Asymmetric Models of Sales

David P. Myatt
London Business School
www.david-p-myatt.org
dmyatt@london.edu

David Ronayne
ESMT Berlin
www.davidronayne.com
david.ronayne@esmt.org

Preliminary and Incomplete. Last compiled: 31 May 2023.[1]

Abstract. We broaden the classic captive-and-shopper model of sales. Firstly, we allow for asymmetric marginal costs as well as asymmetric captive audiences. These asymmetries jointly determine the identities of the firms who compete (via randomized sales) to serve shoppers, and there can be more than two such firms. In a leading case of interest, the prices paid by shoppers fall in response to a cost rise for the firm that serves most of them. Secondly, we study asymmetric price adjustment opportunities via a two-stage game in which firms may cut but not raise their initial prices. In this setting (and also in a scenario with endogenous move order) we predict the play of pure strategies and that a unique firm serves the shoppers. Welfare properties depend on whether firm asymmetry is on the supply side (costs) or on the demand side (captive audiences). Finally, we allow firms to choose endogenously their production technologies via process innovations. One firm innovates distinctly more than others, attains a lower marginal cost, and ultimately serves the shoppers. We relate the distinctive asymmetric pattern of innovations to demand-side asymmetries and the shape of technology opportunity.

Keywords: model of sales, captives, shoppers, price dispersion, clearinghouse models.

In the classic "model of sales" competition amongst firms (via low prices) for the business of "shoppers" (who consider every price) sacrifices profits earned (via higher prices) from exploiting "captive" customers (who are locked in to a single firm). Varian (1980) constructed a symmetric equilibrium of a symmetric single-stage game in which firms continuously mix over an interval of prices. Others (Narasimhan, 1988; Baye, Kovenock, and de Vries, 1992) studied firms with different captive-audience sizes: the two firms with the fewest captives play mixed strategies (often interpreted as random sales) to attract shoppers, while other firms charge a monopoly price to their captives. Firms share the same marginal cost, and so from a welfare perspective it does not matter who serves the shoppers. Furthermore, a restriction to cost symmetry prevents the investigation of situations in which individual firms take endogenous steps, perhaps via process and product innovations, to lower costs or to improve their products. This is important if equilibrium innovation pushes toward asymmetric costs.

[1]This paper builds upon an auxiliary result (Lemma 2) drawn from an earlier version (Myatt and Ronayne, 2019) of our paper "A Theory of Stable Price Dispersion" which now (Myatt and Ronayne, 2023a) assumes that firms have symmetric marginal costs. Julian Wright (thank you) suggested that we develop this result separately; it now forms Proposition 1 of this paper. We also thank many people (including seminar partcipants) for comments on our companion work that influenced this new paper. As noted elsewhere these include Simon Anderson, Mark Armstrong, Dan Bernhardt, Yongmin Chen, Alex de Cornière, Andrea Galeotti, Justin Johnson, Jeanine Miklós-Thal, José Moraga-González, Volker Nocke, Martin Obradovits, Martin Pesendorfer, Régis Renault, Michael Riordan, Robert Somogyi, Greg Taylor, Juuso Välimäki, Hal Varian, and Chris Wilson.

We expand the classic captive-and-shopper simultaneous-move pricing game to incorporate asymmetric costs as well asymmetric captive audiences. We also allow for asymmetric price adjustment via a two-stage pricing game in which, after their initial price positions are chosen, firms may adjust their prices downward (but not upward). Relatedly, we study asymmetric move order via a Stackelberg-style game which gives each firm an opportunity to make an early-mover commitment to an advertised price position. The use of an asymmetric cost specification allows us to examine the (interesting) impact of changes to firms' individual costs. Finally, we add and study a prior stage in which firms endogenously choose production technologies via fixed-cost innovations which lower their marginal costs of production.

A distinctive feature throughout is a move away from symmetric costs. Asymmetric cost specifications for the captive-shopper model have received only limited attention. In a recent exception, Shelegia and Wilson (2021) considered a very rich generalized model of (advertised) sales. This deserves a much fuller discussion to which we return later in our paper.[2] Nevertheless, we do not know of a full treatment of the simultaneous-move captive-and-shopper pricing game with asymmetric costs. In response, a simple first contribution (in Section 1) is to allow for asymmetric marginal costs in a model of sales. The equilibrium involves mixing between the two most aggressive firms where a firm's aggression is measured by the lowest price that it would be willing to charge in order to capture shoppers. A firm is more aggressive if it has fewer captive customers (this is demand-side asymmetry) or if its marginal cost is low (this is supply-side asymmetry). In leading cases the other firms exploit their captive customers by charging the monopoly price. Nevertheless, there are also (plausible) cases in which more than two firms are required to participate in randomized sales.

Given that firms have different marginal costs, welfare is determined by the identities of the firms who serve the shoppers; (full) efficiency requires shoppers to be served by the lowest-cost firm. The involvement of (at least) two firms who challenge for sales means that (unless their costs are tied and lowest amongst all firms) the equilibrium is necessarily inefficient. It can be less efficient still if the lowest-cost firms have more captives, for in that situation the most aggressive firms can be those with very few captives but relatively high marginal cost.

Cost asymmetry also allows insightful comparative-static exercises. Of most interest is the variation in the cost of the most aggressive firm; in leading cases of interests this is the firm that most often sells to shoppers. In a result that generalizes an insight from Inderst (2002), any increase in this firm's cost pushes down the distribution of prices charged by the second-most-aggressive firm, and so customers pay less. Relatedly, the most aggressive firm benefits distinctly more from cost reductions in comparison to other firms. A consequence is that there are asymmetric incentives for cost-reducing innovations.

[2]Their firms engage in (asymmetrically) costly advertising to reach shoppers; they allow for utility offers à la Armstrong and Vickers (2001); and their model allows for asymmetric costs in a unit-demand setting. Their solution concept carefully chooses tie-break rules as part of the equilibrium. As their specification approaches the classic model (by taking to zero their advertising costs) their prediction coincides with ours. In an earlier paper, Shelegia (2012) observed that if exactly two firms have zero costs then they randomly challenge for shoppers while positive-cost firms charge the monopoly price. Inderst (2002) considered a duopoly model equivalent to one with marginal-cost asymmetry, with the special feature that one firm has no captive customers.

A single-stage captive-and-shopper pricing game has no pure-strategy Nash equilibrium. Any dispersion in prices arises from the realizations of mixed strategies from which there are profitable ex post deviations. Elsewhere (Myatt and Ronayne, 2023a) we argue that the predictions of single-stage models are incompatible with stable price dispersion, that nevertheless such dispersion is an empirical regularity, and that realistically firms find it easy to lower (but difficult to raise) prices in the short run. We suggest (in that paper) a two-stage framework: in a first stage, initial prices are chosen; in a second stage, firms are able to lower (but not raise) their prices. (In essence, there is an asymmetric ex post price adjustment opportunity.) Amongst other results (in Myatt and Ronayne, 2023a) we identify (in the captive-shopper setting) a subgame perfect equilibrium of a two-stage game in which a single profile of prices (one price per firm) is chosen in the first stage and maintained in the second stage. That result and others provide a theory of stable price dispersion. However, that theory restricts attention to firms with the same marginal cost.[3] In response, a second contribution of our current paper (in Section 2) is the analysis of a two-stage model of sales in which firms are fully asymmetric.

We identify a unique profile of prices that are played as pure strategies on the equilibrium path of a subgame perfect equilibrium. In the first stage, the most aggressive firm sets an initial price which is just low enough to deter a second-stage undercut by the other firms. This implies that (in contrast to a single-stage pricing model) shoppers are served by a single firm. If the most aggressive firm is the one with the lowest marginal cost (as it is if the sizes of captive audiences are sufficiently similar, which means that firm asymmetry is driven by the supply side rather than the demand side) then this two-stage equilibrium is efficient; this contrasts with the inefficiency of single-stage equilibrium play. Moreover, the comparative-static result that an increase in the cost of the most aggressive firm reduces prices no longer holds. Instead, a local change in this firm's price has no effect on the prices paid by any customers.

Despite these differences between the outcomes of one-stage and two-stage pricing games, the expected profits of firms are the same in both cases. This, in particular, means that the response of firms' profits to changes in their costs are just as they were before: the most aggressive firm (uniquely serving the shoppers in a two-stage-pricing environment) benefits distinctly more from a marginal-cost reduction (because it serves the shoppers as well as its own captive customers). A maintained theme, therefore, is that there are asymmetric innovation incentives even when firms are symmetric in other ways.

The two-stage-pricing model described above offers a commitment opportunity for firms at the first stage. This is related to the commitment opportunity that can be exploited in Stackelberg-style games when firms move in sequence. A third contribution (in Section 3) of our paper is the study of a model with endogenous asymmetric move order. Specifically, we allow all firms an opportunity to commit voluntarily to an advertised price position (which is then fixed) or instead to delay until the prices of others are observed. In essence, we give all firms opportunities (should they wish) to seek out a position as a Stackelberg early mover.

[3]Instead we allow for a richer set of "consideration sets" than in a "captives and shoppers" setting. In recent work, Armstrong and Vickers (2022) have considered more general consideration sets in single-stage games.

The most aggressive firm emerges as the Stackelberg leader. It commits (at the first opportunity) to the limit price which dissuades others from undercutting. All others firms ultimately charge the captive-exploiting monopoly price. The (unique) equilibrium outcome mimics that of our two-stage play scenario and so our comparative-static and efficiency claims are maintained. In particular (and perhaps most importantly) the equilibrium (expected) profits of firms are the same across our single-stage, two-stage, and endogenous-move-order pricing games.

We have noted (and prove later) that a change in marginal cost impacts the most aggressive firm differently than other firms. This is easy to see in our two-stage and sequential-move environments, where that firm uniquely serves all shoppers, and where any local changes in its marginal cost do not influence the prices of other firms: the most aggressive firm benefits more from a marginal-cost reduction (which we associate with a process innovation) simply because it sells to more customers. As such, we find asymmetric innovation incentives.

To show the consequences of this, and as our final (and perhaps most substantial) contribution, we extend (in Section 4) the model of sales (in any of its variants) to allow for the endogenous exploitation of opportunities to engage in process innovations. We add a pre-pricing stage in which firms choose their production technologies: a firm can pay (via a higher fixed cost) to lower its marginal cost of production. Firms then proceed to play one of the pricing games. (Equilibrium innovation choices do not depend on which pricing game is played.)

Asymmetric capabilities emerge naturally. For example, consider a world in which firms face the same innovation opportunities and where the sizes of their captive audiences are equal. In an equilibrium with pure strategies at the innovation stage, exactly one firm chooses a distinct production technology with more innovation (a higher fixed cost) and lower marginal cost, whereas other (less innovative, and so ultimately with higher marginal cost) firms act symmetrically. Subsequently (in our two-stage and sequential-move models) the innovative firm goes on to set a shopper-capturing low price, while the others simply exploit their captive audiences. Fixing the identity of the innovative firm, the equilibrium outcome is unique. (Given the symmetry of firms, we can choose arbitrarily the innovator.) This result shows that asymmetric marginal costs emerge endogenously, even when firms are otherwise symmetric. This provides a rationale for the study of models of sales with asymmetric costs.

