggressively to maintain the incentive for the (now more costly) firm n to price at $p_{n-1}^\dagger$ rather than the (now more attractive, given the higher cost) higher prices within $[p_{n-1}^\dagger, v)$. This implies that the captive customers of firm $n-1$, as well as the shoppers, benefit from any cost increase suffered by the most aggressive firm that also most often supplies the shoppers (as we confirmed via claim (iv) of Proposition 2).

The cost c_{n-1} of firm $n-1$ has a more conventional impact. A direct effect of an increase in c_{n-1}, by inspection of eq. (5), is to increase $F_n(p)$ and so to push down the prices charged by firm n. (This follows from the logic discussed just above.) However, an increase in c_{n-1} also raises the lower bound of the interval of sales prices charged by both firms, $p_{n-1}^\dagger$. This lowers both $F_{n-1}(p)$ and $F_n(p)$. There are competing effects on $F_n(p)$, but overall the impact (as the proof of the next proposition confirms) is to push up the prices charged by both competitors.

Proposition 4 (The Effect of Costs on Mixed-Strategy Prices). *Suppose that* $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. *Further suppose that eq.* (7) *is satisfied, so that only firms* $n-1$ *and* n *use sales.*

(i) No price changes in response to local changes in the cost c_i *of any firm* $i \in \{1, \ldots, n-2\}$.

(ii) A local increase in c_{n-1} *shifts rightward the distributions of prices charged by* $n-1$ *and* n.

(iii) A local increase in c_n *shifts leftward the distribution of prices charged by* $n-1$.

Claim (iii) implies that firm n disproportionately gains from any reduction in its marginal cost. A reduction in c_n has the usual direct effect on its profit. However, it also prompts a price rise from its competitor (in the market for sales to shoppers) firm $n-1$. This is a positive strategic effect.

(Notably, a negative strategic effect is more common in pricing games.) In fact,

$$\frac{\partial \pi_i}{\partial c_i} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = n, \text{and} \\ \lambda_i & \text{otherwise,} \end{cases} \tag{11}$$

and so a given firm gains more from a cost reduction when it is the most aggressive than when it is not. This suggests that there are asymmetric incentives to engage in innovative activity. We develop this observation when we study endogenous technology choice in Section 2.

Efficiency. The only efficiency-relevant question is: who serves the shoppers? The outcome is efficient only if shoppers are served by the lowest cost firm.[25] Here, however, they are served by the two most aggressive firms. These shopper-serving firms may not have low costs (resulting in greater inefficiency) if the aggression is because (for example) low-cost firms have bigger captive audiences. We return to welfare considerations toward the end of Section 2.

2. Process Innovations and Endogenous Asymmetry

From eq. (11), the most aggressive firm gains distinctly more (relative to another firm with a similar captive audience) from a reduction in its marginal cost. Here we explore such cost reductions via the consideration of a prior-stage innovation game followed by model-of-sales pricing.

We have several reasons for pursuing this avenue of inquiry. Firstly, and pragmatically, our analysis presents us with a natural opportunity to do so. Secondly, the asymmetric incentives to reduce costs suggests that asymmetric costs might arise endogenously. This reinforces the requirement for an asymmetric-cost model. Thirdly, we are able to evaluate the likely cross-sectional relationship between captive-audience size, marginal cost, and the use of sales including whether the conditions from Proposition 3 are met such that three or more firms partake in sales. Finally, we ask: to what extent do the "right" firms serve the shoppers, and is innovation efficient in a model-of-sales world?

We also note here that innovation is equivalent (in a familiar way) to a choice of production technology: a higher fixed cost to achieve a lower marginal cost can be thought of as a shift from a

[25]This is always true if costs are symmetric, and there is no welfare analysis in classic papers (Varian, 1980; Baye, Kovenock and De Vries, 1992). Welfare in a duopoly was considered by Golding and Slutsky (2000).

labor-intensive to a capital-intensive production technique. We can ask, therefore, whether firms that use sales (rather than those who are captive-focused) employ different technologies.

A Model of Process Innovation. Prior to the pricing stage firms choose their production technologies via costly innovations. We study the following two-stage game.

(1) Firms simultaneously choose and observe production technologies, denoted by z_i.
(2) Firms play a Nash equilibrium of the pricing game (Section 1) with the profits of eq. (4).[26]

We interpret a technology choice, $z_i \in [0, \bar{z}_i]$, as a fixed-cost expenditure which lowers the marginal cost of production: it is a (costly) process innovation. It is equivalent to a product innovation that raises valuations, and so what really matters is the net surplus, $v - c_i$, created. We assume that

$$v - c_i = V_i(z_i), \tag{12}$$

where $V_i(z_i)$ is positive, smoothly increasing, and concave. We adopt the regularity conditions $\lambda_i V_i'(0) > 1 > (\lambda_i + \lambda_S)V_i'(\bar{z}_i)$ so that innovation choices satisfy $z_i \in (0, \bar{z}_i)$.

We have (so far) labeled firms n and $n-1$ as most aggressive. Here that status is endogenous because firms choose their technologies. For now, then, we do not allocate such labels.

Deducting z_i from the gross equilibrium expected profit from eq. (4), a firm's net profit is

$$\pi_i = -z_i + \begin{cases} (\lambda_i + \lambda_S)V_i(z_i) - \lambda_S(\lambda_i + \lambda_S)\max_{j \neq i}\left[\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right] & \frac{V_i(z_i)}{\lambda_i + \lambda_S} > \max_{j \neq i}\left\{\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right\}, \\ \lambda_i V_i(z_i) & \text{otherwise.} \end{cases} \tag{13}$$

The first case applies when firm i is the most aggressive (so that $p_i^\dagger < \min_{j \neq i} p_j^\dagger$) and the second case applies if not. Using these expected profit expressions as the outcomes from stage (2) described above, we study a simultaneous-move "innovation game": stage (1) described above.

In contrast to prices, we think of a firm's technology choice as a longer-term decision subject to adjustment following the choices of others. We look, therefore, for choices from which no firm would deviate ex post: a pure-strategy equilibrium Nash equilibrium of the innovation game.

[26]To solve for the technology choices at stage (1) requires only the equilibrium expected profits from stage (2). Whereas the equilibrium characterization for stage (2) is novel to this paper, the equilibrium profit expressions are not: as we have noted, these can be derived in a contest setting from Theorem 1 and Corollary 2 of Siegel (2009).

Nevertheless, there are parameters under which there are multiple pure-strategy equilibria and so there will also be mixed-strategy equilibria. Such mixed equilibria are not our focus.[27]

Asymmetric Equilibrium Innovation. We now consider the possible profiles of equilibrium innovation choices. Inspecting eq. (13), the response of firm i's profit to a local change in z_i depends on whether it expects to be the most aggressive firm. If it does not, then its expected profit $\lambda_i V(z_i) - z_i$ is equal to that obtained from fully exploiting captive customers. For any "wallflower" firms that do not use sales, this is literally true: such a firm expects to act as a monopolist over its captive customers and to extract all surplus from them. This profit expression also applies, however, to other firms (other than the most aggressive firm) that use sales.

