Asymmetric Models of Sales

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Abstract. We broaden and develop the classic captive-and-shopper model of sales. Firstly, we allow for asymmetric marginal costs as well as asymmetric captive audiences. These asymmetries jointly determine the identities of the two or more firms we find compete (via randomized sales) to serve shoppers. In a leading case, the prices paid by shoppers fall following a cost rise for the firm that serves most of them. Secondly, we study asymmetric price adjustment opportunities via a two-stage game in which firms may cut but not raise their initial prices. In this setting (and in scenarios with risk aversion or endogenous move order) we predict the play of pure strategies and that a unique firm serves the shoppers. Despite the different pricing predictions across games, firms' profits are equivalent. Welfare properties depend on whether firm asymmetry is predominantly on the supply side (costs) or on the demand side (captive audiences). Thirdly, we allow firms to choose production technologies via process innovations. One firm innovates distinctly more than others, attains a lower marginal cost, and ultimately serves the shoppers. We connect the distinctive asymmetric pattern of innovations to demand-side asymmetries and the shape of technology opportunity. JEL: D43 L11 M3

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In the classic "model of sales" competition among firms (via low prices) for the business of "shoppers" (who consider every price) sacrifices profits earned (via higher prices) from exploiting "captive" customers (who are locked in to a single firm). Varian (1980) constructed a symmetric Nash equilibrium of a symmetric single-stage game in which firms continuously mix over an interval of prices. Others (Narasimhan, 1988; Baye, Kovenock, and De Vries, 1992) studied firms with different captive-audience sizes: in equilibrium the two firms with fewest captives play mixed strategies (often interpreted as random sales) to attract shoppers, while other firms charge a monopoly price to their captives. Firms share the same marginal cost, and so for welfare it does not matter who serves the shoppers. Furthermore, a restriction to cost symmetry prevents the investigation of situations in which individual firms take endogenous steps, perhaps via process and product innovations, to lower costs or to improve their products. This is important if equilibrium innovation pushes toward distinctly asymmetric costs.

[1]This paper includes a result contained in an earlier version (Myatt and Ronayne, 2019) of "A Theory of Stable Price Dispersion" which now (Myatt and Ronayne, 2023a) assumes firms have symmetric marginal costs. We are grateful to Julian Wright (thank you) suggested that we develop this agenda. We also thank many people (including seminar partcipants) for comments on this paper and our companion work that influenced this new paper. These include Simon Anderson, Mark Armstrong, Dan Bernhardt, Yongmin Chen, Alex de Cornière, Andrea Galeotti, Roman Inderst, Maarten Janssen, Justin Johnson, Jeanine Miklós-Thal, José Moraga-González, Volker Nocke, Martin Obradovits, Martin Peitz, Martin Pesendorfer, Régis Renault, Michael Riordan, Nicolas Schutz, Robert Somogyi, Greg Taylor, Juuso Välimäki, Hal Varian, and Chris Wilson.

We expand the classic captive-and-shopper single-stage pricing game to incorporate asymmetric costs as well asymmetric captive audiences.[2] We also allow for asymmetric price adjustment via a two-stage pricing game in which, after their initial price positions are chosen, firms may adjust their prices downward (but not upward). There, we also study the implications of risk averse pricers and expose the connection to an asymmetric-move-order Stackelberg-style game. For all of these games, we examine the impact of changes to firms' individual costs. Lastly, we add and study a prior stage in which firms endogenously choose production technologies via fixed-cost innovations which lower their marginal costs of production.

Captive-shopper models with asymmetric costs have received only limited attention: a little-known paper by Golding and Slutsky (2000) reports an analysis of a duopoly with asymmetric costs, while a more recent paper by Shelegia and Wilson (2021) considers a very rich generalized model of (advertised) sales.[3] (This second paper deserves a much fuller discussion to which we return later.[4]) We do not know of a full treatment of the classic captive-and-shopper pricing game with asymmetric costs and more than two firms, and so our first contribution (in Section 1) is to provide this treatment.[5] For some parameter values, a common result from the literature holds: in equilibrium the two most "aggressive" firms employ mixed strategies, while the others charge the monopoly price to their captives. Here we measure a firm's aggression by the lowest price that it would be willing to charge in order to sell to shoppers. A firm is more aggressive if it has fewer captive customers (demand-side asymmetry) or if its marginal cost is low (supply-side asymmetry). However, in other cases we differ from the literature: there are reasonable specifications for which more than two firms must participate in randomized sales in equilibrium, and where different pairs of firms do so using distinct price intervals. Within our supplementary material (Appendix B) we provide an algorithm that constructs an equilibrium for any asymmetric captive-shopper game, and does so uniquely for generic parameter values.

Given that firms have different marginal costs, welfare is determined by the identities of the firms who serve the shoppers; (full) efficiency requires shoppers to be served by the lowest-cost firm. The involvement of (at least) two firms who challenge for sales means that (unless their costs are tied and lowest amongst all firms) the equilibrium is necessarily inefficient. It can be less efficient still if the lowest-cost firms have more captives, for in that situation the most aggressive firms can be those with very few captives but relatively high marginal cost.

[2]In our supplementary online material (Appendix C) we also extend our model so shoppers only buy from firms that advertise their price (cf. Shelegia and Wilson, 2021), and where the costs of advertising are also asymmetric.

[3]Golding and Slutsky (2000) also allowed for downward sloping demand. Inderst (2002) considered a model equivalent to a unit-demand version in which one firm has no captive customers.

[4]Shelegia and Wilson (2021) model firms that engage in (asymmetrically) costly advertising to reach shoppers; they model utility offers à la Armstrong and Vickers (2001); and they allow for asymmetric costs in a unit-demand setting. Their solution concept does not allow an advertising firm to undercut non-advertising competitors, it places specific conditions on how shoppers' demand is allocated when no firm advertises, and it restricts to equilibria in which all firms who mix do so over the same price interval. (In Appendix C we provide a very full discussion.) As their advertising costs fall to zero their prediction sometimes coincides with ours.

[5]The asymmetric contest setting of Siegel (2009, 2010) can incorporate an asymmetric model of sales, but omits a full treatment. Our Proposition 1 is covered by Siegel (2009), but our other results are not. Siegel (2010) derived equilibrium strategies for contests with m (homogeneous) prizes and $m+1$ players, and so covers models of sales (where shoppers are the single "prize") only in the case of duopoly; as studied by Golding and Slutsky (2000). See also Shelegia and Wilson (2023) for a discussion of the connections between the settings.

Cost asymmetry also allows insightful comparative-static exercises. Of most interest is the variation in the cost of the most aggressive firm; in leading cases of interest this is the firm that most often sells to shoppers. In a result that generalizes findings under duopoly (Golding and Slutsky, 2000; Inderst, 2002), any increase in this firm's cost pushes down the distribution of prices charged by the second-most-aggressive firm, and so customers pay less. Relatedly, the most aggressive firm benefits distinctly more from cost reductions in comparison to other firms. A consequence is that there are asymmetric incentives for cost-reducing innovations.