Connecting back to the literature, we know only of one paper (Shelegia and Wilson, 2021) that deals extensively with asymmetric costs in a captive-shopper world, albeit with costly advertising. Our explicit equilibrium characterization is also accompanied by novel welfare and comparative-static results, including one (the effect of costs on price) that build upon an insight that we credit to Inderst (2002). Our consideration of two-stage (and multi-stage, more generally) settings extends and complements our own companion work (Myatt and Ronayne, 2023a,b). Finally, our finding that asymmetric costs endogenously arise ex post (even if firms are symmetric ex ante) resonates with papers that find endogenously asymmetric product awareness (Ireland, 1993; McAfee, 1994) and captive customer bases (Chioveanu, 2008).

1. A Single-Stage Model of Sales with Asymmetric Firms

In this section we study a standard single-stage model of sales with fully asymmetric firms.

Model. There are n firms who simultaneous choose their prices, where $p_i \in [0, v]$ is the price chosen by firm $i \in \{1, \dots, n\}$ and v is customers' (common) maximal willingness to pay.

A mass of $\lambda_i > 0$ customers are captive to firm i, and so always buy from it. A mass of $\lambda_S > 0$ customers are shoppers who buy from the cheapest firm, or from one of the cheapest (in the event of a tie). For technical convenience we break any ties in favor of a lowest-cost firm.[4]

Firm i faces a constant marginal cost c_i. We assume (as is standard in this literature) that the cost of serving captive customers and shoppers is the same. However, equilibrium pricing strategies are unaffected if the cost of serving captive customers is changed to some $\hat{c}_i \neq c_i$.

Firm i earns profit $\lambda_i(p_i - c_i)$ from its captive customers and additionally $\lambda_S(p_i - c_i)$ if it wins the business of the shoppers. The summation of these profit components generates a firm's payoff, where all firms are assumed (for now) to be risk neutral. (In a later section we consider what happens if a firm is risk averse.) This specification corresponds to that of Varian (1980) if $\lambda_i = \lambda$ and $c_i = c$ for every i, so that firms are symmetric.[5]

Equilibrium Play. We begin by identifying the undominated prices for each firm.

Firm i guarantees a profit of at least $\lambda_i(v - c_i)$ by setting $p_i = v$ and selling only to captive customers. The lowest price it would be willing to set in order to capture the business of shoppers is $p_i^\dagger$ satisfying $\lambda_i(v - c_i) = (\lambda_i + \lambda_S)(p_i^\dagger - c_i)$, or equivalently

$$p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i}{\lambda_i + \lambda_S}. \tag{1}$$

This lowest undominated price is a measure of how aggressive (in terms of pricing) a firm is willing to be. It is higher when a firm has more captive customers (because it is more costly to lose money on sales to them by lowering price) and when the marginal cost of serving shoppers is higher (making it less tempting to capture those shoppers).[6] Firm j is strictly more aggressive than firm i and if and only if $p_j^\dagger < p_i^\dagger$ which (following re-arrangement) is

$$\underbrace{(c_i - c_j)}_{\text{cost adv. } j \text{ vs. } i} > \frac{v - (c_i + c_j)/2}{\lambda_S + (\lambda_i + \lambda_j)/2} \underbrace{(\lambda_j - \lambda_i)}_{\text{captive adv. } j \text{ vs. } i}. \tag{2}$$

If firm j has an advantage over firm i on the supply side (lower costs) and a disadvantage on the demand side (fewer captives) then this holds. However, if a firm has advantages on both sides of the market (lower costs, more captives) then this inequality can break either way.

[4]This allows us to apply an off-the-shelf equilibrium-existence result (from Dasgupta and Maskin, 1986).

[5]Baye, Kovenock, and de Vries (1992) and Kocas and Kiyak (2006) allowed for asymmetry in the "captives" parameters so that $\lambda_i \neq \lambda_j$ for some $i \neq j$, but retained common (and normalized to zero) marginal costs.

[6]The cost of serving captives is irrelevant to decision making, simply because they are served no matter what. This means that if the cost of serving captives is $\hat{c}_i \neq c_i$ then the expression for $p_i^\dagger$ remains unchanged.

Without loss of generality, we order firms by increasing aggressiveness: $p_1^\dagger \geq p_2^\dagger \geq \cdots \geq p_n^\dagger$.

Baye, Kovenock, and de Vries (1992, Section V) found a unique Nash equilibrium when firms with the same marginal cost ($c_i = c$ for all i) have differently sized captive audiences: ordering firms so that $\lambda_1 > \cdots > \lambda_n$, the equilibrium involves mixing (the "tango" of their paper, which describes the competition by two firms for sales to shoppers via randomized sales) by firms $n-1$ and n while firms $i \in \{1, \ldots, n-2\}$ set $p_i = v$. In other situations (including symmetry) there can be other equilibria, all of which generate the same expected profits.

The "two to tango" result extends here, but only partially. Any such tango is danced by the two firms that are most aggressive according to the minimum undominated prices of eq. (1) in the sense that they play continuous mixed strategies (which compete for sales to shoppers) over an interval of prices ranging upward from $p_{n-1}^\dagger$. However, there is a possibility that (for higher prices) other firms step on to the dance floor. In fact, for strictly asymmetric firms, we find situations in which more than two must tango (cf. Baye, Kovenock, and de Vries, 1992).

We can also find conditions so that only two tango. For example, if firms with smaller captive audiences also have lower costs, then we can readily pin down a unique equilibrium.

Proposition 1 (Single-Stage Nash Equilibria). *Consider a single-stage model of sales with asymmetric marginal costs and captive audiences sizes. Order firms so that* $p_1^\dagger \geq \cdots \geq p_n^\dagger$.

(i) There is at least one Nash equilibrium. In any equilibrium, the expected profit of firm i *is*

$$\pi_i = \underbrace{\lambda_i(v - c_i)}_{\text{captive-only profit}} + \begin{cases} (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger) & \text{if } i = n, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

so that only the most aggressive firm n *earns (weakly) more than its captive-only profit.*

Now suppose that the two most aggressive firms are uniquely defined so that $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$.

(ii) All firms $i \in \{1, \ldots, n-1\}$ *place an atom at* $p_i = v$, *whereas firm* n *does not. There is some* $p^\star \in (p_{n-2}^\dagger, v]$ *such that all firms* $i \in \{1, \ldots, n-2\}$ *do not price below* $p^\star$ *while firms* $n-1$ *and* n *mix continuously over* $[p_{n-1}^\dagger, p^\star)$ *with distributions*

$$F_{n-1}(p) = \frac{(p - p_{n-1}^\dagger)(\lambda_n + \lambda_S)}{\lambda_S(p - c_n)} \quad \text{and} \quad F_n(p) = \frac{(p - p_{n-1}^\dagger)(\lambda_{n-1} + \lambda_S)}{\lambda_S(p - c_{n-1})}. \tag{4}$$

Now suppose, additionally, that one of the most aggressive firms also has the smallest captive audience size, so that $\min\{\lambda_n, \lambda_{n-1}\} \leq \lambda_i$ *for all* $i \in \{1, \ldots, n-2\}$.

(iii) There is a unique equilibrium. This satisfies $p^\star = v$ *and so all firms* $i \in \{1, \ldots, n-2\}$ *chooses* $p_i = v$ *and serve only captives, while firms* $n-1$ *and* n *mix continuously on* $[p_{n-1}^\dagger, v)$, *with firm* $n-1$ *placing an atom at* v. $F_{n-1}(p)$ *first order stochastically dominates* $F_n(p)$.

(The formal proof, together with the proofs of other results, is contained within Appendix A.)

The first claim of this proposition establishes unambiguously the equilibrium expected profits of firms. At most one firm strictly benefits from its access to shoppers.[7] A special case is when demand conditions are symmetric (so that $\lambda_i = \lambda$ for all λ) and in this case profits are the same as they would be if firms offered a discriminatory price to shoppers. Specifically, firm n earns $\lambda(v - c_n) + \lambda_S(c_{n-1} - c_n)$, which is equivalent to what it would earn if the shoppers are served by it (the lowest-cost firm) at a price equal to the second lowest cost.[8] More generally, by pricing below $p^\dagger_{n-1}$ the most aggressive firm n can guarantee to capture all shoppers. Any price above this invites an "undercut" from the next-most-aggressive firm $n-1$, and for prices ranging upward from $p^\dagger_{n-1}$ we see the familiar mixing from (at least) two firms.

The second claim (which uses the assumption $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$ primarily to streamline exposition) describes the "tango" that is familiar from Baye, Kovenock, and de Vries (1992) and other papers in which two of the firms compete with randomized sales. Only firms $n-1$ and n are willing to price below $p^\dagger_{n-2}$, and the solutions reported in eq. (4) are obtained from solving familiar indifference conditions. Consider, for example, firm $n-1$. It can earn its (equilibrium) captive-only profit of $\lambda_{n-1}(v - c_{n-1})$ by charging v. It is indifferent to charging $p < v$ if

$$\underbrace{(v-p)\lambda_{n-1}}_{\text{loss on captives}} = \underbrace{(p - c_{n-1})\lambda_S(1 - F_n(p)),}_{\text{gain from shopper sales}} \tag{5}$$

which solves for $F_n(p)$. The desire to compete for shoppers via a lower price is lessened if a firm has more captives, and sales to shoppers are less valuable if its marginal cost is higher. Any lower-indexed (and so less aggressive) firm $i \in \{1, \ldots, n-2\}$ that has both more captives ($\lambda_i > \lambda_{n-1}$) and higher costs ($c_i > c_{n-1}$) has, very clearly, a strictly weaker incentive to charge a price $p < v$. Such a firm does not wish to "step on to the dance floor" and so (if this is true for all $i \in \{1, \ldots, n-2\}$) we can construct an equilibrium in which firms $n-1$ and n "tango" by using the distributions reported in eq. (4) all of the way up to v.

However, it is possible for a lower-indexed (and so less aggressive) firm i to have both higher costs (making it less aggressive) but fewer captives (making it more aggressive). This combination can result in a higher $p^\dagger_i$ (so that firm i is not one of the most aggressive pair) but a greater temptation to charge some intermediate price p. To see this explicitly, let us construct a "two to tango" equilibrium in which firms n and $n-1$ mix over the entire interval $[p^\dagger_{n-1}, v)$ according to the distributions reported in eq. (4), while other firms charge $p_i = v$. For this to be an equilibrium, we must be sure that for all p and each firm $i \in \{1, \ldots, n-2\}$

$$(v-p)\lambda_i \geq (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)). \tag{6}$$

Suppose, however, we set $c_i \in (p^\dagger_{n-1}, v)$. This guarantees that $p^\dagger_i > p^\dagger_{n-1}$, and so firm i is not one of the two most aggressive firms. We can now choose λ_i sufficiently small such that the eq. (6) fails. This means that there is a price at which firm i wishes to join the dance floor.