If a firm does expect to be the most aggressive, however, then its expected profit is instead equal to $(\lambda_i + \lambda_S)V_i(z_i) - z_i$ minus a term that does not depend on z_i. Such a firm faces an incentive to act as if it fully exploits a customer base of size $\lambda_i + \lambda_S$, giving it (relative to another firm with a similarly sized actual captive audience) a heightened incentive to innovate. In fact, we find that

$$\frac{\partial \pi_i}{\partial z_i} = -1 + V_i'(z_i) \begin{cases} \lambda_i + \lambda_S & \text{if } \frac{V_i(z_i)}{\lambda_i+\lambda_S} > \max_{j \neq i} \left\{ \frac{V_j(z_j)}{\lambda_j+\lambda_S} \right\} \text{ (so that } i \text{ is most aggressive)} \\ \lambda_i & \text{if } \frac{V_i(z_i)}{\lambda_i+\lambda_S} < \max_{j \neq i} \left\{ \frac{V_j(z_j)}{\lambda_j+\lambda_S} \right\}, \end{cases} \tag{14}$$

implying that the profit of firm i has a (convex) kink when i becomes the most aggressive firm. This kink implies that firm i will never optimally choose z_i such that $p_i^\dagger = \min_{j \neq i}\{p_j^\dagger\}$ and so, in equilibrium, the most aggressive firm is uniquely identified. Furthermore, the expressions for $\partial \pi_i / \partial z_i$ (and so any potential first-order condition solutions) do not depend on z_j for $j \neq i$.

Drawing these observations together, we identify two possible solutions for firm i's innovation choice: z_i^H for when it is the most aggressive firm, and z_i^L for when it is not. These solutions are uniquely determined by the two respective conditions

$$1 = \lambda_i V_i'(z_i^L) \quad \text{and} \quad 1 = (\lambda_i + \lambda_S) V_i'(z_i^H), \tag{15}$$

they satisfy $z_i^H > z_i^L$, and they do not depend on the innovation choices of any other firms.

[27] This contrasts with the mixed equilibria of our pricing game. In fact, mixing is not crucial for generating the expected payoffs of Proposition 1. In an older working version of this paper we generated pure-strategy play on the equilibrium path of natural pricing games (using the ideas in Myatt and Ronayne, 2025*b*) in the same asymmetric models of sales environment. The expected payoffs to firms in those games coincide with what we report here.

There are n candidate equilibrium profiles. For each candidate we choose a firm $i \in \{1, \ldots, n\}$ to be most aggressive, set $z_i = z_i^H$, and $z_j = z_j^L$ for $j \neq i$. Two checks are then needed. Firstly, firm i must be the most aggressive firm. This requires $p_i^\dagger < \min_{j \neq i}\{p_j^\dagger\}$ or equivalently

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j \neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\} \quad \Leftrightarrow \quad \frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j \in \{1,\ldots,n\}}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}. \tag{16}$$

Secondly, we need to check that firm i does not wish to deviate back to z_i^L, and that no $j \neq i$ wishes to deviate from z_j^L to z_j^H. For example, if a firm i uniquely maximizes (across firms) both $V_j(z_j^L)/(\lambda_j + \lambda_S)$ and also $V_j(z_j^H)/(\lambda_j + \lambda_S)$ then it can always take the "most aggressive" role in the innovation game. Proposition 5 establishes more generally one firm takes this role.[28]

Proposition 5 (Asymmetric Innovation Equilibria). *There exist pure-strategy equilibria. In any, there is a uniquely most aggressive firm at the pricing stage. If firms are symmetric ex ante then exactly one firm innovates more, operates with a lower marginal cost, uses sales, and most often supplies the shoppers, while other firms make symmetric innovation choices.*

If firms are ex-ante symmetric (with the same sized captive audiences and the same technological opportunities) then innovation and pricing choices are asymmetric. This provides a distinct rationale for opening up models of sales to cost asymmetry. It is especially relevant because (with innovation) it arises ex post even when firms are ex ante symmetric.

In that symmetric scenario, the lead innovator and one other firm use sales while all other firms sit out (as "wallflowers"). This further emphasizes the necessity of two-sided asymmetry (in both captive audiences and marginal costs) in order to rationalize more than two firms using such sales.

From Demand-Side to Supply-Side Asymmetry. In the presence of endogenous innovation, which firms ultimately engage in sales? To address this, we specify symmetric technological opportunities, so that $v - c_i = V(z_i)$ for all i, and we place firms in size order, $\lambda_1 < \cdots < \lambda_n$.

We first identify key properties of this innovation production function to determine how changes in expected output determines technology choice. Consider the socially optimal technology choice z_Q when Q units are supplied. This satisfies the first-order condition $QV'(z_Q) = 1$. As output

[28] We provide a further characterization of equilibria of the innovation game in our Supplemental Appendix; see Proposition S1.

rises, the optimally higher choice of z_Q raises the per-customer surplus $V(z_Q)$. We say that the technological opportunity for innovation is *strong* if this surplus reacts elastically to output.

Definition. *The* technological opportunity for innovation *is* strong *if the socially optimal per-customer surplus always responds elastically to an increase in output. Similarly, that opportunity is* weak *if the socially optimal surplus always responds inelastically.*

This notion is relevant because a firm with a larger captive audience has a greater incentive to innovate. These features (more captives but lower marginal cost) have opposing effects on its desire to charge a lower price. Whether the second factor outweighs the first factor turns on whether the (endogenously chosen) per-customer surplus reacts elastically to an increase in output.[29]

We suppose that the opportunity for innovation is strong (unless noted otherwise). If firms had the same (exogenous) marginal cost then firm 1 would be the most aggressive. Here, however, larger (captive audience) firms innovate sufficiently to be able to become more aggressive. In fact,

$$\frac{V(z_1^H)}{\lambda_1+\lambda_S} < \cdots < \frac{V(z_n^H)}{\lambda_n+\lambda_S} \quad \text{and also} \quad \frac{V(z_1^L)}{\lambda_1+\lambda_S} < \cdots < \frac{V(z_n^L)}{\lambda_n+\lambda_S}. \tag{18}$$

Firm i can obtain the most aggressive position if the inequality of eq. (16) holds. Here this is

$$\frac{V(z_i^H)}{\lambda_i+\lambda_S} > \frac{V(z_n^L)}{\lambda_n+\lambda_S}. \tag{19}$$

The left-hand side is increasing in λ_i and so the firms satisfying this inequality are $\{k,\ldots,n\}$ for some $k \in \{1,\ldots,n\}$; these are the firms with larger captive bases. Moreover, the left-hand side of eq. (19) is increasing in λ_S while the right-hand side is decreasing in λ_S. This means as the population of shoppers falls (λ_S shrinks), k rises. For $i<n$,

$$\lim_{\lambda_S\downarrow 0}\frac{V(z_i^H)}{\lambda_i+\lambda_S} = \frac{V(z_i^L)}{\lambda_i} < \frac{V(z_n^L)}{\lambda_n} = \lim_{\lambda_S\downarrow 0}\frac{V(z_n^L)}{\lambda_n+\lambda_S}, \tag{20}$$

[29] Per-customer surplus reacts elastically to output if the elasticity of $V(z)$ exceeds its curvature:

$$\partial[V(z_Q)/Q]/\partial Q > 0 \Leftrightarrow z_Q V'(z_Q)/V(z_Q) > -z_Q V''(z_Q)/V'(z_Q) \tag{17}$$