A single-stage captive-and-shopper pricing game has no pure-strategy Nash equilibrium. Any dispersion in prices arises from the realizations of mixed strategies from which there are profitable ex post deviations. Elsewhere (Myatt and Ronayne, 2023a) we argue that the predictions of single-stage models are incompatible with stable price dispersion, that nevertheless such dispersion is an empirical regularity, and that realistically firms find it easy to lower (but difficult to raise) prices in the short run. We suggest (in that paper) a two-stage framework: in a first stage, initial prices are chosen; in a second stage, firms are able to lower (but not raise) their prices. (In essence, there is an asymmetric ex post price adjustment opportunity.) Amongst other results (in Myatt and Ronayne, 2023a) we identify (in the captive-shopper setting) a subgame perfect equilibrium of a two-stage game in which a single profile of prices (one price per firm) is chosen in the first stage and maintained in the second stage. That result and others provide a theory of stable price dispersion. However, that theory restricts attention to firms with the same marginal cost.[6] In response, a second contribution here (in Section 2) is the analysis of a two-stage model of sales in which firms also have asymmetric costs.

We identify a unique profile of prices that are played as pure strategies on the equilibrium path of a subgame perfect equilibrium. In the first stage, the most aggressive firm sets an initial price which is just low enough to deter a second-stage undercut by the other firms. This implies that (in contrast to a single-stage pricing model) shoppers are served by a single firm, which deterministically sets an "on-sale" price starkly lower than its rivals' high "regular" prices. If the most aggressive firm is the one with the lowest marginal cost (as it is if the sizes of captive audiences are sufficiently similar, which means that firm asymmetry is driven by the supply side rather than the demand side) then this two-stage equilibrium is efficient; this contrasts with the inefficiency of single-stage equilibrium play. Moreover, the comparative-static result that an increase in the cost of the most aggressive firm reduces prices no longer holds. Instead, a local change in this firm's price has no effect on the prices paid by any customers.

Despite these differences between the outcomes of one-stage and two-stage pricing games, the expected profits of firms are the same in both cases and so the response of such profits to changes in costs are as before: the most aggressive firm (uniquely serving the shoppers in a two-stage-pricing environment) benefits distinctly more from a marginal-cost reduction (because it serves the shoppers as well as its own captive customers). A maintained theme, therefore, is that there are asymmetric innovation incentives even when firms are symmetric in other ways.

[6]Instead we allow for richer "consideration sets" than in a "captives and shoppers" setting. In recent work, Armstrong and Vickers (2022) elegantly considered more general consideration sets in single-stage games.

The two-stage-pricing model described above offers a commitment opportunity for firms at the first stage. This is related to the commitment opportunity that can be exploited in Stackelberg-style games when firms move in sequence. A third contribution (in Section 3) of our paper is the study of a model with endogenous asymmetric move order. Specifically, we allow all firms an opportunity to commit voluntarily to an advertised price position (which is then fixed) or instead to delay until the prices of others are observed. In essence, we give all firms opportunities (should they wish) to seek out a position as a Stackelberg early mover.

The most aggressive firm emerges as the Stackelberg leader. It commits (at the first opportunity) to the limit price which dissuades others from undercutting. All other firms ultimately charge the captive-exploiting monopoly price. The (unique) equilibrium outcome matches that of our two-stage play scenario and so our comparative-static and efficiency claims are maintained. In particular (and perhaps most importantly) the equilibrium (expected) profits of firms are the same across our single-stage, two-stage, and endogenous-move-order pricing games.

We have noted (and prove later) that a change in marginal cost impacts the most aggressive firm differently than other firms. This is easiest to see in our two-stage and sequential-move environments, where that firm uniquely serves all shoppers, and where any local changes in its marginal cost do not influence the prices of other firms: the most aggressive firm benefits more from a marginal-cost reduction (which we associate with a process innovation) simply because it sells to more customers than the others. As such, we find asymmetric innovation incentives.

To show the consequences of this, and as our final contribution, we extend (in Section 4) the model of sales to allow for the endogenous exploitation of opportunities to engage in process innovations. We add a pre-pricing stage in which firms choose their production technologies: a firm can pay (via a higher fixed cost) to lower its marginal cost of production. Firms then proceed to play one of the pricing games. (Innovation choices do not depend on which pricing game is played owing to their profit equivalence.)

Asymmetric capabilities emerge naturally. For example, consider a world in which firms face the same innovation opportunities and where the sizes of their captive audiences are equal. In an equilibrium with pure strategies at the innovation stage, exactly one firm chooses a distinct production technology with more innovation (a higher fixed cost) and lower marginal cost, whereas other (less innovative, and so ultimately with higher marginal cost) firms act symmetrically. Subsequently (in our two-stage and sequential-move models) the innovative firm goes on to set a shopper-capturing low price, while the others simply exploit their captive audiences. The equilibrium outcome is unique, up to the identity of the innovative firm (if firms are symmetric then we can arbitrarily choose the innovator). This result shows that asymmetric marginal costs emerge endogenously, even when firms are otherwise symmetric. Furthermore, this also provides a rationale for the study of models of sales with asymmetric costs. If firms are asymmetric because of demand-side differences in captive populations, then we can further pin down the nature of their (asymmetric, of course) innovation choices.

In what follows, we first (Section 1) examine the classic model of sales with single-stage pricing allowing for asymmetric captive shares and marginal costs, and relate our work to that in the literature. We then proceed to two-stage pricing (Section 2) before introducing a sequential-move model (Section 3). We then analyse endogenous marginal costs via innovative investment choices (Section 4), before offering several discussion points (Section 5).

1. A Single-Stage Model of Sales with Asymmetric Firms

Here we extend the classic single-stage model of sales to allow for fully asymmetric firms.

Model. There are n firms who simultaneously choose their prices, where $p_i \in [0, v]$ is the price chosen by firm $i \in \{1, \ldots, n\}$ and $v > 0$ is customers' (common) maximal willingness to pay. Firm i faces a constant marginal cost $c_i \in [0, v)$ to serve any customer.[7]

A mass of $\lambda_i > 0$ customers are "captive" to firm i. A mass of $\lambda_S > 0$ customers are "shoppers" who buy from the cheapest firm, or from one of the cheapest (in the event of a tie).[8] Shoppers see all firms' prices. However, in an extension (contained in Appendix C) we also study shoppers who only buy from a firm that advertises its price, and where such advertising is costly.