[7] This means that this equilibrium as stated does not hold when there is a positive exogenous (Shelegia and Wilson, 2021) or endogenous (Baye and Morgan, 2001) cost to access a "clearinghouse" for shoppers.

[8] As usual, such a Bertrand construction requires the careful treatment of tie-break rules; for example by breaking a tie favor of a lowest-cost firm. Our reference to discriminatory pricing refers to unit-demand customers. A more general analysis of captive-vs-shoppers discrimination was reported by Armstrong and Vickers (2019).

In essence, this argument adds a third firm to disrupt the tango danced by two existing firms. By choosing this firm's marginal cost to be sufficiently high, we guarantee that the two existing firms will still compete together (in a lower price range) for randomized sales. However, if this third firm has a negligible captive customer base then it will wish to join the action in order to make at least some sales from shoppers. We summarize this as a simple proposition.

Proposition 2 (Three to Tango). *Suppose that there is an equilibrium of a single-stage model of sales in which two firms n and $n-1$ mix over $[p^{\dagger}_{n-1}, v)$ while all other firms charge the monopoly price. We can modify the game either by adding an extra firm or by modifying an existing firm $i \in \{1, \dots, n-2\}$ such that an equilibrium involves mixing by a third firm.*

The "additional firm" can be thought of here as a new entrant to the market. Such an entrant might often be thought to have relatively higher costs, and a smaller established base of locked-in captive customers. Indeed, if that captive population is small then such a firm has an incentive to use a production technology with a relatively high marginal cost if that results in a lower fixed cost. (This corresponds to the choice of production technology in the spirit of Dasgupta and Stiglitz (1980) that we consider in Section 4.) This implies that the presence of such a firm (and the use of randomized sales by more than two competitors) is not necessarily a theoretical curiosity. In Appendix A we construct explicitly an equilibrium in an asymmetric captive-and-shopper triopoly in which all firms play (and must play) mixed strategies. A rough description of the equilibrium structure is that there is some price $p^{\star}$ at which firm $n-1$ "steps off the dance floor" and switches remaining mass to the monopoly price, while the third firm then begins mixing together with firm n.

Efficiency. Proposition 1 contributes to a complete and contained statement of equilibrium pricing in captive-shopper pricing games. However, there are also new economic findings that emerge only when there is cost asymmetry. A first concern is efficiency.

In a model of sales all customers are served. The only efficiency-relevant question is this: who serves the shoppers? This is uncertain (in a mixed-strategy equilibrium) and undetermined (if there are multiple equilibria). However, from an efficiency standpoint, this does not matter if the cost of serving those shoppers is the same for everyone. This changes when costs differ: the outcome is efficient only if shoppers are served by one of the lowest cost firms. If there is a unique such firm then efficiency requires it to be the only supplier of shoppers.

The equilibria described in Proposition 1 allocate output across (at least) two firms. If there is strict cost asymmetry (more generally, if the lowest-cost firm is unique) then necessarily some output is produced at a higher (and inefficient) cost. Efficiency can only be restored if there is cost symmetry, at the lowest level, of all firms that participate in a Nash equilibrium. This condition is stringent. We summarize our observations a corollary to Proposition 4.[9]

[9]We can map out more fully the conditions for efficiency, but at the expense of adding complexity to our statements. We return to the topic of efficiency when we study two-stage pricing.

Corollary (Efficiency of Single-Stage Equilibria). *If the lowest-cost firm is unique then any equilibrium of a single-stage pricing game is inefficient. If the equilibrium involves mixing by only firms $n-1$ and n, then it is efficient if and only if $c_n = c_{n-1} = \min_{i\in\{1,\ldots,n\}}\{c_i\}$.*

Changing Costs. The asymmetric solution allows us to vary the costs of individual firms. For simplicity of exposition, suppose that the three most aggressive firms are strictly asymmetric, so that $p^\dagger_{n-2} > p^\dagger_{n-1} > p^\dagger_n$, and consider local changes that do not change these inequalities. We also consider situations under which the equilibrium involves mixing (or the "tango") by only the two most aggressive firms. Each firm $i \in \{1,\ldots,n-2\}$ sets $p_i = v$, and this is unaffected by any local changes in costs. Moreover, the costs of such firms do not influence prices.

The interesting comparative-static properties concern the marginal costs of firms n and $n-1$.

Inspecting eq. (4) from Proposition 1, c_n enters only into the solution for firm $n-1$. $F_{n-1}(p)$ is increasing in c_n: an increase in the marginal cost of firm n pushes downward the prices charged by firm $n-1$ in the usual sense of first-order stochastic dominance. This is because firm $n-1$ prices more aggressively to maintain the incentive for the (now more costly) firm n to price at $p^\dagger_{n-1}$ rather than the (now more attractive, given the higher cost) higher prices in the interval $[p^\dagger_{n-1}, v)$. This (naturally) implies that the captive customers of firm $n-1$, as well as the shoppers, benefit from any cost increase suffered by the most aggressive firm. If the final claim of Proposition 1 applies then this is also the firm that most often supplies the shoppers.)

The cost c_{n-1} of firm $n-1$ has a more conventional effect. This is a direct effect of an increase in c_{n-1}, by inspection of eq. (4), to increase $F_n(p)$, and so push down the prices charged by firm n. (This follows from the logic discussed just above.) However, an increase in c_{n-1} also raises the lower bound $p^\dagger_{n-1}$ to the interval of sales prices charged by both firms. This lowers both $F_{n-1}(p)$ and $F_n(p)$. There are competing effects on $F_n(p)$, but overall the impact (as the proof of the next proposition confirms) is to push up the prices charged by both competitors.

Proposition 3 (The Effect of Costs on Prices). *Suppose that $p^\dagger_{n-2} > p^\dagger_{n-1} > p^\dagger_n$, and suppose that conditions hold such the (unique) equilibrium involves mixing by only $n-1$ and n.*

(i) Prices do not change in response to local changes in the cost c_i of any firm $i \in \{1,\ldots,n-2\}$.

(ii) A local increase in c_{n-1} shifts upward the distributions of prices charged by $n-1$ and n.

(iii) A local increase in c_n shifts downward the distribution of prices charged by $n-1$.

Claim (iii) implies that firm n disproportionately gains from any reduction in its marginal cost. A reduction in c_n has the usual direct effect on its profit. However, it also prompts a price rise from its competitor (in the market for sales to shoppers) firm $n-1$. This is a positive strategic effect. (A negative strategic effect is more common in pricing games.) In fact,

$$\frac{\partial \pi_i}{\partial c_i} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = n, \text{ and} \\ \lambda_i & \text{otherwise,} \end{cases} \tag{7}$$

and so the most aggressive firm gains distinctly more from a reduction in marginal cost than do other firms. This effect (which is maintained within our two-stage and sequential-move models in subsequent sections) suggests that that there are asymmetric incentives to engage in marginal-cost-reducing process innovations. This is a theme that we take up in Section 4.

Related Literature. We now relate claim (iii) of Proposition 3 to an insightful result from Inderst (2002, Proposition 2). He studied duopoly and triopoly pricing games; we focus on his duopoly case here.[10] (In a captive-shopper oligopoly the main "action" is within the duopoly of firms $n-1$ and n; the other firms are captive-focused bystanders.) For that duopoly case ($n = 2$) his firms are labelled as "incumbent" ($i = n - 1 = 1$) and "entrant" ($i = n = 2$) firms. He specifies zero marginal cost for both. However, the entrant has no captive customers and shoppers face a strictly positive "cost of substitution" to buy from the entrant. This is equivalent (following a relabelling of prices) to setting a positive marginal cost for the entrant. Using our notation, this corresponds to $\lambda_1 > 0$, $\lambda_2 = 0$, $c_1 = 0$, $c_2 > 0$.[11] For the entrant to be the aggressive firm we set $p_2^\dagger < p_1^\dagger$ or equivalently $c_2 < v\lambda_1/(\lambda_1 + \lambda_S)$.

Inderst (2002, p. 451) summarized his key finding to be "the expected price of an incumbent firm may increase in response to increasing competition as it may become more profitable to exploit a rather immobile fraction of consumers instead of capturing a larger but more contested segment of the market." The "increasing competition" is a lower cost-of-substitution parameter c, which is equivalent to the cost of the second firm. Claim (iii) of Proposition 3 recasts this insight of Inderst (2002, Proposition 2, p. 457) by considering costs, and extends to a more general environment in which all firms have captive audiences.

Turning to other literature, very few papers in the model-of-sales tradition have considered asymmetric marginal costs. One exception is a paper by Shelegia (2012) which characterized a duopoly version of Varian (1980) with asymmetric marginal costs but symmetric captive populations. Referencing a doctoral dissertation, Shelegia (2012) also noted that a symmetric-captive but asymmetric-marginal-cost model in which two firms have zero costs while the other firms have marginal cost $c > 0$ has a unique equilibrium in which (as verified by Proposition 1 here) the zero-cost firms randomize.[12] However, we have shown (via Proposition 2) does that this is not always true when there is demand-side asymmetry.

[10] His triopoly model stands outside the captive-shopper framework. Elsewhere (Myatt and Ronayne, 2023a, Appendix A) we reach a different equilibrium characterization to that of Inderst (2002, Lemma 3).

[11] Technically our model description specifies strictly positive captive audiences for every firm. However, we can straightforwardly handle the case where a firm has no captives; it corresponds to the case where $\lambda_2 \downarrow 0$. For Inderst (2002), customers are willing to pay r, there is an adjustment cost of buying for the entrant of c, a mass α of customers are captive the incumbent, and the remaining mass $1 - \alpha$ are shoppers. This maps to our model by setting $n = 2$, $c_1 = 0$, $c_2 = c$, $v = r$, $\lambda_1 = \alpha$, $\lambda_2 = 0$, $\lambda_S = 1 - \alpha$, and so $\lambda_1 + \lambda_S = 1$.

[12] Shelegia (2012) also considered asymmetric costs in the costly search model of Burdett and Judd (1983).

In an important and more recent exception, Shelegia and Wilson (2021) considered a substantially generalized model of sales. We spend time to discuss this here.

Shelegia and Wilson (2021) broadened the model of sales in three ways: (i) their firms (at least for symmetric marginal costs) make utility offers à la Armstrong and Vickers (2001) which allows for downward-sloping demand; (ii) firms pay (fixed) advertising costs for their prices to reach shoppers; and (iii) their model allows for asymmetric marginal costs in a unit-demand setting.[13] Their solution concept carefully specifies tie-break rules as part of the equilibrium.

When marginal costs are asymmetric, the Shelegia and Wilson (2021) model corresponds to ours but where firm i pays a fixed cost $A_i > 0$ to reach the shoppers. (Equivalently, we set $A_i = 0$ in their world to obtain our model.) Beyond their duopoly analysis, their main oligopoly ($n > 2$) finding (Shelegia and Wilson, 2021, Proposition 2; p. 209) is that a unique equilibrium can involve (even with strict asymmetry) mixing (advertised sales to tempt the shoppers) by more than two firms. They also show (as their Corollary 1) that as symmetric advertising costs fall to zero (so that their specification approaches our zero-cost-of-advertising version) then (with asymmetric marginal costs) only two firms mix (or "use sales" in their parlance).