The elasticity measures the impact of innovation on per-customer surplus; the curvature is the elasticity of the slope of $V(z)$ and so measures how quickly the incentive to innovate falls as innovation increases. These measures jointly determine how the (endogenously chosen) impact of innovation reacts to output. The first set of equalities in eq. (18) is implied by eq. (17), and the second set follows straightforwardly. Specifically, for firm i we write $V(z_i^L)/(\lambda_i+\lambda_S) = [\lambda_i/(\lambda_i+\lambda_S)] \times [V(z_i^L)/\lambda_i]$. The first term is (trivially) increasing in λ_i (and so increasing in i given the ordering of the firms) and the second term is also increasing given that the technological opportunity for innovation is strong.

and so if λ_S is sufficiently small then eq. (19) must fail for any firm $i < n$. This leads to the finding that the firm with the smallest captive audience is uniquely able to take the most-aggressive position when there are sufficiently few shoppers.[30]

For our statements in the following proposition we write $z_i^\star$ and $c_i^\star = v - V(z_i^\star)$ for the equilibrium innovation choices and final marginal costs of firms in equilibrium.

Proposition 6 (The Most Aggressive Firms). *Suppose firms have the same technological opportunities for innovation, that the opportunity is strong, that $\lambda_1 < \cdots < \lambda_n$, and that there is a unique equilibrium (for example, if λ_S is small). Then, larger firms innovate more, so that $z_1^\star < \cdots < z_n^\star$, operate with lower marginal costs, so that $c_1^\star > \cdots > c_n^\star$, and are more aggressive: $p_1^\dagger > \cdots > p_n^\dagger$. The shoppers are served by (at least) the two largest firms. Prices satisfy*

$$\mathrm{E}[p_n] < \mathrm{E}[p_{n-1}] < \mathrm{E}[p_{n-2}] \leq \cdots \leq \mathrm{E}[p_1]. \tag{21}$$

With strong technological opportunities for innovation, firms with larger captive audiences have disproportionately lower costs, which is sufficient to make them more aggressive than other (smaller, in terms of captives) firms. The condition ($c_n \leq c_{n-1}$) needed for claim (iii) of Proposition 2 is met, and we can rank (expected) prices: shoppers are most often served by the lowest-cost and (in expectation) cheapest firm. However, the condition for claim (iv) of Proposition 2 fails, and so the ingredients are in place for a possible pricing equilibrium in which three (or more) firms engage in randomized sales to win the shoppers. As Proposition 3 revealed, equilibrium features three or more firms engaging in randomized sales if and only if eq. (7) is violated for some firm $i \leq n - 2$.

In contrast, with weak technological opportunities for innovation a smaller captive audience is the dominant factor driving aggression.[31] This brings the situation closer to the classic symmetric-cost

[30] More generally, we find the existence of a unique pure-strategy equilibrium does not depend on the strength of innovation opportunities (see Proposition S1 in our Supplemental Appendix). However, in the case of weak opportunities other parts of the analysis are different. For example, we can no longer be sure that the second set of inequalities from eq. (18) hold. We cover the case of weak opportunities fully in our Supplemental Appendix.

[31] See Proposition S2 in our Supplemental Appendix for the analog of Proposition 6 for the case of weak opportunities. These cases are cleanly illustrated when $V(z)$ takes a constant-elasticity form, so that $V(z) = \beta z^{\bar\gamma}$ where $\bar\gamma \in (0,1)$. (This specification is reminiscent the constant-elasticity relationship between production cost and research and development expenditure specified by Dasgupta and Stiglitz (1980, p. 273).) The elasticity and curvature of this function are $\bar\gamma$ and $1-\bar\gamma$ respectively. Technological opportunity is strong if and only if $\bar\gamma > \frac{1}{2}$. When $\bar\gamma > \frac{1}{2}$ (so that the innovation production function is sufficiently elastic) then the claims of Proposition 6 hold, so that the largest firm is the most aggressive; but if $\bar\gamma < \frac{1}{2}$ then the smallest firm is the most aggressive.

model in which the most aggressive firms are the smallest, and so the condition for claim (iv) of Proposition 2 holds. The unique equilibrium of the single-stage pricing game involves the classic "tango" and no dancing by other firms. However, the most aggressive firm does not have the lowest cost, and so claim (iii) of our Proposition 2 does not rank the pricing distributions of the dancers.

Efficiency. In our model all customers are supplied, and captives are restricted to be served by their respective captors. There are, therefore, two requirements for efficiency: to allocate the shoppers to the "right" firm, and to choose each firm's innovation appropriately.

Suppose firms all have access to the same technological opportunities and order them in order of their captive bases so that $\lambda_1 < \cdots < \lambda_n$. The efficient solution is to allocate all shoppers to the largest (captive audience) firm n, and ask firms to innovate optimally in response to this demand.[32] Efficient technology choices are then z_n^H and z_i^L for $i < n$, from eq. (15). Relative to this benchmark, there can be two kinds of inefficiency: shoppers are incorrectly allocated to firms; and the technology choices of (some) firms are not efficient given their expected sales.

To streamline discussion, suppose that the largest firm takes the most aggressive position.[33] We know that the two largest firms use sales and more generally k firms do so. In this situation, $n - k$ firms price at v and innovate efficiently: they focus solely on serving captive customers and use the correct technology. However, shoppers are allocated inefficiently amongst those who offer sales. To nudge output to the lowest-cost (and largest) firm the planner might use subsidies and taxes. Or, it could seek to increase the innovation of the larger firm and reduce that of smaller firms.

However, any intervention to influence technology choices faces a second problem: those choices are themselves inefficient. The innovation of the largest firm is z_n^H, which is optimal for a firm that serves shoppers with certainty; however, this firm does not. Similarly, the innovation of a firm $i < n$ that mixes is z_i^L, which is optimal for a firm that serves only its captives; but such a firm does sometimes sell to the shoppers. Fixing the allocation of shoppers, this means that the largest firm over-innovates and its competitors that mix under-innovate. This presents the planner with a

[32]We can think of allocating n different production technologies. To exploit the lowest marginal cost, we allocate the best technology to both the shoppers and the largest captive audience, and so firm n serves the shoppers.

[33]From Proposition S1 part (ii) such an equilibrium is unique when λ_S is sufficiently small.

tension: fixing the allocation of shoppers it would be preferable to shift innovation from firm n to firm i; but to induce a better allocation of those shoppers it would be better to induce the opposite.

Discriminatory Pricing. It is also instructive to compare the outcome here to when firms use discriminatory pricing. To do this, we simplify by assuming equal captive-audience sizes so that $\lambda_i = \lambda$ for all i. Dropping the subscript i, we write z^L and z^H for the optimal innovation choices that are associated with serving only captives, so that $1 = \lambda V'(z^L)$, or also shoppers, so that $1 = (\lambda + \lambda_S)V'(z^H)$, respectively. We then allow firms to set different captive and shopper prices.