Firm i earns $\lambda_i(p_i - c_i)$ from its captive customers and $\lambda_S(p_i - c_i)$ if it sells to the shoppers. These components sum to form a (risk neutral) firm's payoff. This specification is covered by Varian (1980) if $\lambda_i = \lambda$ and $c_i = c$ for every i, so that firms are symmetric.[9]

Equilibrium Play. Firm i guarantees a profit of at least $\lambda_i(v - c_i)$ by setting $p_i = v$ and selling only to captive customers. The lowest price it would be willing to set in order to win the business of shoppers is $p_i^\dagger$ satisfying $\lambda_i(v - c_i) = (\lambda_i + \lambda_S)(p_i^\dagger - c_i)$, or explicitly

$$p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i}{\lambda_i + \lambda_S}. \tag{1}$$

This lowest undominated price is a measure of how aggressive (in terms of pricing) a firm is willing to be. It is higher when a firm has more captive customers (because it is more costly to lose money on sales to them by lowering price) and when the marginal cost of serving shoppers is higher (making it less tempting to serve those shoppers).[10] Firm j is strictly more aggressive than firm i and if and only if $p_j^\dagger < p_i^\dagger$ which (following re-arrangement) is

$$\underbrace{(c_i - c_j)}_{\text{cost adv. } j \text{ vs. } i} > \frac{v - (c_i + c_j)/2}{\lambda_S + (\lambda_i + \lambda_j)/2} \underbrace{(\lambda_j - \lambda_i)}_{\text{captive adv. } j \text{ vs. } i}. \tag{2}$$

[7] We assume (as is standard) that the cost of serving captive customers and shoppers is the same. However, we note that equilibrium pricing strategies are unaffected if the cost of serving captive customers is different from that of serving shoppers. In particular, those strategies would only include the cost of serving shoppers.

[8] For technical convenience we break any ties in favor of a lowest-cost firm. This allows us to apply an off-the-shelf equilibrium-existence result (Dasgupta and Maskin, 1986, Theorem 5).

[9] Baye, Kovenock, and De Vries (1992) and Kocas and Kiyak (2006) allowed for asymmetry in "captives" so that $\lambda_i \neq \lambda_j$ for $i \neq j$, but retained common (and so without loss of generality, zero) marginal costs.

[10] If the cost of serving captives is $\hat{c}_i \neq c_i$ then the expression for $p_i^\dagger$ remains unchanged.

If firm j has an advantage over firm i on the supply side (lower costs) and a disadvantage on the demand side (fewer captives) then this holds. However, if a firm has advantages on both sides of the market (lower costs, more captives) then this inequality can break either way.

We choose labels for the firms (without loss of generality) so that the three highest-indexed firms n, $n-1$, and $n-2$ are the most aggressive: $p_n^\dagger \le p_{n-1}^\dagger \le p_{n-2}^\dagger \le \min_{i\in\{1,\ldots,n-3\}} p_i^\dagger$.

Baye, Kovenock, and De Vries (1992, Section V) found a unique Nash equilibrium when firms with the same marginal cost ($c_i = c$ for all i) have differently sized captive audiences: ordering firms so that $\lambda_n < \lambda_{n-1} < \lambda_{n-2}$, the equilibrium involves mixing (the "tango" of their paper, which describes the competition for the business of shoppers via randomized sales) by firms $n-1$ and n while firms $i \in \{1, \ldots, n-2\}$ set $p_i = v$. In other situations (including symmetry) there can be many other equilibria, all of which generate the same expected profits. A firm's expected profit is equal to its "captive-only" profit, $\lambda_i(v - c_i)$, for all firms $i < n$. The exception is firm n. It always has the option to set the lowest undominated price of its closest competitor, $p_n = p_{n-1}^\dagger$, sell to all shoppers, and so earn $(p_{n-1}^\dagger - p_n^\dagger)(\lambda_n + \lambda_S)$ on top of its captive-only profit. Our first result proves that this remains true when marginal costs are asymmetric.

Proposition 1 (Nash Equilibrium and Profits). *For any parameter values, there exists a Nash equilibrium of the single-stage game in which i's expected profit is given by*

$$\pi_i = \underbrace{\lambda_i(v - c_i)}_{\text{captive-only profit}} + \begin{cases} (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger) & \text{if } i = n, \text{ and} \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

so only a most aggressive firm earns (weakly) more than its captive-only (expected) profit.

For generic parameter values such that the two most aggressive firms are uniquely defined, i.e., $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$*, the profits in* (3) *are those in any Nash equilibrium.*

The proof, together with the proofs of other results, is contained within Appendix A. In Appendix C we show that this result also holds for an extended model in which firms must pay for (asymmetrically) costly advertising for their prices to be considered by shoppers.[11]

Proposition 1 tells us that the profits given by (3) arise in the non-generic case of $p_n^\dagger = p_{n-1}^\dagger$ or $p_{n-1}^\dagger = p_{n-2}^\dagger$, but leaves scope for other profit levels too. However, we deem any other equilibria to be "pathological" in the sense that a small perturbation to parameters away from such a case would cause a discontinuous change to equilibrium profits—back to those given by (3).

Definition. *An equilibrium is* pathological *if it gives payoffs that differ from those of* (3)*.*

For the remainder of the main paper, we do not consider pathological equilibria further.[12]

[11]Models of sales have been extended to include a fixed cost for access to a "clearinghouse" for shoppers (e.g., Baye and Morgan, 2001, 2009; Baye, Gao, and Morgan, 2011; Shelegia and Wilson, 2021). Garrod, Li, and Wilson (2023) consider a clearinghouse that charges otherwise-captive customers for access to all firms' prices.

[12]Our results in Appendix B show that profits in any pathological equilibrium differ from (3) for only and exactly one firm. Later in that appendix we also provide an example of a pathological equilibrium. A statement

Note that (at most) one firm strictly benefits from its access to shoppers. If demand conditions are symmetric ($\lambda_i = \lambda$ for all i) then profits are the same as if firms offered a discriminatory price to shoppers: firm n earns $\lambda(v - c_n) + \lambda_S(c_{n-1} - c_n)$, which is what it would earn if the shoppers are served by it (the lowest-cost firm) at a price equal to the second lowest cost.[13]

More generally, by pricing below $p^\dagger_{n-1}$ the most aggressive firm n is sure to sell to all shoppers. Any price above this invites an "undercut" from the next-most-aggressive firm, $n-1$, and for prices ranging upward from $p^\dagger_{n-1}$ we see the (familiar) mixing from (at least) two firms. In the classic setting (when firms have asymmetric captive shares) only two firms are involved in such randomized sales. However, this "two to tango" result does not extend fully here. Any such tango is danced by the two most aggressive firms according to the minimum undominated prices of (1) in the sense that they play continuous mixed strategies (which compete for shoppers) over an interval ranging upward from $p^\dagger_{n-1}$. However, there is a possibility that (for higher prices) other firms step on to the dance floor. For strictly asymmetric firms, there are situations (we describe these later) in which more than two tango (cf. Baye, Kovenock, and De Vries, 1992).