Our Proposition 1 fully characterizes the "tango" between two firms when there is such an equilibrium. This complements Shelegia and Wilson (2021, Corollary 1) by characterizing their zero-advertising-cost limit. However, we do state circumstances (as Proposition 2 makes clear) in which an equilibrium necessarily involves participation by at least three firms in randomized sales. This is different from the claim of Shelegia and Wilson (2021, Corollary 1).

Our results also identify (once again, only when such an equilibrium exists) the identities of firms that sell to shoppers. This is consistent with further results of Shelegia and Wilson (2021, pp. 211–212) which suggest that those should be the firms with smaller captive audiences (Corollary 2) and (given symmetry of captive audiences) those with lower costs (Corollary 3). We offer a more precise statement of which firms chase the shoppers. Additionally, we offer efficiency and comparative-static results.

Summarizing, we see our paper as offering a complementary contribution to that of Shelegia and Wilson (2021) by offering a clear characterization of randomized sales when firms have different costs but when (as in the classic model) shoppers see all prices for free. Our predictions are largely consistent with those offered in their (very rich) framework with costly advertising; the only notable exception is when they indicate that an equilibrium (with vanishing advertising costs) has the "two to tango" property, when we show that this might not be true.

[13] Their firms make "utility offers" as elegantly suggested by Armstrong and Vickers (2001): firm i offers surplus u_i in exchange for profit $\pi_i(u_i)$. Under a unit-demand specification, $u_i = v - p_i$ and so $\pi_i(u_i) = v - u_i - c_i$; a linear trade-off. The approach is more general: it includes situations in which each customer has downward-sloping (multi-unit) demand. However, Shelegia and Wilson (2021, p. 202) clearly explain a restriction (their Assumption U) that is commonly used by others: the consumer surplus that maps to the maximum profit for a firm is constant across firms. Under unit demand, this consumer surplus is zero (from $p_i = v$) and this holds. Under downward-sloping demand the consumer surplus from monopoly pricing changes with the firm's marginal cost. This rules out downward-sloping demand when marginal costs are asymmetric, and so this means that the added benefit from the broader utility-offer specifications is more muted in an asymmetric-cost setting.

2. A Two-Stage Model of Sales

In related work (Myatt and Ronayne, 2023a) we observe that disperse prices do not change at every price-revision opportunity. This is inconsistent (at least for those firms who offer sales to shoppers) with the repeated play of a mixed-strategy Nash equilibrium. We also observe that, once established, there are barriers to upward price adjustments. In response, we study (also in Myatt and Ronayne, 2023a) firms that establish initial price positions, and then subsequently have an asymmetric-direction opportunity to cut (but not raise) those prices. In settings that extend beyond the captive-shopper world we find stable disperse prices: initial (and different, across firms) price positions are chosen, and not subsequently adjusted. However, firms share the same marginal cost. Here we apply the approach to asymmetric-cost models of sales.

Model. We retain the supply-side (firms' costs) and demand-side (captives and shoppers) specifications from the previous section. For simplicity of exposition, we assume that the most aggressive firm is uniquely defined, so that $p_n^\dagger < p_{n-1}^\dagger$. (Everything that we say holds, with suitably adjusted statements, if this is not true.) Now, however, we consider two-stage pricing:

(1) firms simultaneously choose and observe initial price positions $\bar{p}_i \in [0, v]$; and then
(2) firms simultaneously choose final retail prices that must satisfy $p_i \in [0, \bar{p}_i]$.

Shoppers respond to final retail prices (as usual) by choosing the cheapest firm. For technical convenience we (as before) break ties in favor of a lowest-cost firm.

If all firms choose $\bar{p}_i = v$ in the first stage, then the second stage corresponds to the conventional simultaneous-move captive-shopper game (but with asymmetric costs) studied in Section 1.

The sequence described here specifies (equating firm profits to payoffs) an extensive-form game, and the natural solution concept is subgame-perfect equilibrium. We seek such an equilibrium that predicts stable prices. By this we mean a specific set of prices that are chosen and maintained, so that no mixing is observed on the equilibrium path.

Definition. *A profile of prices is* supported in equilibrium by the on-path play of pure strategies *if there is a subgame perfect equilibrium in which (i) those prices are set as initial prices in the first stage; and (ii) on the equilibrium path firms do not lower prices in the second stage.*

This definition allows for sales in the sense that a firm may set initially and maintain a price strictly below v. However, the on-path play of pure strategies means that such sales are not randomized. There is price dispersion (not all prices are the same) and this dispersion is stable (the price profile can be observed from pure strategy play).

Equilibrium Play. Following the definition above, we now seek a profile of prices (an initial price $\bar{p}_i$ for each firm i) that is supported by the on-path equilibrium play of pure strategies.

Only one price profile can meet this definition: firm n prices low enough at the initial stage, by setting $\bar{p}_n = \bar{p}^{\dagger}_{n-1}$, to deter others from undercutting. Those other firms simply set $\bar{p}_i = v$. The following four-step argument shows that this is the unique profile of interest.

Firstly, prices must be undominated: $\bar{p}_i \in [p^{\dagger}_i, v]$ for all i, and so all prices are strictly positive.

Secondly, the lowest price must be unique: if it were not then one of the tied firms would gain from undercutting. We conclude that there is a unique firm i satisfying $\bar{p}_i < \min_{j \neq i} \bar{p}_j$.

Thirdly, all other firms $j \neq i$ sell only to their captive customers. Anticipating that this is so, they charge the captive-exploiting monopoly price: $\bar{p}_j = v$ for $j \neq i$.

Fourthly, no firm j must want to undercut firm i, and so $p^{\dagger}_j \geq \bar{p}_i \geq p^{\dagger}_i$ which means that $i = n$.[14]

By construction the strategy profile in which all firms maintain their prices ($p_i = \bar{p}_i$ for all i) is a (unique) Nash equilibrium at the second stage. Furthermore, no firm wishes to cut its initial price at the first stage: were it to be profitable to do so, then it could execute that price cut at the second stage. (A first-stage price cut could influence the behavior of competitors in the second stage, but only by inducing them to cut their own prices.)

Given that no firm has a profitable deviation in the on-path subgame, and that there is no profitable first-stage deviation downward, it remains for us to check for any possible first-stage deviation upward. This is possible only for firm n. If $\bar{p}_n < p^{\dagger}_{n-1}$ then firm n could deviate upwards (at least locally) without prompting a second-stage undercut, and so earn greater profit. From this we conclude that $\bar{p}_n = p^{\dagger}_{n-1}$. In fact, this strategy profile maximizes industry profit (and is uniquely Pareto efficient) amongst all price profiles that are "undercut proof" in the sense that no firm wishes to undercut any other firm.

If firm n deviates upward to strictly above $p^{\dagger}_{n-1}$ in the first stage, then it is no longer true that firms maintain their prices in the corresponding subgame: at least one of the other firms $i < n$ has an incentive to undercut the (now higher) initial price of firm n. Nevertheless, we can construct an equilibrium of this subgame in which firm n earns its equilibrium-path payoff. (For all other subgames we can specify any equilibrium play.)

Proposition 4 (Stable Prices under Two-Stage Play). *Suppose that the most aggressive firm is unique. The profile of prices in which $\bar{p}_n = p^{\dagger}_{n-1}$ and $\bar{p}_j = v$ for $j < n$ is the unique price profile that is supported in equilibrium by the on-path play of pure strategies. The equilibrium profits of firms are equal to the expected profits reported in Proposition 1.*

Efficiency. We noted that the efficient outcome is for the shoppers to be served by the lowest-cost firm. Under single-stage pricing this is (if the lowest-cost firm is unique) impossible, simply because (at least) two firms compete for the sales to shoppers. Here, however, full efficiency

[14]Of course, if the most aggressive firm is not unique then we construct a profile in which any other most-aggressive firm i (where $p^{\dagger}_i = p^{\dagger}_n$) take adopts the low-price position.

is possible. However, this is true only if the most aggressive firm is also the lowest cost firm. This can fail if a firm with high costs has relatively few captives.[15]

Putting aside complete efficiency, we can also compare the efficiencies of single-stage and two-stage play. Single-stage play (at least under the conditions needed for claim (ii) of Proposition 1) allocates shoppers to firms $n-1$ and n, whereas two-stage play allocates output to only one of those firms. Clearly, two-stage play is more efficient if and only if the second-most-aggressive firm has higher costs than the most aggressive firm.

Proposition 5 (Efficiency of Two-Stage Pricing). *The equilibrium on-path play of pure strategies of a two-stage pricing game is efficient if and only if the most aggressive firm has the lowest marginal cost of production. Comparing to single-stage pricing, label the two most aggressive firms who will fill the positions $n-1$ and n as L and H where $c_L < c_H$. Further assume conditions such that only these two firms compete for shoppers under single-stage pricing.*

Two-stage pricing is more efficient (so that firm L takes the position as the most aggressive firm n) and generates higher consumer surplus than single-stage pricing if and only if

$$\lambda_L - \lambda_H < (c_H - c_L)\frac{\lambda_S + (\lambda_H + \lambda_L)/2}{v - (c_H + c_L)/2}, \tag{8}$$

which means that any offsetting demand-side asymmetry, in terms of captive-audience sizes, must be small relative to the cost advantage of the least-cost firm.

The inequality reported above is, of course, a re-arranged version of eq. (2). It always holds when there is symmetry on the demand side, so that captive audiences are the same size.

Proposition 5 also refers to the outcome for consumers: two-stage pricing (relative to single-stage pricing) is better for consumers if and only if it is more efficient. This is because industry profit is constant across the two settings: any gain in efficiency is captured by customers.

Changing Costs. The effect of changing costs is more straightforward in the two-stage setting, and the (perhaps surprising) effect of an increase in the cost of the most aggressive firm is absent here. All but one of the firms charge the monopoly price (equal to v) to their captive audiences, and so they do not respond to cost changes. The only responsive price, $\bar{p}_n = p^{\dagger}_{n-1}$, charged by the most aggressive firm, is chosen as a limit price to deter the undercut from the nearest competitor firm $n-1$, and so depends on the marginal cost c_{n-1}. This fact, and the responses of firms' profits to changes in their costs, are reported here.

[15] A specification with this feature is the duopoly setting of Inderst (2002) that we discussed in Section 1. His entrant firm has no captives and so (using our terminology) is always more aggressive than his incumbent firm. However, his entrant firm also has higher costs. This means that equilibrium play using on-path pure strategies of a two-stage-pricing version of Inderst (2002) is necessarily inefficient.