At the second stage, let us order firms by innovation so that $z_1 \leq \cdots \leq z_n$, and so higher indexed firms have lower marginal cost. All firms charge v to their captive customers, and compete à la Bertrand for shoppers. That Bertrand game has (for $n \geq 3$) many pure-strategy equilibria, but all involve the lowest-cost firm n supplying all of the shoppers at a price equal to the marginal cost of the second-lowest-cost firm $n-1$. Firms' payoffs are $\lambda_i V(z_i) - z_i$ for $i < n$ and $\lambda_n V(z_n) + \lambda_S(V(z_n) - V(z_{n-1})) - z_n$. These payoffs are equal precisely to the expected payoffs obtained in an innovation game that is followed by uniform pricing.[34]

Three conclusions follow. Firstly, there are n equilibria in which one firm chooses innovation z^H and ultimately serves all shoppers while others choose z^L and serve only captives. Secondly, this is fully efficient: a single (lowest cost) firm supplies the shoppers, and all technologies are optimized. Finally, expected profits equal those obtained in a model of sales with uniform pricing, and so the efficiency gain from a shift to discriminatory pricing results in a higher consumer surplus.

3. The Demand Side: Shifting and Accessing Customers

In a symmetric-cost context, researchers have considered actions related to the demand side. Here we consider three demand-related extensions but within our asymmetric-marginal-cost framework.

[34]For an in-depth duopoly treatment of price discrimination with captive customers, see Armstrong and Vickers (2019).

Acquiring Captive Customers. Asymmetric technologies emerge because the expected profit of the most aggressive firm reacts in a distinctive way to a change in its costs. Chioveanu (2008) found a related effect when costly "persuasive" advertising determines captivity, or "brand loyalty."[35]

Her insights hold when costs are asymmetric. From eq. (13), if a firm i is not the most aggressive then its profit, $\pi_i = \lambda_i V_i(z_i) - z_i$, does not depend on the characteristics of competitors, and the incentive to acquire new captives corresponds exactly to the per-customer surplus that such a captive customer generates. In contrast, if a firm i is the most aggressive, then its expected profit is

$$\pi_i = (\lambda_i + \lambda_S)V_i(z_i) - \frac{\lambda_S(\lambda_i + \lambda_S)V_k(z_k)}{\lambda_k + \lambda_S} - z_i \quad \text{where} \quad k = \arg\max_{j \neq i} \left\{ \frac{V_i(z_j)}{\lambda_j + \lambda_S} \right\}, \qquad (22)$$

The first term (just as before) provides an incentive to acquire captive customers equal to the surplus $V_i(z_i)$. However, the second (negative) term is decreasing in λ_i and increasing in λ_k: it encourages the most aggressive firm to push captives to its nearest competitor. These effects were identified by Chioveanu (2008), and mean that a firm's profit as a function of its captive audience has a convex kink at the point where it becomes the most aggressive firm.[36] If we add a formal stage at which firms take actions to acquire captives, then the presence of this kink means that in equilibrium the most aggressive firm will be distinct. This feature is shared with our endogenous-innovation model. A difference between cost-reducing innovations and captive-audience acquisition of is that the distinct firm (the aggressive firm that expects to serve shoppers) faces a stronger incentive to innovate but a weaker incentive to recruit captives. However, the underlying force is the same: that firm faces a stronger incentive to become more aggressive. That aggression is achieved by over-investment in one case, and under-investment in the other.

Firms might also make longer-term decisions to jointly influence captive audiences and costs. To illustrate this idea, suppose that each symmetric firm has a budget, $\bar{z}$, to spend on process innovations via a production function $V(z_i)$ or on captive acquisition so that $\lambda_i = \Lambda(\bar{z} - z_i)$ where $\Lambda(\cdot)$ has natural properties. If firms simultaneously choose z_i at a pre-pricing stage then there are n equilibria in which $n - 1$ firms allocate the same budget, $z^\star$, to cost-reducing innovation,

[35] A similar finding, but in a setting with a comparison site that advertises alongside sellers for its captive base can be found in an earlier version of Ronayne and Taylor (2022): Ronayne and Taylor (2020, Appendix W.3).

[36] The findings of Chioveanu (2008) also resonate with Ireland (1993) and McAfee (1994), which considered firms that send ads that inform customers randomly and independently à la Butters (1977) and Grossman and Shapiro (1984) before choosing prices. There, one firm advertises distinctly more than others, even if they are ex ante symmetric.

where this allocation satisfies $V'(z^\star)\Lambda(\bar{z} - z^\star) = V(z^\star)\Lambda'(\bar{z} - z^\star)$. However, the remaining (and endogenously most aggressive) firm devotes more to cost reduction by diverting resources away from acquiring captive customers; it pursues heightened aggression in two dimensions.

Captive Customers who Switch Firms. The forces pushing toward asymmetry come from the decisions of firms. This contrasts with the extended model of Baye, Kovenock and De Vries (1992, Section V) in which captive customers (interpreted as "uninformed" about the identity of the lowest-price firm, but have correct expectations) switch between firms. If marginal costs are symmetric then the firm with the fewest captives has the lowest expected price, and so captives seeking low prices move toward the smallest firm, which (under a reasonable game form) can result in equal captive-audience sizes. This underpinned an argument from Baye, Kovenock and De Vries (1992, Theorem 3) to focus on the symmetric play of a symmetric game.

If firms endogenously choose their technologies then asymmetry (rather than symmetry) can be reinforced by the endogenous decisions of captive customers to switch suppliers: the argument can be reversed. If the opportunity for innovation is strong then there is an equilibrium in which the firm with the most captives is (endogenously) the most aggressive, and becomes the most frequent supplier of shoppers. If we were to extend our innovation and pricing stages to allow captive customers to choose their captors then an equilibrium outcome (depending on the exact specification) can involve a single dominant firm with a large captive audience.

Clearinghouses. One interpretation of captive customers is that they are the local locked-in clientele of a firm, whereas (remotely located) shoppers compare all prices via a "clearinghouse" that acts as fee-levying "information gatekeeper" (Baye and Morgan, 2001) for firms' prices.[37] We take this up here, supposing that firms must pay a fee to advertise their prices via the clearinghouse. Shoppers buy from the cheapest advertising supplier, but have no other access to the firms' prices.

Specifically, firm i can earn $\lambda_i(v - c_i)$ from restricting to its captive customers. Otherwise, it pays $a_i > 0$ to a clearinghouse to advertise a price $p_i \in [0, v]$. Shoppers buy from a firm advertising the lowest price; if no firm advertises via the clearinghouse then they do not buy at all.

[37] Such informative price advertising is typically modeled as a channel to reach shoppers. This contrasts with the persuasive advertising we studied in our first demand-side application, which typically induces brand loyalty (captivity). Baye and Morgan (2009) bring these ideas together in their study of firms that engage in both forms of advertising.

We write $F_i(p)$ for the distribution of prices chosen by firm i across $p \in [0, v]$ so that $1 - F_i(v)$ is the probability that firm i does not join the clearinghouse. Clearly, if $a_i > \lambda_S(v - c_i)$ then firm i will never advertise, and so we suppose that $a_i \leq \lambda_S(v - c_i)$ for all i.[38]

We can define for firm i the minimum undominated (advertised) price, $p_i^\dagger$, satisfying

$$(p_i^\dagger - c_i)(\lambda_i + \lambda_S) = \lambda_i(v - c_i) + a_i \quad \Leftrightarrow \quad p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i + a_i}{\lambda_i + \lambda_S}. \tag{23}$$

For this extension, we strictly rank the most aggressive firms so that $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. In Section 1 we discussed a more general notion of aggression via the "minimum win probability" (the required probability of selling to shoppers) for firm i contemplating charging a price p instead of simply exploiting captive customers. That quantity is central to our algorithm that solves for equilibrium (detailed in our Supplemental Appendix). Here that expression, from eq. (3), becomes

$$\underline{w}_i(p) = \frac{\lambda_i(v - p) + a_i}{\lambda_S(p - c_i)}, \tag{24}$$

where (as before, and by construction) these functions satisfy $\underline{w}_i(p_i^\dagger) = 1$ and $\underline{w}_i(v) > 0$ for all firms. These minimum win probability functions can readily cross. The extra degree of freedom provided by the clearinghouse cost parameter a_i makes it easier to construct examples of this.