We also find conditions so that only two tango. For example, if the most aggressive firms have the smallest captive audiences then we can readily pin down such an equilibrium.

In Appendix B we describe an explicit algorithm that constructs an equilibrium for any parameters, and which for generic parameters is unique.[14] Here, however, to ease exposition of the equilibrium description, we report properties of the equilibrium strategies for cases in which the second-most aggressive firm is uniquely identified: $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.

Proposition 2 (Nash Prices I: When Two Tango). *Suppose that the second most aggressive firm is uniquely identified: $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$. In any Nash equilibrium all firms $i < n$ place an atom at $p_i = v$, while n mixes continuously over all $p \in [p^\dagger_{n-1}, v]$. There is $p^\ddagger \in (p^\dagger_{n-2}, v]$ such that for $p \in [p^\dagger_{n-1}, p^\ddagger)$, $F_i(p) = 0$ for $i \leq n-2$, while $i \in \{n, n-1\}$ mix via*

$$F_n(p) = \frac{(p - p^\dagger_{n-1})(\lambda_{n-1} + \lambda_S)}{\lambda_S(p - c_{n-1})} \quad \textit{and} \quad F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda_n + \lambda_S)}{\lambda_S(p - c_n)}. \tag{4}$$

If firm $n-1$ has fewer captives than those less aggressive: $\lambda_{n-1} \leq \min_{i \in \{1,\ldots,n-2\}}\{\lambda_i\}$, then there is a unique Nash equilibrium in which $p^\ddagger = v$ and so all firms $i \in \{1, \ldots, n-2\}$ choose $p_i = v$ and serve only captives, while firms n and $n-1$ mix via (4) over prices $p \in [p^\dagger_{n-1}, v)$.

Equation (4) characterizes the equilibrium mixed-strategy "tango" danced by the firms $n-1$ and n that are willing to price below $p^\dagger_{n-2}$. Consider firm $n-1$. It can earn its captive-only

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equivalent to Proposition 1 in a contest setting is found in Theorem 1 and Corollary 2 of Siegel (2009). In Appendix C we provide a mapping between Siegel's asymmetric contest setting and models of sales.

[13] As usual, such a Bertrand construction requires the careful treatment of tie-break rules; for example by breaking a tie favor of a lowest-cost firm. Our reference to discriminatory pricing refers to unit-demand customers. A more general analysis of captive-vs-shoppers discrimination was reported by Armstrong and Vickers (2019).

[14] This is in the spirit of Siegel (2010) who described an algorithm to construct an equilibrium in a related class of contest games. Applied to a model of sales, his approach can be used for the duopoly ($n = 2$) case.

profit of $\lambda_{n-1}(v - c_{n-1})$ by charging v. It is indifferent to charging $p < v$ if

$$\underbrace{(v-p)\lambda_{n-1}}_{\text{loss on captives}} = \underbrace{(p - c_{n-1})\lambda_S(1 - F_n(p)),}_{\text{gain from shopper sales}} \tag{5}$$

which solves for $F_n(p)$. The desire to compete for shoppers via a lower price is lessened if a firm has more captives, and sales to shoppers are less valuable if its marginal cost is higher. Any lower-indexed (and so less aggressive) firm $i \in \{1, \dots, n-2\}$ that has both more captives ($\lambda_i > \lambda_{n-1}$) and higher costs ($c_i > c_{n-1}$) has, very clearly, a strictly weaker incentive to charge a price $p < v$. Such a firm does not wish to "step on to the dance floor" and so (if this is true, and in fact under weaker conditions) we can construct an equilibrium in which firms $n-1$ and n "tango" by using the distributions reported in (4) all of the way up to v.

However, it is possible for a less aggressive firm $i \leq n-2$ to have both higher costs (making it less aggressive) but fewer captives (making it more aggressive). This combination can result in a higher $p_i^\dagger$ but a greater temptation to charge some intermediate price p. To demonstrate this explicitly, we construct a "two to tango" equilibrium in which firms n and $n-1$ mix over the entire interval $[p_{n-1}^\dagger, v)$ according to the distributions reported in (4), while other firms charge v. For this to be an equilibrium, we must be sure that for all $p \in [p_{n-1}^\dagger, v)$ and $i \leq n-2$

$$(v-p)\lambda_i \geq (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)). \tag{6}$$

Suppose, however, $c_i \in (p_{n-1}^\dagger, v)$. This guarantees that $p_i^\dagger > p_{n-1}^\dagger$, and so firm i is not one of the two most aggressive firms. We can now choose λ_i sufficiently small such that (6) fails. This means that there is a price at which firm i wishes to join the dance floor.

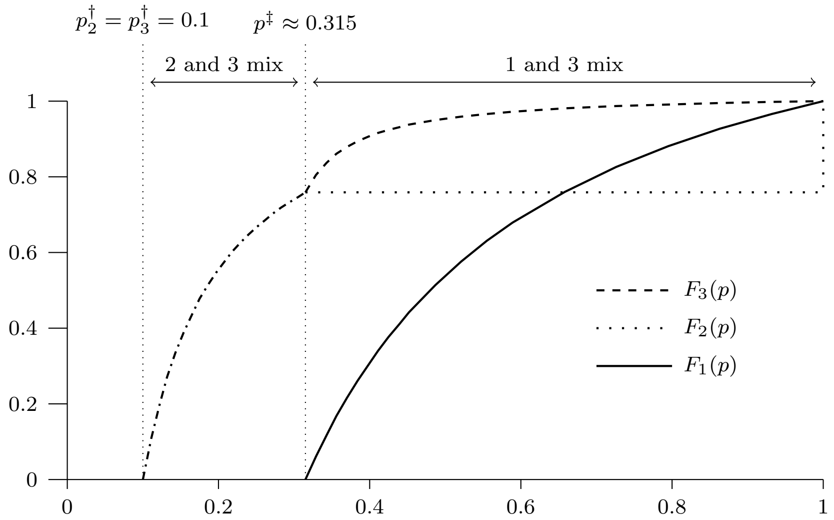
In essence, this argument adds a third firm to disrupt the tango danced by two existing firms. By choosing this firm's marginal cost to be sufficiently high, we guarantee that the two existing firms will still compete together (in a lower price range) for randomized sales. However, if this third firm has a sufficiently small captive customer base then it wishes to join the action at some point, to sell to shoppers at least sometimes. This implies any equilibrium must involve mixing from that third firm. It also prevents the equilibrium with only two dancers. We prove this logic extends such that for any number of firms $k \geq 2$ playing mixed strategies in equilibrium there are parameters such that a $k+1^{\text{th}}$ firm wants to join the dance. We summarize here.