Proposition 6 (The Effect of Costs on Two-Stage Pricing). *Under the equilibrium on-path play of pure strategies of a two-stage pricing game, an increase in the cost of the second most aggressive firm raises the price (charged to shoppers) of the most aggressive firm. Local changes in other costs have no effect on prices. The effect of costs on firms' profits are*

$$\frac{\partial \pi_i}{\partial c_j} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = j = n, \\ \lambda_i & \text{if } i = j \neq n, \text{ and} \\ 0 & \text{if } i \neq j, \end{cases} \tag{9}$$

and so the most aggressive firm benefits distinctly more from cost reductions than other firms.

Risk Aversion and Initial Price Positions. The traditional single-stage game à la Varian (1980) and the on-path play of pure strategies in our two-stage version generate the same expected profits. This is a strength in the sense that either game can be used as part of a deeper model without influencing early-stage decisions. However, it does mean that firm n is indifferent between taking the initial undercut-deterring position $\bar{p}_n = p^{\dagger}_{n-1}$ or instead choosing $\bar{p}_n = v$.[16] Indeed, firm n achieves the same expected profit from any intermediate initial price. An argument in favor of the conventional single-stage model (with its mixed-strategy play) is that firm n might just "wait and see" rather than making an early move.

A response is that an initial low-price position results in a certain profit outcome, whereas a higher initial price results in uncertain profits. Risk neutrality means that firm n is indifferent between these options. Nevertheless, a reasonable suggestion is that a desire for a predictable outcome might push firm n to be in favor of the first option.

An easy way to illustrate this is to modify our model to include risk aversion. To do this we develop an approach that we suggested, albeit briefly, in a supplement to Myatt and Ronayne (2019, Appendix A). Suppose that we split each firm into two players: a manager, and an operational pricing agent. We define an extensive-form game with $2n$ players in which

(1) the firms' managers simultaneously choose initial price positions $\bar{p}_i \in [0, v]$; and then
(2) the firms' agents simultaneously choose their firms' retail prices $p_i \in [0, \bar{p}_i]$.

Agents' payoffs are profits, and they are assumed to be risk neutral. The manager of firm i, however, has payoff $u_i(\pi_i)$, which is a smoothly increasing and concave function of the firm's profit. The important assumption is that the manager is more risk averse than the pricing agent. The manager has the ability and incentive to constrain the agent in the second stage: doing so can induce the play of a preferred equilibrium (from the perspective of the manager).

Equilibrium play in any subgame is unaffected by this "$2n$ player" scenario. If firms' managers choose the initial prices reported in Proposition 4 then they obtain payoffs $u_i(\pi_i)$ where π_i the profit of firm i. Any upward deviation by firm n to $\bar{p}_n > p^{\dagger}_{n-1}$ leads to a subgame with the

[16]Our two-stage pricing game has a subgame perfect equilibrium in which all firms choose $\bar{p}_i = v$ then go on (on the equilibrium path) to play the unique mixed-strategy Nash equilibrium of Proposition 1.

same expected profit, but strictly lower expected utility. This means that manager n's choice of $\bar{p}_n = p^\dagger_{n-1}$ is the unique best reply to the initial price choices $\bar{p}_i = v$ for all $i \neq n$. Our (subgame perfect) equilibrium (with the on-path play of pure strategies) is strict.

This argument rules out a replication of single-stage equilibrium play. (This would require firms' managers to all choose $\bar{p}_i = v$ at the first stage, and we have shown that firm n would deviate from this.) In fact, it rules out all other price profiles at the initial stage.

Proposition 7 (Two-Stage Pricing with Risk-Averse Players). *Consider the model of two-stage pricing in which initial prices are chosen by risk-averse managers, whereas second-stage final retail prices are chosen by risk-neutral pricing agents. In any subgame perfect equilibrium firm n chooses $\bar{p}_n = p^\dagger_{n-1}$ and firms $i \neq n$ choose $\bar{p}_i = v$ at the initial stage; these prices are maintained, so that $p_i = \bar{p}_i$ for all i, at the second stage on the equilibrium path.*

3. Sequential-Move Models of Sales

Two-stage pricing offers a first-stage commitment opportunity that is related to Stackelberg-style settings. Here we briefly discuss exogenous move order, before we build a game in which firms choose if and when to commit to advertised price positions.

Exogenously Sequential Play. We retain our supply-side and demand-side specifications and, to ease exposition, we assume that $p^\dagger_1 > \cdots > p^\dagger_n$ and we break ties in favor of more aggressive firms. Now suppose that the n firms choose their prices in some set sequence.

An easy case is when firms n and $i \neq n$ are the last two firms to move, and when all other firms $j \notin \{i, n\}$ have already chosen $p_j = v$. If firm i moves before n then it recognizes that any price $p_i \in [p^\dagger_i, v]$ will be undercut by player n, and so it focuses on shoppers by setting $p_i = v$. Firm n (moving last) then then captures the shoppers at pruce v. If instead firm n moves before firm i then it chooses $p_n = p^\dagger_i$ to deter any final-period shopper-stealing undercut by firm i.

More generally, firm n (no matter when it moves in the sequence) always captures the sales to shoppers. Given this, a firm i moving before firm n recognizes that it will only sell to captives and so sets $p_i = v$. Firm n needs to deter any undercut by a later-moving firm, and so sets p_i equal to the lowest $p^\dagger_j$ amongst all firms j that follow it.

Proposition 8 (Equilibrium with Exogenous Sequencing). *On the equilibrium path of any subgame perfect equilibrium, each firm $i < n$ chooses $p_i = v$ when called upon to move. If firm n moves at the last period then it chooses $p_n = v$. If it moves earlier then it chooses $p_n = p^\dagger_j$ where j is the most aggressive firm that has yet to move at that time.*

The allocation of shoppers here is unambiguous. However, the price that they (and the captives of firm n) pay (and so the division of surplus between customers and the most aggresive firm) depends on the exogenous move order. Clearly, firm n would rather move later in that order.

Nevertheless, an exogenous-move-order scenario is sensitive to manipulation. Suppose, for example, that we see a firm's decision to set a price as the act of advertising to make that price accessible to price-comparing shoppers. Any firm $i < n$ has no reason to rush to advertise and do this; ultimately, it sells only to captive customers. On the other hand, firm n wants to move early to bring in those shoppers. We now build a model to incorporate these considerations.

Endogenously Sequential Play. What if firms endogenously choose when to move? We now study firms who decide when and at what price to advertise to shoppers. Such a decision commits a firm to that price. However, shoppers can only be reached via advertised prices.

Formally, this is a multi-stage game that takes place over T discrete periods. Each period firms must chooses price positions and whether to advertise those positions, where the decision to advertise locks in a firm in all future periods. Each time $t \in \{1, 2, \ldots, T\}$ proceeds as follows.

(1) All firms observe the entire history of the game so far, and then
  (a) a firm that advertised in period $t-1$ advertises the same price in period t; but
  (b) other firms simultaneously choose a price and whether to advertise that price.
(2) Shoppers buy from (one of) the cheapest advertised prices. Captives buy as usual.

Profits accrue without discounting across all T periods where the mass of sales at time t is scaled by $1/T$.[17] Amongst sales to shoppers, we retain the freedom to choose appropriate tie-break rules in the event of tied prices. Firms are strictly ordered so that $p_1^\dagger > \cdots > p_n^\dagger$. These assumptions help us to cope with unimportant technicalities and to facilitate exposition.

Firm n can guarantee a profit of $(p_{n-1}^\dagger - c_n)(\lambda_n + \lambda_S)$, which is equal to its captive-only profit plus $(p_{n-1}^\dagger - p_n^\dagger)(\lambda_n + \lambda_S)$. It can do this by advertising immediately (at time $t = 1$) at a price $p_n = p_{n-1}^\dagger$. If it does this, then other firms are willing to sit back, charge v to their captives, and refrain from advertising. The possible deviation here is upward by firm n to a higher advertised price at time $t = 1$. Doing so, however, firm n will be undercut and loses shopper sales for the remaining $T - 1$ periods. If T is sufficiently large, then firm n prefers not to do this.

Proposition 9 (Equilibrium with Endogenous Sequencing). *If T is sufficiently large then, in a subgame-perfect equilibrium with suitable tie-break rules, the most aggressive firm n chooses and advertises a shopper-serving low-price position at the first opportunity. All other firms set the monopoly price in every period, and do not advertise in the first period.*

This outcome replicates (of course) the outcome from two-stage play that we characterized in Section 2 in which one firm (this is firm n here) stands apart from the others and uniquely sells to the shoppers. In essence, a desire to avoid delay gives us this outcome here; under two-stage play risk aversion had a similar effect. These two familiar forces (risk aversion and impatience) push us towards a stable price profile in which the "tango" is danced by a single firm. Nevertheless, the profit outcomes match those of a familiar single-stage pricing game.

[17]We can interpret the demand as either a flow of new customers, or as repeated sales. We can also handle several other specifications, including an infinite time horizon with discounting or play in continuous time.

4. Process Innovations and Endogenous Asymmetry

Our comparative-static results reveal the different incentives that firms face to engage in cost reduction. (The same findings would apply to any actions which increase the value offered by a firm's product.) Here we consider situations in which firms' costs (or customers' valuations for their products) arise endogenously from their innovative activities.

A Model of Endogenous Process Innovation. We retain the core specifications from the previous sections. However, prior to the pricing stage (or stages) firms endogenously choose their production technologies via costly innovations. We study the following game.

(1) Firms simultaneously choose and observe production technologies indexed by z_i.
(2) Firms proceed either to (a) single-stage or (b) two-stage pricing where:
   (a) firms play a Nash equilibrium of the single-stage pricing game (of Section 1); or
   (b) firms play a subgame-perfect equilibrium of the two-stage pricing game (of Section 2) that supports the on-path play of pure strategies.

We interpret a firm's technology choice $z_i \in [0, \bar{z}_i]$ as a fixed cost expenditure which lowers its marginal cost of production: it is a (costly) process innovation. It would (as usual) be entirely equivalent to think of a product innovation that raises customers' willingness to pay for that firm's product. What really matters is the net surplus $v - c_i$ offered. We assume that

$$v - c_i = V_i(z_i) \tag{10}$$

where $V_i(z_i)$ positive, smoothly increasing, concave, and satisfies $\lambda_i V_i'(0) > 1 > (\lambda_i + \lambda_S) V_i'(\bar{z}_i)$.

For either version (a) or (b) of pricing from the sequence above, the gross equilibrium expected profit of a firm is taken from the expression reported in eq. (3) of Proposition 1. It does not matter (for the purposes of innovation) which version we use. However, eq. (3) assumes that firms are labelled (without loss of generality for Proposition 1) so that they are ordered by their aggressiveness. Here, however, that order is endogenous given that the firms actively choose their technologies. For now, then, we do not label firms according to any order. The general expression for a firm's net expected profit is

$$\pi_i = \lambda_i (v - c_i) + (\lambda_i + \lambda_S) \max\left\{0, \min_{j \neq i}\{p_j^\dagger\} - p_i^\dagger\right\} - z_i \quad \text{where} \quad v - c_i = V_i(z_i)$$

$$\text{and where, as before,} \quad p_j^\dagger = \frac{\lambda_j v + \lambda_S c_j}{\lambda_j + \lambda_S} = v - \frac{\lambda_S V_j(z_j)}{\lambda_j + \lambda_S}. \tag{11}$$

Using these n expected profit expressions as the outcomes from stage (2) described above, we have specified a simultaneous-move innovation game. We look for pure-strategy Nash equilibria.