In the presence of a clearinghouse, key properties of an equilibrium are maintained. Notably, the profit prediction of Proposition 1 (which itself can be attributed to Siegel, 2009) is maintained.

[38]Our clearinghouse model features some different assumptions from the classic Baye and Morgan (2001) contribution. For example, they modeled (i) downward-sloping demand such that (ii) ex-ante homogeneous buyers get enough surplus at the monopoly price to suffer a search cost so that (iii) when no firm advertises, they buy from a random firm. In contrast, we maintain our assumptions of unit demand, exogenously specified captive and shopper buyer-types, and that shoppers do not buy at all when no firm advertises. We can readily replace the final assumption by supposing that potential buyers face a small shopping cost to visit a supplying firm, which would then dissuade any visit to any non-advertising firm via the familiar paradox of Diamond (1971). We can also use other assumptions (for example, by supposing that shoppers are allocated to firms in some other manner when no firm advertises) but we streamline our exposition by maintaining the specification described here.

Proposition 7 (Pricing via a Clearinghouse). *In an equilibrium with a clearinghouse the joint support of firms' pricing strategies is* $[p^\dagger_{n-1}, v]$. *Suppose that* $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.

(i) Every firm $i \in \{1, \dots, n-1\}$ *earns its captive-only profit. With positive probability it chooses not to use the clearinghouse and charges* v *to its captive-only customers. Any such firm that does use the clearinghouse advertises prices that are strictly below* v.

(ii) Firm n *earns* $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$ *more than its captive-only profit. This (most aggressive) firm always joins the clearinghouse, and advertises the price* v *with strictly positive probability.*

It remains for us to sketch the steps that can be used to construct a (generically, unique) equilibrium. We set the required win probability for each firm $i \in \{1, \dots, n-1\}$ to equal its minimum win probability: $w_i(p) = \underline{w}_i(p)$. We then set a different required win probability for firm n to reflect its additional expected profit of $\Delta_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. These win probabilities satisfy $w_n(p^\dagger_{n-1}) = w_{n-1}(p^\dagger_{n-1}) < w_i(p^\dagger_{n-1})$ for $i \in \{1, \dots, n-2\}$.

We now follow a procedure that we used for our main model (in Section 1) and which is described in full detail in our Supplemental Appendix: firms n and $n-1$ continuously mix from $p^\dagger_{n-1}$ upwards using distribution functions $1-F_n(p) = w_{n-1}(p)$ and $1-F_{n-1}(p) = w_n(p)$, so that the distribution $F_S(p)$ of the cheapest price satisfies $1 - F_S(p) = w_n(p)w_{n-1}(p)$. One possibility (and a leading case of interest) is that this solution satisfies $w_n(p)w_{n-1}(p) < w_i(p)$ for all $i \notin \{n-1, n\}$ and $p \in [p^\dagger_{n-1}, v)$. If so, then firms $n-1$ and n mix over the whole interval: this is a (classic) de facto duopoly. Firm n then places remaining mass at v, while firm $n-1$ places remaining mass on the act of not advertising. All other firms refrain from joining the clearinghouse. This is an equilibrium.

The other possibility (just as in our Supplemental Appendix, and as we showed by our "thrango" example in Section 1) is that there is some price, $p^\ddagger < v$, at which some other firm j satisfies $w_j(p^\ddagger) = w_n(p^\ddagger)w_{n-1}(p^\ddagger)$. If so, then we execute a "partner swap" at this price, just as we have done in our earlier constructions. It is straightforward to confirm (again, as we do in our Supplemental Appendix) that there are situations in which such partner-swapping necessarily occurs, and the equilibria in such circumstances involve active mixing (in this case, this means the active use of the clearinghouse) by more than two firms. Generically, we only see two firms dancing within any interval of prices.

4. Concluding remarks

A classic model predicts that sales are offered by the two smallest (captive audience) firms. This picture now changes in three ways. Firstly, those using sales may be larger or established firms with lower marginal costs. Secondly, a "thrango" of three or more (even all) firms can use sales; although we offer a sufficient condition (satisfied if the two most aggressive firms have the smallest captive audiences) for the traditional tango. Thirdly, the pattern of sales prices consists of multiple distinct pairwise battles. A single firm (the most "aggressive") mixes with a succession of partners. This predicts marked dispersion when many firms engage in randomized sales: those firms' prices can be starkly spread out because each mixes over a distinct range of prices. Those not engaging in sales sit back and charge a high "regular" or monopoly price.

Asymmetric costs can endogenously arise, even if firms are ex ante symmetric. The endogenous supply-side asymmetry resonates with papers that find endogenously asymmetric demand structures. A common feature to all this work is that there is one firm with distinct incentives. We predicted a uniquely most aggressive firm emerges. With strong technological opportunities, we revealed that firms with more captive customers innovate more, obtain lower marginal costs and set lower prices on average. In terms of pricing patterns, an inverse relation between captive base sizes and marginal costs is in fact that most conducive to (a necessary condition for) multiple firms to engage in randomized sales in equilibrium of a model of sales pricing game.

Appendix A. Core Omitted Proofs

Proof of Proposition 1. Theorem 5 of Dasgupta and Maskin (1986, p.14) applies. Specifically, that theorem asks for the sum of players' payoffs (the total industry profit here) to be upper semi-continuous in actions (here, the profile of prices) and this holds if ties are broken in favor of lower-cost firms. (From our work later on, the equilibria are the same for any tie-break rule.)

Turning to payoffs, if $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ then claim (v) of Lemma S1 in our Supplemental Appendix shows that any Nash equilibrium gives the payoffs specified in the proposition. In the Supplemental Appendix we provide an algorithm to construct an equilibrium with the required payoffs for any parameter values. □

Proof of Proposition 2. Standard properties of any equilibrium (see Lemma S1 in our Supplemental Appendix) are that all firms mix continuously up to v with any atoms at v. Firm n earns more than its captive-only profit, requiring others to play atoms at v while firm n does not. This completes claim (i).

If the lower bound of all prices were to strictly exceed $p_{n-1}^\dagger$ then (at least) firms n and $n-1$ could (by pricing just above $p_{n-1}^\dagger$) sell to all shoppers and strictly exceed their equilibrium profit. We conclude that the joint support of firms' mixed strategies extends down to $\min_i \underline{p}_i = p_{n-1}^\dagger$. Prices below $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ are strictly dominated for firms $i \in \{1,\ldots,n-2\}$, and so (given the absence of gaps; Lemma S1) firms $n-1$ and n must mix continuously over $[p_{n-1}^\dagger, \min_{j\in\{1,\ldots,n-2\}} p_j^\dagger)$. Given that they both price below $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ with strictly positive probability, a price at or just above $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ will not be chosen by any firm $i \in \{1,\ldots,n-2\}$, and so there is some $p^\ddagger > \min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ such that firms $n-1$ and n mix on the interval $[p_{n-1}^\dagger, p^\ddagger)$.