Proposition 3 (Nash Prices II: When Three or More Tango). *For any $k \in \{3, \dots, n\}$ there is an open set of parameters such that k firms play mixed strategies in any equilibrium.*

If three or more firms mix in equilibrium, then for generic parameter values the set consisting of the supports (excluding any atoms at v) of each firm $i < n$ that mixes is a partition of $[p_{n-1}^\dagger, v]$.

If $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$, then at least three firms mix in any equilibrium if and only if (6) holds with equality for some $i \leq n-2$ at some $p \in (p_{n-2}^\dagger, v)$, where $F_n(p), F_{n-1}(p)$ are given by (4).

A firm that joins a dance "late" (rather than at the "start", i.e., at $p_{n-1}^\dagger$) may be a new entrant with higher costs and a smaller captive base. Indeed, if its captive base is small then it has an



This illustrates the mixed-strategy distribution functions for a triopoly in which three firms use randomized sales. The specification is from the text, where $c_2 = c_3 = 0$, $\lambda_2 = \lambda_3 = \lambda_H = 0.1$, $\lambda_S = 0.9$, and so $p_2^\dagger = p_3^\dagger = 0.1$. For the less-aggressive firm, $\lambda_1 = \lambda_L = 0.005$ and $c_1 = c = 0.25$. There are two distinct "dance floor" segments, with firm 3 "partner swapping" from firm 2 to firm 1 at $p^\ddagger \approx 0.315$.

FIGURE 1. Mixing CDFs in a Three-Firm "Thrango" Example

incentive to use a production technology with a relatively high marginal cost if that results in a lower fixed cost. (This corresponds to the choice of technology in the spirit of Dasgupta and Stiglitz, 1980, which we use in Section 4.) This suggests that the presence of such a firm (and so randomized sales by more than two competitors) is not only a theoretical curiosity.

Randomized Sales from Multiple Firms. In Appendix B we offer an equilibrium characterization that is unique except for knife-edge cases (such as when some firms share the same characteristics, as they do under symmetry). If k firms play mixed strategies then in equilibrium the interval $[p_{n-1}^\dagger, v)$ is partitioned into $k-1$ sub-intervals. The most aggressive firm n mixes over the entire range of prices. However, each of the other $k-1$ firms mixes only over a single sub-interval. Moving upward through prices, the "dance partner" of firm n switches. More than two firms mix; but only two firms mix within any particular interval of prices.

Propositions 2 and 3 confirm existence and properties of equilibria with $k > 2$. Here we build an illustrative "thrango" example with $k = n = 3$. To simplify exposition, two firms are symmetric, but this is easily modified so that the three firms are (generically) different.

Suppose that $n = 3$. Firms 2 and 3 have the same characteristics, comprising low costs and larger captive audiences: $c_2 = c_3 = 0$ and $\lambda_2 = \lambda_3 = \lambda_H$. Firm 1 has higher costs, but a smaller captive audience: $c_1 = c > 0$ and $\lambda_1 = \lambda_L$ where $\lambda_L < \lambda_H$. By setting $c > (\lambda_H - \lambda_L)v/(\lambda_H + \lambda_S)$ we guarantee that firm 1 is less aggressive than the others. An equilibrium involves the two jointly most aggressive firms engaged in a "tango" over the lower interval of prices, with

$$1 - F_3(p) = 1 - F_2(p) = \frac{\lambda_H(v-p)}{\lambda_S p} \tag{7}$$

The inequality in (6) holds for all relevant p if and only if $\lambda_L \geq [\lambda_H^2(v-c)^2]/[4\lambda_S vc]$. Our "thrango" situation arises when this fails, as it does, at a price $p^\ddagger$, for $\lambda_L > 0$ sufficiently small. If so, then the "tango" between firms 2 and 3 ends at the price $p^\ddagger$, which satisfies (6):

$$p^\ddagger = \frac{v + c - \sqrt{(v+c)^2 - 4\left(1 + [\lambda_L \lambda_S / \lambda_H^2]\right) vc}}{2\left(1 + [\lambda_L \lambda_S / \lambda_H^2]\right)} \tag{8}$$

At this point there is a partner swap: firm 2 shifts all remaining weight to price at v, while firm 1 then begins mixing.[15] Over the interval $[p^\ddagger, v)$ the relevant mixing distributions are

$$1 - F_3(p) = \frac{\lambda_L}{\lambda_H} \frac{v-p}{v-p^\ddagger} \frac{p^\ddagger}{p-c} \quad \text{and} \quad 1 - F_1(p) = \frac{v-p}{v-p^\ddagger} \frac{p^\ddagger}{p}. \tag{9}$$

The equilibrium CDFs of this example are illustrated in Figure 1 for suitable parameter choices.

Efficiency. The presence of cost asymmetry now raises the issue of efficiency.

In a model of sales all customers are served. The only efficiency-relevant question is this: who serves the shoppers? This is uncertain (in a mixed-strategy equilibrium) and undetermined (if there are multiple equilibria). However, from an efficiency standpoint, this does not matter if the cost of serving those shoppers is the same for everyone. This changes when costs differ: the outcome is efficient only if shoppers are served by one of the lowest cost firms. If there is a unique such firm then efficiency requires it to be the only supplier of shoppers.

The equilibria described in Propositions 2 and 3 allocate output across (at least) two firms. If there is strict cost asymmetry (more generally, if the lowest-cost firm is unique) then necessarily some output is traded at a higher (and inefficient) cost. Efficiency can only be restored if there is cost symmetry, at the lowest level, amongst all firms that play mixed strategies in a Nash equilibrium. This condition is stringent. We summarize our observations as a corollary.[16]

Corollary 1 (Efficiency of Single-Stage Equilibria). *If the lowest-cost firm is unique then any equilibrium of a single-stage pricing game is inefficient. Let K be the set of firms mixing in equilibrium so that $|K| \geq 2$ and $n \in K$; the equilibrium is efficient if and only if $c_n = c_k$ $\forall k \in K$.*

Changing Costs. The asymmetric solution allows us to vary the costs of individual firms. To ease exposition, we maintain the strict ranking $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ and consider local changes that do not change these inequalities. We also assume (a sufficient condition is $\lambda_{n-1} \leq \min_{i \in \{1,\ldots,n-2\}}\{\lambda_i\}$) that the equilibrium involves mixing by only the two most aggressive firms (Proposition 2). Each firm $i \in \{1, \ldots, n-2\}$ sets $p_i = v$, which is unaffected by any local changes in costs, and the costs of such firms do not influence prices. The interesting comparative-static exercises concern the marginal costs of firms n and $n-1$.

[15] Of course, because firms 2 and 3 are completely symmetric in this example, there is a second equilibrium in which 3 (instead of 2) leaves the dance floor and firms 1 and 2 mix over $[p^\ddagger, v)$.

[16] We can map out more fully the conditions for efficiency, but at the expense of adding complexity to our statements. We return to the topic of efficiency when we study two-stage pricing.