Asymmetric Equilibrium Innovation. Consider the response of firm i's profit to a local increase in z_i. Firm i is the most aggressive firm if and only if $p_i^\dagger < p_j^\dagger$, or equivalently

$$\frac{V_i(z_i)}{\lambda_i+\lambda_S} > \max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j+\lambda_S}\right\} \;\Rightarrow\; \frac{\partial \pi_i}{\partial z_i} = -1 + V_i'(z_i)\begin{cases}\lambda_i+\lambda_S & \frac{V_i(z_i)}{\lambda_i+\lambda_S} > \max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j+\lambda_S}\right\}\\ \lambda_i & \frac{V_i(z_i)}{\lambda_i+\lambda_S} < \max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j+\lambda_S}\right\}\end{cases} \quad (12)$$

This says that the profit of firm i has a (convex) kink at the point where i becomes the most aggressive firm. It implies that firm i will never optimally choose z_i such that $p_i^\dagger = \min_{j\neq i}\{p_j^\dagger\}$ and so, in equilibrium, the most aggressive firm is distinct.

We can, therefore, characterize two possible solutions for firm i's choice of technology. We write z_i^H for its optimal innovation when it is the most aggressive firm, and z_i^L for when it is not. Given the regularity conditions, these solutions are uniquely determined by the conditions

$$1 = \lambda_i V_i'(z_i^L) = (\lambda_i+\lambda_S)V_i'(z_i^H), \quad (13)$$

satisfy $z_i^H > z_i^L$, and do not depend on the innovation choices of any other firms.

To find an equilibrium of our innovation game we follow this recipe: (i) pick a firm $i \in \{1,\dots,n\}$ to be the aggressive firm; (ii) set $z_i = z_i^H$ for that firm; and then (iii) set $z_j = z_j^L$ for all other firms $j \neq i$. This is a candidate for an equilibrium. However, two checks are needed.

Firstly, this works only if firm i is the most aggressive firm. This requires $p_i^\dagger < \min_{j\neq i}\{p_j^\dagger\}$ or

$$\frac{V_i(z_i^H)}{\lambda_i+\lambda_S} > \max_{j\neq i}\left\{\frac{V_i(z_j^L)}{\lambda_j+\lambda_S}\right\}. \quad (14)$$

Secondly, we need to check that no firm j wishes to deviate from z_j^L to z_j^H. For this deviation to be feasible z_j^H needs to be high enough to displace firm i as the most aggressive firm. For example, this is not possible (and so we do have an equilibrium) if firm i maximizes $V(z_i^H)/(\lambda_i+\lambda_S)$ across the set of firms. (This argument establishes the existence of at least one equilibrium.) Otherwise, we need to check. We report our findings formally here.

Proposition 10 (Asymmetric Innovation Equilibria). *Consider the innovation game.*

(i) There is at least one pure-strategy Nash equilibrium, and there are at most n such equilibria.

(ii) In any equilibrium there is some firm k that, at the pricing stage, is the uniquely most aggressive firm. One such equilibrium is when $k \in \arg\max_j V(z_j^H)/(\lambda_j+\lambda_S)$.

(iii) If $\arg\max_j V(z_j^L)/\lambda_j$ is unique then there is unique equilibrium if λ_S is sufficiently small.

(iv) If firms are symmetric then there are exactly n pure-strategy equilibria.

The final claim of the proposition (finding n equilibria, with any one of the symmetric firms eventually taking the aggressive sell-to-the-shoppers position) also holds when firms are sufficiently similar. It is also of interest because it implies that exogenously symmetric firms become endogenously asymmetric. One of the firms (it could be any) behaves distinctly differently.

Corollary. *Suppose that firms have the same sized captive audiences and the same technological opportunities. In a pure-strategy equilibrium their innovation choices and pricing behavior are asymmetric. One of the firms chooses strictly higher innovation, and so ultimately has a strictly lower marginal cost, whereas the other $n-1$ firms chooses the same (and lower) innovation.*

We emphasize this as a corollary because it provides a distinct rationale for opening up the model of sales to cost asymmetry. Such asymmetry is a relevant case simply because (with endogenous innovation) it arises ex post even when firms are symmetric ex ante.

From Demand-Side to Supply-Side Asymmetry. Proposition 10 establishes that there may be circumstances (when λ_S is small) in which there are fewer than n equilibria. However, it does not specify which firms are able to claim the shopper-supplying position.

Here we develop further results when firms have symmetric innovation opportunities, but when their captive audiences can differ in size. Let us order firms (strictly, for simplicity of exposition) by such sizes, so that $\lambda_1 > \cdots > \lambda_n$ which means that these firms would also be ordered by aggressiveness if they shared the same marginal cost. Give this demand-side asymmetry, we maintain technological opportunities that are constant across firms so that $v - c_i = V(z_i)$ for all i. Moreover, we also adopt a convenient functional form here:

$$V(z) = \beta z^{\gamma} \quad \text{where} \quad \gamma \in (0,1). \tag{15}$$

The parameter $\gamma = zV'(z)/V(z)$ is the elasticity of a firm's per-customer surplus with respect to its fixed-cost innovation outlay. It represents, therefore, a measure of the shape of technological opportunity to a firm, whereas the parameter β scales that opportunity.[18]

Equations (13) and (15) allow us to express in closed form the innovation decisions of firms:

$$z_i^L = (\gamma\beta\lambda_i)^{1/(1-\gamma)} \quad \text{and} \quad z_i^H = (\gamma\beta(\lambda_i + \lambda_S))^{1/(1-\gamma)}. \tag{16}$$

Very naturally in each case the innovative fixed-cost expenditure of a firm increases with the size of its captive audience, and the elasticity of that relationship is determined by the shape (via its elasticity γ) of the technological opportunity.

We are more interested, however, in how the size of a firm's captive audience (this is concerned with demand-side asymmetry) influences the surplus $v - c_i$ generated by its product (this is endogenous supply-side asymmetry). This is (depending on whether it expects to serve captives)

$$V(z_i^L) = \beta\,(\gamma\beta\lambda_i)^{\gamma/(1-\gamma)} \quad \text{or} \quad V(z_i^H) = \beta\,(\gamma\beta(\lambda_i + \lambda_S))^{\gamma/(1-\gamma)}. \tag{17}$$

The elasticity of per-consumer surplus $v - c_i = V(z_i^L)$ with respect to the captive-audience size λ_i (for a firm that expects to serve only its captives) is, by inspection, $\gamma/(1-\gamma)$. This strictly exceeds one, and so is an elastic relationship, if and only if $\gamma > \frac{1}{2}$.

[18] This specification is reminiscent (albeit different from) the classic constant-elasticity relationship between production cost and research-and-development expenditure in Dasgupta and Stiglitz (1980, p. 273). Here the constant elasticity is between the per-unit surplus and any fixed-cost expenditure.

To pin down the equilibrium of the innovation game we need to identify a firm that ultimately becomes the most aggressive competitor and so serves the shoppers. The key criterion for whether this is possible for a firm i is the inequality eq. (14). We can evaluate this explicitly given the solutions reported here. Moreover, we know that the criterion always holds (and indeed there is always an equilibrium) for the firm k that maximizes $V(z_i^H)/(\lambda_i + \lambda_s)$. (This is claim (ii) of Proposition 10.) In fact, straightforward substitution yields

$$\frac{V(z_i^H)}{\lambda_i + \lambda_s} = \beta^{1/(1-\gamma)}\gamma^{\gamma/(1-\gamma)} \left(\lambda_i + \lambda_S\right)^{(2\gamma-1)/(1-\gamma)} \tag{18}$$

By inspection this is constant if $\gamma = \frac{1}{2}$. This means that every one of the n firms maximizes this expression. In turn, this means that for every firm we can (for this parameter choice) find an equilibrium of the innovation game in which that firm becomes the most aggressive competitor.

If $\gamma > \frac{1}{2}$ then the per-customer surplus is relatively responsive to the innovation expenditure. This means that a firm with a larger captive audience endogenously chooses surplus that is disproportionately larger. The implicit cost advantage (which makes it more aggressive) more than offsets its larger captive audience (which makes it less aggressive). Ultimately, this means that it is the lower-indexed firms (with more captives) that can take the lead as the (endogenously) most aggressive firm that eventually serves the shoppers. If $\gamma < \frac{1}{2}$ then of course the logic reverses, and the higher-indexed firms (with fewer captives) are candidates for the role of the most aggressive firm. In both cases, there can be more than one possible firm to take the role, but the list of possible candidates shrinks with the population of shoppers.

Proposition 11 (Innovation and Captive Audience Sizes). *Consider the innovation game in which all firms share the same technological opportunity via* $v - c_i = V(z_i) = \beta z_i^\gamma$, *and recall that firms are ordered by captive-audience sizes so that* $\lambda_1 > \cdots > \lambda_n$.

(i) If $\gamma = \frac{1}{2}$ *then there are* n *equilibria. Any firm can take the most aggressive position.*

(ii) If $\gamma > \frac{1}{2}$, *so that innovation opportunities are relatively strong, then there is some* k *such that any firm* $i \in \{1, \ldots, k\}$ *can take the most aggressive position.*

(iii) If $\gamma < \frac{1}{2}$, *so that innovation opportunities are relatively weak, then there is some* k *such that any firm* $i \in \{n-k+1, \ldots, n\}$ *can take the most aggressive position.*

For $\gamma \neq \frac{1}{2}$ *and cases (ii) and (iii),* $k = 1$ *if the mass of shoppers* λ_S *is sufficiently small.*

Our study of two-stage pricing identified a unique firm that serves shoppers, but did so based on exogenously specified supply-side and demand-side asymmetries. Proposition 11 reveals how those demand-side asymmetries (the pattern of captive-customer populations) can intereact with the shape of technological opportunities to determine both supply-side asymmetry and to decide which firm serves the shoppers. Nevertheless, endogenous actions can also influence the demand side and so we comment briefly on this next before offering concluding remarks.

Endogenous Captive Audiences. A asymmetric innovation decisions emerge because the profit of the most aggressive firm reacts in a distinctly different way when we change its costs. Chioveanu (2008) identified a related effect. She specified firms where costly advertising (equivalent to our innovation expenditure here) influences the size of captive audiences (where an increase in λ_i from action by firm i can also influence λ_j for $j \neq i$). The advertising stage is followed by a conventional single-stage pricing game. Chioveanu (2008, Proposition 3) identified an equilibrium in which one firm advertises strictly less than others, and so (p. 68) "despite ex ante symmetry, the equilibrium profile of advertising outlays is asymmetric."