The expected profit earned by firm $n-1$ from charging a price $p \in [p_{n-1}^\dagger, p^\ddagger)$ is

$$\pi_{n-1}(p) = (p - c_{n-1})\left(\lambda_{n-1} + \lambda_S\left(1 - F_n(p)\right)\right) = \lambda_{n-1}(v - c_{n-1}), \tag{25}$$

where the final term is its captive-only profit. The expected profit of firm n from $p \in [p_{n-1}^\dagger, p^\ddagger)$ is

$$\pi_n(p) = (p - c_n)\left(\lambda_n + \lambda_S\left(1 - F_{n-1}(p)\right)\right) = \lambda_n(v - c_n) + (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger), \tag{26}$$

where the final expression is the profit of firm n from (v) of Lemma S1. These equations solve:

$$F_n(v) = 1 - \frac{\lambda_{n-1}(v-p)}{\lambda_S(p - c_{n-1})} \quad \text{and} \quad F_{n-1}(v) = 1 - \frac{\lambda_n(v-p)}{\lambda_S(p-c_n)} - \frac{(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)}{\lambda_S(p - c_n)}. \tag{27}$$

These are valid distribution functions that strictly and continuously increase from $F_{n-1}(p_{n-1}^\dagger) = F_{n-1}(p_{n-1}^\dagger) = 0$, and they can be rewritten to obtain eq. (5), and completing claim (ii).

For claim (iii), we use the properties $c_n \leq c_{n-1}$ and $p_n^\dagger < p_{n-1}^\dagger$. From rearrangement, $\lambda_i + \lambda_S = \lambda_S(v - c_i)/(v - p_i^\dagger)$ and so substituting for $\lambda_{n-1} + \lambda_S$ and $\lambda_n + \lambda_S$ in $F_n(p)$ and $F_{n-1}(p)$ respectively,

$$F_{n-1}(p) < F_n(p) \quad \Leftrightarrow \quad \frac{v - c_n}{(v - p_n^\dagger)(p - c_n)} < \frac{v - c_{n-1}}{(v - p_{n-1}^\dagger)(p - c_{n-1})}, \tag{28}$$

which holds if $p_n^\dagger < p_{n-1}^\dagger$ and (given that $v \geq p$) $c_n \leq c_{n-1}$.

For claim (iv), any equilibrium involves mixing by $n-1$ and n up to some $p^\ddagger$. One possibility is that $p^\ddagger = v$, so that all other firms choose $p_i = v$. By inspection, $\lim_{p \uparrow v} F_n(p) = 1$ and $\lim_{p \uparrow v} F_{n-1}(p) \leq 1$ (the latter inequality is strict if $p_{n-1}^\dagger > p_n^\dagger$) which means that we have valid distributions. Firms $n-1$ and n cannot improve by deviating. If there is an equilibrium in which only these two firms mix then (by construction) it is unique. We must check to see if some other firm i might wish to deviate to $p_i < v$. Firm $i \in \{1, \ldots, n-2\}$ earns $\lambda_i(v - c_i)$. By deviating to $p \in [p_{n-1}^\dagger, v)$, and assuming that $\lambda_{n-1} \leq \lambda_i$ (the condition in the proposition) it earns

$$\begin{aligned}
\pi_i(p) &= (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_{n-1}(p)\right)\left(1 - F_n(p)\right)\right) && (29)\\
&< (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_n(p)\right)\right) && (30)\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_{n-1}(v - p)}{(p - c_{n-1})}\right) && (31)\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_{n-1}(v - p)}{\lambda_{n-1}(v - p_{n-1}^\dagger) + \lambda_S(p - p_{n-1}^\dagger)}\right) && (32)\\
&\leq (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_i(v - p)}{\lambda_i(v - p_i^\dagger) + \lambda_S(p - p_i^\dagger)}\right) && (33)\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_i(v - p)}{(p - c_i)}\right) = \lambda_i(v - c_i). && (34)
\end{aligned}$$

The third line is from substitution of $F_n(p)$. The fourth line is obtained by writing c_{n-1} in terms of λ_{n-1} and $p_{n-1}^\dagger$ and rearranging. The fifth line holds because $\lambda_{n-1} \leq \lambda_i$ and $p_{n-1}^\dagger \leq p_i^\dagger$. The final line is from substituting $p_i^\dagger$ and rearranging. Firm i performs strictly worse by deviating.

We have a (unique, within this class) "tango" equilibrium. Any other equilibrium requires another firm "step onto the dance floor" at some $p^\ddagger < v$. The argument above demonstrates (given the lack of atoms below v, and continuity properties) that this would be strictly suboptimal. □

Proof of Proposition 3. This follows from the preceding argument in the main text. To see a specific example of this, consider a "two to tango" equilibrium in which (for simplicity of exposition) firms n and $n-1$ satisfy $c_n = c_{n-1} = 0$ and $\lambda_n = \lambda_{n-1} = \lambda$. This means that

$$p_n^\dagger = p_{n-1}^\dagger = \frac{\lambda v}{\lambda + \lambda_S} \quad \text{and} \quad F_n(p) = F_{n-1}(p) = \frac{(p - p_{n-1}^\dagger)(\lambda + \lambda_S)}{p\lambda_S} = 1 - \frac{(v-p)\lambda}{p\lambda_S}. \quad (35)$$

The condition required for this to be an equilibrium is that there is no price at which another lower-indexed firm wishes to join. Equation (7) from the main text here requires

$$(v-p)\lambda_i \geq (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)) = (p - c_i)\lambda_S \left(\frac{(v-p)\lambda}{p\lambda_S}\right)^2$$
$$\Leftrightarrow \quad \lambda_i \geq \frac{(p - c_i)(v-p)\lambda^2}{p^2\lambda_S} \quad \forall\, p \in \left(\frac{\lambda v}{\lambda + \lambda_S}, v\right). \quad (36)$$

By inspection, if $p > c_i$ then this fails if λ_i is small. Pushing further, suppose that this holds for all $i \in \{1, \ldots, n-3\}$, but let us choose λ_{n-2} so that this fails for some p. For firm $n-2$ set $c_{n-2} = p_{n-1}^\dagger = \lambda v/(\lambda + \lambda_S)$, which guarantees that $p_{n-2}^\dagger > p_{n-1}^\dagger$ for any $\lambda_{n-2} > 0$, no matter how small. Firm $n-2$ will wish to step on to the dance floor at the lowest price $p^\ddagger$ which satisfies

$$\lambda_{n-2} = \frac{(p^\ddagger - c_{n-2})(v - p^\ddagger)\lambda^2}{(p^\ddagger)^2\lambda_S} = \frac{\left(p^\ddagger(\lambda + \lambda_S) - \lambda v\right)(v - p^\ddagger)\lambda^2}{(p^\ddagger)^2\lambda_S(\lambda + \lambda_S)}. \quad (37)$$

Explicitly, this is the lower solution to

$$(\lambda + \lambda_S)\left(1 + \frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)(p^\ddagger)^2 - (2\lambda + \lambda_S)vp^\ddagger + \lambda v^2 = 0$$
$$\Rightarrow \quad p^\ddagger = \frac{v(2\lambda + \lambda_S) - v\sqrt{\lambda_S^2 - \frac{4\lambda_{n-2}\lambda_S(\lambda + \lambda_S)}{\lambda}}}{2(\lambda + \lambda_S)\left(1 + \frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)}, \quad (38)$$

where this solution satisfies $p^\ddagger \downarrow \lambda v/(\lambda + \lambda_S)$ as $\lambda_{n-2} \downarrow 0$.