Inspecting (4), c_n enters only into the solution for firm $n-1$. The distribution $F_{n-1}(p)$ is increasing in c_n: an increase in the marginal cost of firm n lowers the prices charged by firm $n-1$. This is because firm $n-1$ prices more aggressively to maintain the incentive for the (now more costly) firm n to price at $p^\dagger_{n-1}$ rather than the (now more attractive, given the higher cost) higher prices within $[p^\dagger_{n-1}, v)$. This implies that the captive customers of firm $n-1$, as well as the shoppers, benefit from any cost increase suffered by the most aggressive firm that also most often supplies the shoppers (as we confirm via claim (iv) of Proposition 4).

The cost c_{n-1} of firm $n-1$ has a more conventional impact. A direct effect of an increase in c_{n-1}, by inspection of (4), is to increase $F_n(p)$ and so to push down the prices charged by firm n. (This follows from the logic discussed just above.) However, an increase in c_{n-1} also raises the lower bound $p^\dagger_{n-1}$ to the interval of sales prices charged by both firms. This lowers both $F_{n-1}(p)$ and $F_n(p)$. There are competing effects on $F_n(p)$, but overall the impact (as the proof of the next proposition confirms) is to push up the prices charged by both competitors.

Proposition 4 (The Effect of Costs on Prices). *Suppose that $p^\dagger_n < p^\dagger_{n-1} < \min_{i\in\{1,\dots,n-2\}} p^\dagger_i$. and that conditions hold such that the equilibrium involves mixing by only $n-1$ and n.*

(i) Prices do not change in response to local changes in the cost c_i of any firm $i \in \{1, \dots, n-2\}$.

(ii) A local increase in c_{n-1} shifts upward the distributions of prices charged by $n-1$ and n.

(iii) A local increase in c_n shifts downward the distribution of prices charged by $n-1$.

(iv) If $c_n \le c_{n-1}$, then $F_{n-1}(p)$ first order stochastically dominates $F_n(p)$.

Claim (iii) implies that firm n disproportionately gains from any reduction in its marginal cost. A reduction in c_n has the usual direct effect on its profit. However, it also prompts a price rise from its competitor (in the market for sales to shoppers) firm $n-1$. This is a positive strategic effect. (Notably, a negative strategic effect is more common in pricing games.) In fact,

$$\frac{\partial \pi_i}{\partial c_i} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = n \text{, and} \\ \lambda_i & \text{otherwise,} \end{cases} \tag{10}$$

and so the most aggressive firm gains distinctly more from a reduction in marginal cost than do other firms. This effect on profits (which is maintained within our two-stage and sequential-move models in subsequent sections) suggests that that there are asymmetric incentives to engage in marginal-cost-reducing process innovations. We take up that theme in Section 4.

Related Literature. We know of very few contributions within the literature that have considered asymmetric cost specifications in a classic captive-shopper model of sales.

Although not widely cited, the under-appreciated paper by Golding and Slutsky (2000) is the earliest contribution (to our knowledge) that properly allows for asymmetric costs. Their model also allows for downward-sloping demand, but restricts to duopoly. Their comparative-static results consider changes in the total number of captives, shifts in captives between firms, and changes in firms' marginal costs. Most notably, they report (Golding and Slutsky, 2000, p. 139) that "increasing the low cost store's marginal cost can lower average prices and raise the expected utilities of both types of consumers." This is claim (iii) of our Proposition 4.

Relatedly, and independently, Inderst (2002) studied a duopoly with firms labelled as "incumbent" and "entrant" respectively.[17] He specified zero marginal cost for both. However, the entrant has no captive customers and shoppers face a strictly positive "cost of substitution" to buy from the entrant. This is equivalent (following a relabeling of prices) to setting a positive marginal cost for the entrant.[18] Inderst (2002, p. 451) summarized his key finding to be "the expected price of an incumbent firm may increase in response to increasing competition as it may become more profitable to exploit a rather immobile fraction of consumers instead of capturing a larger but more contested segment of the market." The "increasing competition" is a lower cost-of-substitution parameter, which is equivalent to the cost of the second firm. This insight, then, is equivalent to Theorem V of Golding and Slutsky (2000, p. 149). Claim (iii) of our Proposition 4 extends it to a context with more than two firms.

Shelegia (2012) also discussed a duopoly with asymmetric marginal costs but symmetric captive populations. Referencing a doctoral dissertation, Shelegia (2012) commented that a symmetric-captive model in which two firms have zero costs while the other firms have positive marginal cost has a unique equilibrium in which (as verified by Proposition 2 here) the zero-cost firms randomize.[19] In contrast, this is not always true in a more general setting (Proposition 3).

Armstrong and Vickers (2022) made substantial progress in characterizing Nash equilibria under various constellations of consumer consideration sets (including arbitrary sets for triopoly), revealing the "patterns of competitive interaction" that result in equilibrium. They focused on the demand side and assumed cost symmetry. In contrast, our work in this paper assumes a simple demand-side specification (captives and shoppers; so that consideration sets are singletons and the set of all firms, respectively) but allows for asymmetric costs. Our characterization shows the importance of supply-side asymmetries for the patterns of competitive interaction resulting from the single-stage pricing game. Suppose that firms have asymmetric captive bases. If firms have symmetric costs, then there is a unique equilibrium in which only two firms vie for shoppers (Proposition 2); but if firms have asymmetric costs then any number of firms may fight for them via a sequence of pairings of the most aggressive firm with others (Proposition 3).

[17]He also described a triopoly model within the same paper, but that model stands outside the captive-shopper framework. Elsewhere (in Myatt and Ronayne, 2023a, Appendix A) we describe, for his triopoly case, a different equilibrium characterization to his (from Inderst, 2002, Lemma 3).

[18]Using our notation, this corresponds to $\lambda_1 > 0$, $\lambda_2 = 0$, $c_1 = 0$, $c_2 > 0$. To make the entrant (firm $i = 2$) the more aggressive firm we set $p_2^\dagger < p_1^\dagger$, or equivalently $c_2 < v\lambda_1/(\lambda_1 + \lambda_S)$. Technically our model description specifies strictly positive captive audiences for every firm. However, we can straightforwardly handle the case where a firm has no captives; it corresponds to the case where $\lambda_2 \downarrow 0$. For Inderst (2002), customers are willing to pay r, there is an adjustment cost of buying from the entrant of c, a mass α of customers are captive to the incumbent, and the remaining mass $1 - \alpha$ are shoppers. This maps to our model with $n = 2$, $c_1 = 0$, $c_2 = c$, $v = r$, $\lambda_1 = \alpha$, $\lambda_2 = 0$, $\lambda_S = 1 - \alpha$, and so $\lambda_1 + \lambda_S = 1$.

[19]Shelegia (2012) also considered asymmetric costs in the costly search model of Burdett and Judd (1983).