We can confirm that this important insight also holds when costs are asymmetric. If firm i is not the most aggressive, then very straightforwardly

$$\frac{\partial \pi_i}{\partial \lambda_i} = v - c_i \quad \text{and} \quad \frac{\partial \pi_i}{\partial \lambda_j} = 0 \quad \text{for} \quad j \neq i. \tag{19}$$

On the other hand, if firm i is the most aggressive, so that eq. (14) holds, then

$$\frac{\partial \pi_i}{\partial \lambda_i} = v - c_i - \lambda_S \max_{j \neq i} \left\{ \frac{v - c_j}{\lambda_j + \lambda_S} \right\} \quad \text{and} \quad \frac{\partial \pi_i}{\partial \lambda_k} > 0 \quad \text{if} \quad k = \arg\max \left\{ \frac{v - c_j}{\lambda_j + \lambda_S} \right\}. \tag{20}$$

Once again there is a kink in the response of a firm's profit, which (if we were to view the acquisition of captives as costly) pushes towards an asymmetric equilibrium.

The result of Chioveanu (2008) also resonates with early work by Ireland (1993) and McAfee (1994) which considered firms that independently advertise to customers à la Butters (1977) and Grossman and Shapiro (1984) before choosing prices.[19] In those papers one firm advertises distinctly more than others, even if they are symmetric ex ant. In our related work we obtain the same result using two-stage pricing (Myatt and Ronayne, 2023a).

A key difference between the cost-reducing process innovations here and the captive-audience-enhancing advertising of Chioveanu (2008) is that here the distinct firm (that is, the aggressive firm that expects to serve shoppers) faces a stronger incentive to innovate, whereas the standalone firm faces a weaker incentive to advertise. However, the underlying force is the same: that firm faces a stronger incentive than others to become more aggressive. The difference is that aggression is achieve by over-investment in one case, and under-investment in the other.

The forces which push toward asymmetry here come from the decisions of firms. This contrasts the extended model of Baye, Kovenock, and de Vries (1992, Section V) in which captive consumers are able to switch between firms. They identified an incentive for customers to shift toward (in our language) the most aggressive firm, which corresponds (in their setting) to the firm with fewest captives. This underpinned their argument (and their Theorem 3) that the play of a symmetric equilibrium by symmetric firms is most reasonable. Here we join Chioveanu (2008) and others in suggesting that asymmetric outcomes may be more likely.

[19] A customer sees the price of firm i with probability α_i. Given a unit mass of potential customers, the mass of shoppers is $\lambda_S = \prod_{i=1}^{n} \alpha_i$, the captive audience of firm i has size $\lambda_i = \alpha_i \prod_{j \neq i}(1 - \alpha_j)$, and (for example) there are $\lambda_{ij} = \alpha_i \alpha_k \prod_{k \notin \{i,j\}}(1 - \alpha_k)$ customers who compare the pair of i and j.

Shelegia and Wilson (2021) also considered the incentives of firms to engage in costly activities that influence the marketplace. They studied (Shelegia and Wilson, 2021, Section III.B, pp. 214–216) a game in which costly effort increases the size of a firm's captive audience (and possibly reduces the captive audiences of competitors). This is the approach of Chioveanu (2008). However, the result (their Proposition 3) refers to the properties of a symmetric equilibrium (in which all firms make the same captive-enhancing effort choices) when such an equilibrium exists. Here, however, we align with Chioveanu (2008) in suggesting that an equilibrium will involve asymmetric choices, which precludes the existence of the symmetric equilibrium. Similarly, their study of the comparative-static effects of firm profitability (here this is equivalent to $v - c_i$) also begins from a symmetric situation whereas we suggest (as our Proposition 10) that asymmetric outcomes may be more likely.

5. Concluding Remarks

$\star$ **To be completed** $\star$

Appendix A. Omitted Proofs

Before proving Proposition 1, we state several (relatively standard) properties that must hold for the equilibria of a single-stage model of sales. Abusing (by recycling) notation from our two-stage model (in one case) we write $\bar{p}_i = \inf\{p : F_i(p) = 1\}$ and $\underline{p}_i = \sup\{p : F_i(p) = 0\}$ for the upper and lower bounds of the support of firm i. As usual we say that a firm plays an atom if it chooses a price with strictly positive probability. We also say that firm i actively mixes at some price $p_i < v$ if its distribution is strictly increasing at that price.

Lemma 1 (Atoms). *There can be no atoms strictly below v, and at most $n - 1$ atoms at v.*

Proof. A firm will choose a price strictly below v with positive probability only if that price can win shoppers. However, no other firm would then choose that price or just above it: it would be better to undercut and capture the atom. This means that the atom-playing firm can safely raise its price locally (strictly gaining profit from captives) without losing any sales. This contradiction proves for the first claim. Turning to the second claim, if there were n atoms at v then at least one firm would undercut the others and so (by capturing the joint atom of the other $n - 1$ firms) strictly increase expected profit; again a contradiction. □

Lemma 2 (Highest Prices). *The upper bound of the support of prices for firm i is $\bar{p}_i = v$.*

Proof. If $\bar{p}_i < v$ then no firm $j \neq i$ would choose $p_j \in [\tilde{p}_i, v)$, because such a price would sell only to captives and j would strictly prefer $p_j = v$. This means that firm i can strictly gain from raising its price from $p_i = \bar{p}_i$ (and for slightly lower prices). This is a contradiction. □

Lemma 3 (Gaps). *There is no gap in the joint support of firms' strategies. Relatedly, if any interval is in the support for some firm i then it is in the support for some other firm $j \neq i$.*

Proof. Suppose that there is such gap. Expand the gap downward to its lowest possible price to find a price in the support of some firm i. Noting that there are no atoms, firm i can strictly gain (it loses no sales) by shifting that price upward into the gap. Relatedly, suppose that a firm i prices within an interval that is not in the support of any other firm, so that it is in gap in everyone else's support. Firm i could again raise that price without losing any sales. □

Lemma 4 (Captive-Only Profits). *At least $n-1$ of the firms earn their captive-only profit.*

Proof. At least one firm i does not place an atom at v (from Lemma 1) so that $F_i(p_i)$ increases continuously to $F_i(v) = 1$. Each firm $j \neq i$ is willing to price at or arbitrarily close to v (given that $\bar{p}_j = v$ for all j) and when doing so loses shoppers to firm i. This means that j earns the captive-only profit $\lambda_j(v_j - c_j)$ from charging v to its captive customers. □

Lemma 5 (Profits). *If the two most aggressive firms $n-1$ and n satisfy $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ then: (i) each firm $i < n$ earns its captive-only profit; (ii) firm n earns $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n) > \lambda_n(v - c_n)$; (iii) every firm $i < n$ places an atom at*

Proof. Firm n can guarantee a profit strictly higher than its captive-only profit by setting $p_n = p_{n-1}^\dagger$ and so earning $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n)$. We have established that $n-1$ firms earn captive-only profits, and so this must apply to all firms $i \in \{1, \ldots, n-1\}$. This is claim (i).

If firm n were to earn strictly more than $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n)$, then $\underline{p}_n > p_{n-1}^\dagger$. Firm $n-1$ could then set a price $p_{n-1} \in (p_{n-1}^\dagger, \min\{\underline{p}_n, p_{n-2}^\dagger\})$ which would capture shoppers and earn strictly more than its captive-only profit; a contradiction. We have established claim (ii). □

Proof of Proposition 1. An equilibrium exists because (given the use of our tie-break rule in which ties are broken in favor of a lowest-cost firm) the conditions of Theorem 5 of Dasgupta and Maskin (1986, p. 14) are satisfied. Specifically, that theorem asks for the sum of players' payoffs (the total industry profit here) to be upper semi-continuous in actions (here, this is the profile of prices) and this holds if ties are broken in favor of lower-cost firms.

The expected profits of firms in equilibrium are established by Lemmas 4 and 5.

Turning to claim (ii), if the lower bound of all prices were to strictly exceed $p_{n-1}^\dagger$ then (at least) firms n and $n-1$ could (by pricing just above $p_{n-1}^\dagger$) sell to all shoppers and achieve a profit strictly exceeding their equilibrium profit. We conclude that the joint support of firms' mixed strategies extends down to $\min_i \underline{p}_i = p_{n-1}^\dagger$. Prices below $p_{n-2}^\dagger$ are strictly dominated for firms $i \in \{1, \ldots, n-2\}$, and so (given the absence of gaps, from Lemma 3) firms $n-1$ and n must mix continuously over $[p_{n-1}^\dagger, p_{n-2}^\dagger]$. Given that they both price below $p_{n-2}^\dagger$ with strictly positive probability, price at or just above $p^\dagger_{n-2}$ will not be played by any firm $i \in \{1, \ldots, n-2\}$, and so there is some $p^\star > p^\dagger_{n-2}$ such that firms $n-1$ and n mix on the interval $[p^\dagger_{n-1}, p^\star)$.

The expected profit earned by firm $n-1$ from charging a price $p \in [p^\dagger_{n-1}, p^\star)$ is

$$\pi_{n-1}(p) = (p - c_{n-1})\left(\lambda_{n-1} + \lambda_S\left(1 - F_n(p)\right)\right) = \lambda_{n-1}(v - c_{n-1}), \tag{21}$$

where the final term is its captive-only profit. Similarly, the expected profit earned by firm n from charging a price $p \in [p^\dagger_{n-1}, p^\star)$ is

$$\pi_n(p) = (p - c_n)\left(\lambda_n + \lambda_S\left(1 - F_{n-1}(p)\right)\right) = \lambda_n(v - c_n) + (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n) \tag{22}$$

where the final expression is the profit of firm n from Lemma 5. These equations solve:

$$F_n(v) = 1 - \frac{\lambda_{n-1}(v-p)}{\lambda_S(p - c_{n-1})} \quad \text{and} \quad F_{n-1}(v) = 1 - \frac{\lambda_n(v-p)}{\lambda_S(p-c_n)} - \frac{(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)}{\lambda_S(p - c_n)}. \tag{23}$$

These are valid cumulative distribution functions that strictly and continuously increase from $F_{n-1}(p^\dagger_{n-1}) = F_{n-1}(p^\dagger_{n-1}) = 0$, and they can be re-written to obtain eq. (4).