The argument given is that there can be parameters under which a third firm must participate. In our Supplemental Appendix we provide a characterization of equilibria. The same approach taken here applies: if there are k firms that engaged in randomized sales then we can add an additional firm with relatively high marginal cost, but few captives, that wishes to participate in sales. □

Proof of Proposition 4. Claim (i) holds because firms $i \in \{1, \ldots, n-2\}$ play $p_i = v$ as pure strategies, and their costs do not enter the solutions reported in eq. (5). For claim (ii), $F_{n-1}(p)$ is decreasing in $p^\dagger_{n-1}$ which itself is increasing in c_{n-1}. For $F_n(p)$,

$$\frac{\partial F_n(p)}{\partial c_{n-1}} = \frac{\lambda_{n-1} + \lambda_S}{\lambda_S} \left(\frac{(p - p^\dagger_{n-1})}{(p - c_{n-1})^2} - \frac{1}{p - c_{n-1}} \frac{\partial p^\dagger_{n-1}}{\partial c_{n-1}} \right) \tag{39}$$

$$= \frac{\lambda_{n-1} + \lambda_S}{\lambda_S (p - c_{n-1})} \left(\frac{(p - p^\dagger_{n-1})}{(p - c_{n-1})} - \frac{\lambda_S}{\lambda_{n-1} + \lambda_S} \right) = -\frac{(v - p)\lambda_{n-1}}{\lambda_S (p - c_{n-1})^2} < 0 \tag{40}$$

The CDFs are both decreasing in c_{n-1}, which means an increase in c_{n-1} pushes rightward the distributions of prices. Claim (iii) follows an inspection of $F_{n-1}(p)$. □

Proof of Proposition 5. For existence (and further characterization) see Proposition S1 in our Supplemental Appendix. The argument then follows the one made after eq. (14): the profit of firm i has an upward kink as the firm ties to be the most aggressive, and so cannot be the optimal choice of z_i. □

Proof of Proposition 6. Equations (19) and (20) (and the surrounding text) explain that when there are sufficiently few shoppers, only n can be the most aggressive. The order of the $z^\star_i$ terms follows because firms are indexed in order of how many consumers they serve in total in equilibrium. The rankings of $c^\star_i$ and $p^\dagger_i$ terms follow automatically. □

Proof of Proposition 7. The properties of any equilibrium with a clearinghouse are reported in Lemma S2 of our Supplemental Appendix. Firm n guarantees $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$ more than its captive-only profit by choosing $p_n = p^\dagger_{n-1}$. From claim (v) of Lemma S2, other firms earn captive-only profits. If n were to earn strictly more, then the lower bound of its support would exceed $p^\dagger_{n-1}$. This would give $n-1$ an opportunity to earn strictly more than its own captive-only profit; a contradiction. □

References

Armstrong, Mark, and John Vickers. 2001. "Competitive Price Discrimination." *RAND Journal of Economics*, 32(4): 579–605.

Armstrong, Mark, and John Vickers. 2019. "Discriminating Against Captive Customers." *American Economic Review: Insights*, 1(3): 257–72.

Armstrong, Mark, and John Vickers. 2022. "Patterns of Competitive Interaction." *Econometrica*, 90(1): 153–191.

Arnold, Michael A, and Lan Zhang. 2014. "The Unique Equilibrium in a Model of Sales with Costly Advertising." *Economics Letters*, 124: 457–460.

Arnold, Michael A, Chenguang Li, Christine Saliba, and Lan Zhang. 2011. "Asymmetric Market Shares, Advertising and Pricing: Equilibrium with an Information Gatekeeper." *Journal of Industrial Economics*, 59(1): 63–84.

Baye, Michael R., and John Morgan. 2001. "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets." *American Economic Review*, 91(3): 454–474.

Baye, Michael R., and John Morgan. 2009. "Brand and Price Advertising in Online Markets." *Management Science*, 55(7): 1139–1151.

Baye, Michael R., Dan Kovenock, and Casper G. De Vries. 1992. "It Takes Two to Tango: Equilibria in a Model of Sales." *Games and Economic Behavior*, 4(4): 493–510.

Baye, Michael R., Dan Kovenock, and Casper G. De Vries. 1996. "The All-Pay Auction with Complete Information." *Economic Theory*, 8(2): 291–305.

Baye, Michael R, John Morgan, and Patrick Scholten. 2004*a*. "Price Dispersion in the Small and in the Large: Evidence from an Internet Price Comparison Site." *Journal of Industrial Economics*, 52(4): 463–496.

Baye, Michael R., John Morgan, and Patrick Scholten. 2004*b*. "Temporal Price Dispersion: Evidence from an Online Consumer Electronics Market." *Journal of Interactive Marketing*, 18(4): 101–115.

Baye, Michael R, John Morgan, and Patrick Scholten. 2006. "Information, Search, and Price Dispersion." In *Handbook on Economics and Information Systems*. Vol. 1, , ed. T Hendershott. Amsterdam:Elsevier.

Baye, Michael R. John Morgan, Patrick Scholten, and D. Jansen. 2006. "Persistent Price dispersion in Online Markets." *The New Economy and Beyond: Past Present and Future, Edward Elgar, S*, 122–143.

Baye, Michael R, Xiaxun Gao, and John Morgan. 2011. "On the Optimality of Clickthrough Fees in Online Markets." *The Economic Journal*, 121(556): F340–F367.

Bulow, Jeremy, and Jonathan Levin. 2006. "Matching and Price Competition." *American Economic Review*, 96(3): 652–668.

Butters, Gerard R. 1977. "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economic Studies*, 44(3): 465–491.

Chandra, Ambarish, and Mariano Tappata. 2011. "Consumer Search and Dynamic Price Dispersion: An Application to Gasoline Markets." *RAND Journal of Economics*, 42(4): 681–704.

Chioveanu, Ioana. 2008. "Advertising, Brand Loyalty and Pricing." *Games and Economic Behavior*, 64(1): 68–80.

Dasgupta, Partha, and Eric Maskin. 1986. "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory." *Review of Economic Studies*, 53(1): 1–26.

Dasgupta, Partha, and Joseph Stiglitz. 1980. "Industrial Structure and the Nature of Innovative Activity." *The Economic Journal*, 90(358): 266–293.

Diamond, Peter A. 1971. "A Model of Price Adjustment." *Journal of Economic Theory*, 3(2): 156–168.

Galenianos, Manolis, Rosalie Liccardo Pacula, and Nicola Persico. 2012. "A Search-Theoretic Model of the Retail Market for Illicit Drugs." *Review of Economic Studies*, 79(3): 1239–1269.

Garrod, Luke, Ruochen Li, and Chris M Wilson. 2023. "Automated Switching Services." *Economics Letters*, 232: 111351.