In a recent contribution, Shelegia and Wilson (2021) substantially generalized the classic model of sales in three ways: (i) firms (at least for symmetric marginal costs) make utility offers à la Armstrong and Vickers (2001) which allows for downward-sloping demand; (ii) firms pay (fixed) advertising costs for their prices to reach shoppers; and (iii) firms have asymmetric marginal costs in a unit-demand setting.[20] Their solution concept requires a specific tie-break rule (a split of demand) for shoppers when no firm advertises.

When marginal costs are asymmetric, their model corresponds to ours but where firm i pays a fixed cost $A_i > 0$ to reach the shoppers. Their main oligopoly ($n > 2$) finding (Shelegia and Wilson, 2021, Proposition 2; p. 209) is that a unique equilibrium can involve (even with strict asymmetry) mixing (or the use of sales) by more than two firms. They also report (their Corollary 1) that as advertising costs fall to zero (to approach our zero-cost-of-advertising version) then (with asymmetric marginal costs) only two firms mix (or "use sales"). However, we have confirmed that there are (reasonable) circumstances (as Proposition 3 makes explicit) in which an equilibrium must involve participation by at least three firms in randomized sales, but that (for generic parameters) only two firms randomize within any price interval.

The source of the difference appears to be that Shelegia and Wilson (2021) restricted attention to equilibria in which all mixing firms use the same lower bound of support for prices; with the exception of knife-edge cases (such as those with symmetric firms) our equilibria with more that two firms mixing do not have this feature. They recognized (p. 204) that "there may be multiple forms of sales equilibria with firms using different supports." Indeed, we find that firm n mixes over an interval which is partitioned into sub-intervals which correspond to the non-overlapping supports of other firms. They also (again, p. 204) said that their focus was "only on sales equilibria where all advertising firms use the full convex support" which (for our model) rules out by assumption the equilibria that we highlight in our Proposition 3.

The difference in predictions is explored further within Appendix C. In that appendix, we also explain the importance of the special tie-break specifications used by Shelegia and Wilson (2021), and illustrate how our own approach can incorporate positive advertising costs.

Summarizing, we see our Section 1 as offering a complementary contribution to that of Shelegia and Wilson (2021) by offering a clear characterization of randomized sales when firms have different costs but when (as in the classic model) shoppers see all prices for free. Some predictions are consistent with those offered in their (rich) framework with costly advertising; a notable exception is that they indicate that an equilibrium (with vanishing advertising costs) has the "two to tango" property, which we show is sometimes not the case.

[20] Their firms make "utility offers" as elegantly suggested by Armstrong and Vickers (2001): firm i offers surplus u_i in exchange for profit $\pi_i(u_i)$. Under a unit-demand specification, $u_i = v - p_i$ and so $\pi_i(u_i) = v - u_i - c_i$; a linear trade-off. The approach is more general: it includes situations in which each customer has downward-sloping (multi-unit) demand. However, Shelegia and Wilson (2021, p. 202) clearly explain a restriction (their Assumption U) that is commonly used by others: the consumer surplus that maps to the maximum profit for a firm is constant across firms. Under unit demand, this consumer surplus is zero (from $p_i = v$) and this holds. Under downward-sloping demand the consumer surplus from monopoly pricing changes with the firm's marginal cost. This rules out downward-sloping demand when marginal costs are asymmetric, and so this means that the added benefit from the broader utility-offer specifications is more muted in an asymmetric-cost setting.

2. A Two-Stage Model of Sales

In related work (Myatt and Ronayne, 2023a) we observe that disperse prices do not change at every price-revision opportunity. This is inconsistent (at least for those firms who offer sales to shoppers) with the repeated play of a mixed-strategy Nash equilibrium. We also observe that, once established, there are barriers to upward price adjustments. In response, we study (also in Myatt and Ronayne, 2023a) firms that establish initial price positions, and then subsequently have an asymmetric-direction opportunity to cut (but not raise) those prices. In settings that extend beyond the captive-shopper world we find stable disperse prices: initial (and different, across firms) price positions are chosen, and not subsequently adjusted. However, firms share the same marginal cost. Here we apply the approach to asymmetric-cost models of sales.

Model. We retain the supply-side (firms' costs) and demand-side (captives and shoppers) specifications from the previous section. For simplicity of exposition, we assume that the most aggressive firm is uniquely defined, so that $p_n^\dagger < p_{n-1}^\dagger$. We consider a two-stage pricing game:

(1) firms simultaneously choose and observe initial price positions $\bar{p}_i \in [0, v]$; and then
(2) firms simultaneously choose final retail prices $p_i \in [0, \bar{p}_i]$.

Shoppers choose the cheapest firm. As before, we break ties in favor of a lowest-cost firm. If all firms choose $\bar{p}_i = v$ in the first stage, then the second stage corresponds to the conventional simultaneous-move captive-shopper game (but with asymmetric costs) studied in Section 1.

The sequence described here specifies (equating firm profits to payoffs) a complete-information extensive-form game, and the natural solution concept is subgame-perfect equilibrium. We seek such an equilibrium that predicts stable prices. By this we mean a specific set of prices that are chosen and maintained, so that no mixing is observed on the equilibrium path.

Definition. *A profile of prices is* supported in equilibrium by the on-path play of pure strategies *if there is a subgame perfect equilibrium in which (i) those prices are set as initial prices at the first stage; and (ii) on the equilibrium path firms do not change prices at the second stage.*

This definition allows for sales in the sense that a firm may set initially and maintain a price strictly below v.[21] However, the on-path play of pure strategies means that such sales are not randomized. There is price dispersion (not all prices are the same) and this dispersion is stable (the price profile can be observed from pure strategy play).

Equilibrium Play. Following the definition above, we now seek a profile of prices (an initial price $\bar{p}_i$ for each firm i) that is supported by the on-path equilibrium play of pure strategies.

[21]The definition smooths exposition. A more general version would feature on-path play of pure strategies but not force prices to be equal across periods. But as we shall see, the restriction is without loss of generality.

Only one price profile can meet this definition: firm n prices low enough at the initial stage, by setting $\bar{p}_n = \bar{p}^\dagger_{n-1}$, to deter others from undercutting. Those other firms simply set $\bar{p}_i = v$. The following constructive arguments show that this is the unique profile of interest.

Firstly, prices are undominated: $\bar{p}_i \in [p^\dagger_i, v]\ \forall\ i$, and so prices are strictly above respective costs. Secondly, the lowest price must be unique: if it were not then one of the tied firms would gain from undercutting. We conclude that there is a unique firm i satisfying $\bar{p}_i < \min_{j \neq i} \bar{p}_j$. Thirdly, all other firms $j \neq i$ sell only to their captive customers. Anticipating that this is so, they charge the captive-exploiting monopoly price: $\bar{p}_j = v$ for $j \neq i$. Fourthly, no firm j must want to undercut firm i, and so $p^\dagger_j \geq \bar{p}_i \geq p^\dagger_i$ which means that $i = n$.[22]

By construction, this strategy profile is a (unique) Nash equilibrium at the second stage. Furthermore, no firm wishes to cut its initial price at the first stage: were it profitable to do so, then it could execute that price cut at the second stage. (A first-stage cut could influence the behavior of competitors in the second stage, but only by inducing them to cut their own prices.)