Any equilibrium involves mixing by the dance partners $n-1$ and n up to some price $p^\star$. One possibility is that $p^\star = v$, so that only these two firms mix and all other firms chooses $p_i = v$. By inspection, the $\lim_{p\uparrow v} F_n(p) = 1$ and $\lim_{p\uparrow v} F_{n-1}(p) \leq 1$ (the latter inequality is strict if $p^\dagger_{n-1} > p^\dagger_n$) which means that we have valid distributions over the entire interval. Firms $n-1$ and n cannot improve by deviating. If there is an equilibrium in which only these two firms mix then (by construction) it is unique. However, we need to check to see if some other firm i might wish to deviate to $p_i < v$. Firm $i \in \{1, \ldots, n-2\}$ earns its captive-only monopoly profit $\lambda_i(v - c_i)$. By deviating to $p \in [p^\dagger_{n-1}, v)$, and assuming that $\lambda_{n-1} \leq \lambda_i$, it earns

$$\pi_i(p) = (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_{n-1}(p)\right)\left(1 - F_n(p)\right)\right) \tag{24}$$

$$< (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_n(p)\right)\right) \tag{25}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_{n-1}(v-p)}{(p - c_{n-1})}\right) \tag{26}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_{n-1}(v-p)}{\lambda_{n-1}(v - p^\dagger_{n-1}) + \lambda_S(p - p^\dagger_{n-1})}\right) \tag{27}$$

$$\leq (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_i (v-p)}{\lambda_i(v - p^\dagger_i) + \lambda_S(p - p^\dagger_i)}\right) \tag{28}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_i(v-p)}{(p - c_i)}\right) = \lambda_i(v - c_i). \tag{29}$$

The third is obtained by substituting in the expression for $F_n(p)$. The fourth line is obtained by writing c_{n-1} in terms of λ_{n-1} and $p^\dagger_{n-1}$ and then re-arranging. The fifth line holds because $\lambda_{n-1} \leq \lambda_i$ and $p^\dagger_{n-1} \leq p^\dagger_i$. The final line is obtained by substituting back in for $p^\dagger_i$ and then re-arranging. This means that firm i performs strictly worse by deviating. We assumed that $\lambda_{n-1} \leq \lambda_i$, but a similar sequence of steps reaches the same conclusion if $\lambda_n \leq \lambda_i$. (The claim in the proposition assumes that $\min\{\lambda_{n-1}, \lambda_n\} \leq \lambda_i$, because either firm $n-1$ or n is the firm with the fewest captives.) We have established the existence of a (unique, within this class) "two to tango" equilibrium. It remains to check whether than can be other equilibria. This would require another firm to begin mixing (or to "step on to the dance floor") at some price $p^\star < v$. However, the argument above demonstrates (given the lack of atoms below v, and the continuity properties) that that is would be strictly sub-optimal for such a firm. □

Proof of Proposition 2. This follows from the preceding argument in the main text. □

Proof of Proposition 3. Claim (i) holds because firms $i < n-2$ play $p_i = v$ as pure strategies, and their cost parameters do not enter the solutions for the mixed strategies of firms $n-1$ and n which are reported in eq. (4). For claim (ii), by inspection $F_{n-1}(p)$ is decreasing in $p^\dagger_{n-1}$ which itself is increasing in c_{n-1}. For $F_n(p)$,

$$\frac{\partial F_n(p)}{\partial c_{n-1}} = \frac{\lambda_{n-1}+\lambda_S}{\lambda_S}\left(\frac{(p-p^\dagger_{n-1})}{(p-c_{n-1})^2} - \frac{1}{p-c_{n-1}}\frac{\partial p^\dagger_{n-1}}{\partial c_{n-1}}\right) \tag{30}$$

$$= \frac{\lambda_{n-1}+\lambda_S}{\lambda_S(p-c_{n-1})}\left(\frac{(p-p^\dagger_{n-1})}{(p-c_{n-1})} - \frac{\lambda_S}{\lambda_{n-1}+\lambda_S}\right) = -\frac{(v-p)\lambda_{n-1}}{\lambda_S(p-c_{n-1})^2} < 0 \tag{31}$$

The CDFs are both decreasing in c_{n-1}, which means an increase in c_{n-1} pushes up the distributions of prices. Finally, for claim (iii) follows from the observation in the text. □

Proof of Proposition 4. The argument in the text before the statement of the proposition establishes that the profile is the only one that can be supported by the on-path play of pure strategies. It remains to show that we can construct a subgame perfect equilibrium that supports such play. The argument in the text also explains that there firms maintain their initial prices in the subgame on the equilibrium path, and that there is no incentive to deviate downward at the first stage. The only remaining deviation to consider is an upward deviation by player n at the first stage to an initial price $\bar{p}_n > p^\dagger_{n-1}$. If $\bar{p}_n = v$ then the subgame is equivalent to a single-stage games, and we specify that firms play the equilibrium form a single-stage game; such an equilibrium exists (Proposition 1) and the same expected profit for firm n as that earned on the equilibrium path. If $\bar{p}_n \in (p^\dagger_{n-1}, v)$, then in the subgame we use the same strategies as for an equilibrium of the single-stage game, but we truncate those strategies. Specifically, firm n follows its single-stage equilibrium strategy for $p < \bar{p}_n$ and then places any remaining mass at $\bar{p}_n$. Any other firm shifts any mass from $\bar{p}_n$ and above up to the list price v. Straightforwardly, this generates the on-path equilibrium profits for all firms. □

Proof of Proposition 5. As noted in the text, this follows from re-arranging eq. (2) □

Proof of Proposition 6. Straightforward from expressions for equilibrium profits. □

Proof of Proposition 7. ⋆ To be completed ⋆ □

Proof of Proposition 8. ⋆ To be completed ⋆ □

Proof of Proposition 9. ⋆ To be completed ⋆ □

Proof of Proposition 10. ⋆ To be completed ⋆ □

Proof of Proposition 11. ⋆ To be completed ⋆ □

References

Armstrong, M., and J. Vickers (2019): "Discriminating Against Captive Customers," *American Economic Review: Insights*, 1(3), 257–72.

——— (2022): "Patterns of Competitive Interaction," *Econometrica*, 90(1), 153–191.

Armstrong, M., and J. S. Vickers (2001): "Competitive Price Discrimination," *RAND Journal of Economics*, 32(4), 579–605.

Baye, M. R., D. Kovenock, and C. G. de Vries (1992): "It Takes Two to Tango: Equilibria in a Model of Sales," *Games and Economic Behavior*, 4(4), 493–510.

Baye, M. R., D. Kovenock, and C. G. De Vries (1996): "The All-Pay Auction with Complete Information," *Economic Theory*, 8(2), 291–305.

Baye, M. R., and J. Morgan (2001): "Information Gatekeepers on the Internet and the Competitiveness of Homogeneous Product Markets," *American Economic Review*, 91(3), 454–474.

——— (2009): "Brand and Price Advertising in Online Markets," *Management Science*, 55(7), 1139–1151.

Baye, M. R., J. Morgan, and P. Scholten (2004a): "Price Dispersion in the Small and in the Large: Evidence from an Internet Price Comparison Site," *Journal of Industrial Economics*, 52(4), 463–496.

Baye, M. R., J. Morgan, and P. Scholten (2004b): "Temporal Price Dispersion: Evidence from an Online Consumer Electronics Market," *Journal of Interactive Marketing*, 18(4), 101–115.

Baye, M. R., J. Morgan, P. Scholten, and D. Jansen (2006): "Persistent Price dispersion in Online Markets," *The New Economy and Beyond: Past Present and Future, Edward Elgar, S*, pp. 122–143.

Burdett, K., and K. L. Judd (1983): "Equilibrium Price Dispersion," *Econometrica*, 51(4), 955–969.

Butters, G. R. (1977): "Equilibrium Distributions of Sales and Advertising Prices," *Review of Economic Studies*, 44(3), 465–491.

Chioveanu, I. (2008): "Advertising, Brand Loyalty and Pricing," *Games and Economic Behavior*, 64(1), 68–80.

Dasgupta, P., and E. Maskin (1986): "The Existence of Equilibrium in Discontinuous Economic Games, I: Theory," *Review of Economic Studies*, 53(1), 1–26.

Dasgupta, P., and J. Stiglitz (1980): "Industrial Structure and the Nature of Innovative Activity," *The Economic Journal*, 90(358), 266–293.

Gill, D., and J. Thanassoulis (2016): "Competition in Posted Prices with Stochastic Discounts," *The Economic Journal*, 126(594), 1528–1570.

Grossman, G. M., and C. Shapiro (1984): "Informative Advertising with Differentiated Products," *Review of Economic Studies*, 51(1), 63–81.

Inderst, R. (2002): "Why Competition May Drive Up Prices," *Journal of Economic Behavior and Organization*, 47(4), 451–462.

Ireland, N. J. (1993): "The Provision of Information in a Bertrand Oligopoly," *Journal of Industrial Economics*, 41(1), 61–76.

Johnen, J., and D. Ronayne (2021): "The Only Dance in Town: Unique Equilibrium in a Generalized Model of Sales," *Journal of Industrial Economics*, 69(3), 595–614.

Kaplan, G., and G. Menzio (2015): "The Morphology of Price Dispersion," *International Economic Review*, 56(4), 1165–1206.

Kocas, C., and T. Kiyak (2006): "Theory and Evidence on Pricing by Asymmetric Oligopolies," *International Journal of Industrial Organization*, 24(1), 83–105.

Lach, S. (2002): "Existence and Persistence of Price Dispersion: An Empirical Analysis," *Review of Economics and Statistics*, 84(3), 433–444.

Lach, S., and J. L. Moraga-González (2017): "Asymmetric Price Effects of Competition," *Journal of Industrial Economics*, 65(4), 767–803.

McAfee, R. P. (1994): "Endogenous Availability, Cartels, and Merger in an Equilibrium Price Dispersion," *Journal of Economic Theory*, 62(1), 24–47.

Moraga-González, J. L., Z. Sándor, and M. R. Wildenbeest (2017): "Nonsequential Search Equilibrium with Search Cost Heterogeneity," *International Journal of Industrial Organization*, 50, 392–414.

——— (2021): "Simultaneous Search for Differentiated Products: the Impact of Search Costs and Firm Prominence," *The Economic Journal*, 131(635), 1308–1330.

Moraga-González, J. L., and M. R. Wildenbeest (2008): "Maximum Likelihood Estimation of Search Costs," *European Economic Review*, 52(5), 820–848.

Myatt, D. P., and D. Ronayne (2019): "A Theory of Stable Price Dispersion," *Working Paper, University of Oxford*.

——— (2023a): "A Theory of Stable Price Dispersion," *Working Paper*.

——— (2023b): "Two-Stage Pricing with Costly Buyer Search," *Working Paper*.

Narasimhan, C. (1988): "Competitive Promotional Strategies," *Journal of Business*, 61(4), 427–449.

Rosenthal, R. W. (1980): "A Model in which an Increase in the Number of Sellers Leads to a Higher Price," *Econometrica*, 48(6), 1575–1579.

Shelegia, S. (2012): "Asymmetric Marginal Costs in Search Models," *Economics Letters*, 116(3), 551–553.

Shelegia, S., and C. M. Wilson (2021): "A Generalized Model of Advertised Sales," *American Economic Journal: Microeconomics*, 13(1), 195–223.

Shilony, Y. (1977): "Mixed Pricing in Oligopoly," *Journal of Economic Theory*, 14(2), 373–388.

Stahl, D. O. (1989): "Oligopolistic Pricing with Sequential Consumer Search," *American Economic Review*, 79(4), 700–712.

Varian, H. R. (1980): "A Model of Sales," *American Economic Review*, 70(4), 651–659.

——— (1981): "Errata: A Model of Sales," *American Economic Review*, 71(3), 517.