Gill, David, and John Thanassoulis. 2016. "Competition in Posted Prices with Stochastic Discounts." *The Economic Journal*, 126(594): 1528–1570.

Golding, Edward L., and Steven Slutsky. 2000. "Equilibrium Price Distributions in an Asymmetric Duopoly." In *Advances in Applied Microeconomics*. Vol. 8, 139–159.

Gorodnichenko, Yuriy, Viacheslav Sheremirov, and Oleksandr Talavera. 2018. "Price Setting in Online Markets: Does IT click?" *Journal of the European Economic Association*, 16(6): 1764–1811.

Grossman, Gene M., and Carl Shapiro. 1984. "Informative Advertising with Differentiated Products." *Review of Economic Studies*, 51(1): 63–81.

Hagiu, Andrei, and Julian Wright. 2024. "Optimal Discoverability on Platforms." *Management Science*.

Hong, Han, and Matthew Shum. 2010. "Using Price Distributions to Estimate Search Costs." *RAND Journal of Economics*, 37(2): 257–275.

Inderst, Roman. 2002. "Why Competition May Drive Up Prices." *Journal of Economic Behavior and Organization*, 47(4): 451–462.

Ireland, Norman J. 1993. "The Provision of Information in a Bertrand Oligopoly." *Journal of Industrial Economics*, 41(1): 61–76.

Johnen, Johannes, and David Ronayne. 2021. "The Only Dance in Town: Unique Equilibrium in a Generalized Model of Sales." *Journal of Industrial Economics*, 69(3): 595–614.

Kaplan, Greg, and Guido Menzio. 2015. "The Morphology of Price Dispersion." *International Economic Review*, 56(4): 1165–1206.

Kaplan, Greg, Guido Menzio, Leena Rudanko, and Nicholas Trachter. 2019. "Relative Price Dispersion: Evidence and Theory." *American Economic Journal: Microeconomics*, 11(3): 68–124.

Kocas, Cenk, and Tunga Kiyak. 2006. "Theory and Evidence on Pricing by Asymmetric Oligopolies." *International Journal of Industrial Organization*, 24(1): 83–105.

Lach, Saul. 2002. "Existence and Persistence of Price Dispersion: An Empirical Analysis." *Review of Economics and Statistics*, 84(3): 433–444.

Lach, Saul, and José L Moraga-González. 2017. "Asymmetric Price Effects of Competition." *Journal of Industrial Economics*, 65(4): 767–803.

McAfee, R. Preston. 1994. "Endogenous Availability, Cartels, and Merger in an Equilibrium Price Dispersion." *Journal of Economic Theory*, 62(1): 24–47.

Moen, Espen R, Fredrik Wulfsberg, and Øyvind Aas. 2020. "Price Dispersion and the Role of Stores." *The Scandinavian Journal of Economics*, 122(3): 1181–1206.

Moraga-González, José Luis, and Matthijs R. Wildenbeest. 2008. "Maximum Likelihood Estimation of Search Costs." *European Economic Review*, 52(5): 820–848.

Moraga-González, José Luis, and Matthijs R Wildenbeest. 2012. "Comparison Sites." In *The Oxford Handbook of the Digital Economy*. , ed. Martin Peitz and Joel Waldfogel. Oxford, England:Oxford University Press.

Moraga-González, José Luis, Zsolt Sándor, and Matthijs R Wildenbeest. 2017. "Nonsequential Search Equilibrium with Search Cost Heterogeneity." *International Journal of Industrial Organization*, 50: 392–414.

Moraga-González, José Luis, Zsolt Sándor, and Matthijs R Wildenbeest. 2021. "Simultaneous Search for Differentiated Products: the Impact of Search Costs and Firm Prominence." *The Economic Journal*, 131(635): 1308–1330.

Myatt, David P. and David Ronayne. 2025*a*. "Finding a Good Deal: Stable Price Dispersion with Robust Search." *Working Paper*.

Myatt, David P. and David Ronayne. 2025*b*. "Stable Price Dispersion." *Working Paper*.

Nakamura, Emi, and Jón Steinsson. 2008. "Five Facts About Prices: A Reevaluation of Menu Cost Models." *Quarterly Journal of Economics*, 123(4): 1415–1464.

Narasimhan, Chakravarthi. 1988. "Competitive Promotional Strategies." *Journal of Business*, 61(4): 427–449.

Pennerstorfer, Dieter, Philipp Schmidt-Dengler, Nicolas Schutz, Christoph Weiss, and Biliana Yontcheva. 2020. "Information and Price Dispersion: Theory and Evidence." *International Economic Review*, 61(2): 871–899.

Ronayne, David. 2021. "Price Comparison Websites." *International Economic Review*, 62(3): 1081–1110.

Ronayne, David, and Greg Taylor. 2020. "Competing Sales Channels." *Working Paper, University of Oxford*.

Ronayne, David, and Greg Taylor. 2022. "Competing Sales Channels with Captive Consumers." *The Economic Journal*, 132(642): 741–766.

Rosenthal, Robert W. 1980. "A Model in which an Increase in the Number of Sellers Leads to a Higher Price." *Econometrica*, 48(6): 1575–1579.

Shelegia, Sandro, and Chris M. Wilson. 2021. "A Generalized Model of Advertised Sales." *American Economic Journal: Microeconomics*, 13(1): 195–223.

Shelegia, Sandro, and Chris M. Wilson. 2023. "Costly Participation and Default Allocations in All-Pay Contests." *Working Paper*.

Shilony, Yuval. 1977. "Mixed Pricing in Oligopoly." *Journal of Economic Theory*, 14(2): 373–388.

Siegel, Ron. 2009. "All-Pay Contests." *Econometrica*, 77(1): 71–92.

Siegel, Ron. 2010. "Asymmetric Contests with Conditional Investments." *American Economic Review*, 100(5): 2230–2260.

Siegel, Ron. 2012. "Participation in Deterministic Contests." *Economics Letters*, 116(3): 588–592.

Siegel, Ron. 2014. "Asymmetric Contests with Head Starts and Monmonotonic Costs." *American Economic Journal: Microeconomics*, 6(3): 59–105.

Sorensen, Alan T. 2000. "Equilibrium Price Dispersion in Retail Markets for Prescription Drugs." *Journal of Political Economy*, 108(4): 833–850.

Stahl, Dale O. 1989. “Oligopolistic Pricing with Sequential Consumer Search.” *American Economic Review*, 79(4): 700–712.

Varian, Hal R. 1980. “A Model of Sales.” *American Economic Review*, 70(4): 651–659.

Varian, Hal R. 1981. “Errata: A Model of Sales.” *American Economic Review*, 71(3): 517.

Wulfsberg, Fredrik. 2016. “Inflation and Price Adjustments: Micro Evidence from Norwegian Consumer Prices 1975–2004.” *American Economic Journal: Macroeconomics*, 8(3): 175–194.

Xiao, Jun. 2016. “Asymmetric All-Pay Contests with Heterogeneous Prizes.” *Journal of Economic Theory*, 163: 178–221.

Xiao, Jun. 2018. “Equilibrium Analysis of the All-Pay contest with Two Nonidentical Prizes: Complete Results.” *Journal of Mathematical Economics*, 74: 21–34.