Given that no firm has a profitable deviation in the on-path subgame, and that there is no profitable first-stage deviation downward, it remains to check for a first-stage deviation upward by firm n. If $\bar{p}_n < p^\dagger_{n-1}$ then firm n could deviate upwards (locally) without prompting a second-stage undercut, and so earn greater profit. From this we conclude that $\bar{p}_n = p^\dagger_{n-1}$. This strategy profile maximizes industry profit (and is uniquely Pareto efficient) amongst all price profiles that are "undercut proof" in the sense that no firm wishes to undercut any other firm.[23]

If firm n deviates upward to strictly above $p^\dagger_{n-1}$ at the first stage, then it is no longer true that firms maintain their prices in the corresponding subgame: at least one of the other firms $i < n$ has an incentive to undercut the (now higher) initial price of firm n. Nevertheless, we obtain an equilibrium of this subgame in which firm n earns its equilibrium-path payoff. (For all subgames further off path we can specify any equilibrium play.[24])

Proposition 5 (Stable Prices under Two-Stage Play). *Suppose that the most aggressive firm is uniquely defined. The profile of prices in which $\bar{p}_n = p^\dagger_{n-1}$ and $\bar{p}_j = v$ for $j < n$ is the unique price profile that is supported in equilibrium by the on-path play of pure strategies. The equilibrium profits of firms are equal to the expected profits reported in Proposition 1.*

Efficiency. As discussed in Section 1, the efficient outcome is for the shoppers to be served by the lowest-cost firm. Under single-stage pricing this is (if the lowest-cost firm is unique) impossible, simply because (at least) two firms compete for the shoppers via mixed strategies. Here, though, full efficiency is possible. However, this is true only if the most aggressive firm is also the lowest cost firm. This can fail if a firm with high costs has relatively few captives.[25]

[22]Of course, if the most aggressive firm is not unique then we construct a profile in which any other most-aggressive firm i (where $p^\dagger_i = p^\dagger_n$) take adopts the low-price position.
[23]We discuss these properties in the case of symmetric costs elsewhere (Myatt and Ronayne, 2023a, Section 5).
[24]Equilibrium existence in such subgames is guaranteed, see footnote 8.
[25]A specification with this feature is the duopoly setting of Inderst (2002) that we discussed in Section 1. His entrant firm has no captives and so (using our terminology) is always more aggressive than his incumbent firm.

To compare the efficiencies of single-stage and two-stage play, note the former (at least under the conditions needed for claim (ii) of Proposition 2) randomly allocates shoppers to $n-1$ or n, whereas two-stage play allocates output to only one of those firms. Two-stage play is more efficient if and only if the second-most-aggressive firm has higher costs than the most aggressive.

Proposition 6 (Efficiency of Two-Stage Pricing). *The equilibrium on-path play of pure strategies of a two-stage pricing game is efficient if and only if the most aggressive firm has the lowest marginal cost of production. Comparing to single-stage pricing, label the two most aggressive firms who will fill the positions $n-1$ and n as L and H where $c_L < c_H$. Further assume conditions such that only these two firms compete for shoppers under single-stage pricing.*

Two-stage pricing is more efficient (so that firm L takes the position as the most aggressive firm n) and generates higher consumer surplus than single-stage pricing if and only if

$$\lambda_L - \lambda_H < (c_H - c_L)\frac{\lambda_S + (\lambda_H + \lambda_L)/2}{v - (c_H + c_L)/2}, \tag{11}$$

which means that any offsetting demand-side asymmetry, in terms of captive-audience sizes, must be small relative to the cost advantage of the least-cost firm.

The inequality reported above is, of course, a re-arranged version of (2). It always holds when there is symmetry on the demand side, so that captive audiences are the same size.

Proposition 6 also tells us the outcome for customers: two-stage pricing (relative to single-stage pricing) is better for them if and only if it is more efficient. This is because industry profit is constant across the two settings: any gain in efficiency is picked up by customers.

Changing Costs. The effect of changing costs is more straightforward in the two-stage setting, and the (perhaps surprising) effect of an increase in the cost of the most aggressive firm is absent. All but one firm set the monopoly price to their captives, and so do not respond to cost changes. The only responsive price, $\bar{p}_n = p^{\dagger}_{n-1}$, is chosen as a limit price to deter the undercut from the nearest competitor firm $n-1$, and so depends on the marginal cost c_{n-1}. This fact, and the responses of firms' profits to changes in their costs, are reported here.

Proposition 7 (The Effect of Costs with Two-Stage Pricing). *Under the equilibrium on-path play of pure strategies of a two-stage pricing game, an increase in the cost of the second most aggressive firm raises the price of the most aggressive (shopper-serving) firm. Local changes in other costs have no effect on prices. The effect of own costs on firms' profits are*

$$\frac{\partial \pi_i}{\partial c_i} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = n, \text{ and} \\ \lambda_i & \text{otherwise,} \end{cases} \tag{12}$$

and so the most aggressive firm benefits distinctly more from cost reductions than other firms.

However, his entrant firm also has higher costs. This means that equilibrium play using on-path pure strategies of a two-stage-pricing version of Inderst (2002) is necessarily inefficient.

Risk Aversion and Initial Price Positions. The traditional single-stage game à la Varian (1980) and the on-path play of pure strategies in our two-stage version generate the same expected profits. This is a strength in the sense that either game can be used as part of a deeper model without influencing earlier-stage decisions. However, our two-stage game's outcome does mean firm n is indifferent between taking the undercut-deterring position $\bar{p}_n = p^{\dagger}_{n-1}$ or instead choosing $\bar{p}_n = v$.[26] Indeed, firm n achieves the same expected profit from any intermediate initial price. An argument in favor of the conventional single-stage model (with its mixed-strategy play) is that firm n might just "wait and see" rather than making an early move.

A response is that an initial low-price position results in a certain profit outcome, whereas a higher initial price results in uncertain profits. Risk neutrality means that firm n is indifferent between these options. Nevertheless, a reasonable suggestion is that a desire for a predictable outcome might push firm n to be in favor of the first option.

We illustrate this with a modification to include (at least a little) risk aversion. To do this, we develop an approach that we suggested in a supplement to Myatt and Ronayne (2023a, Appendix A). Suppose that we split each firm into two players: a manager, and an operational pricing agent. We define an extensive-form game with $2n$ players in which

(1) the firms' managers simultaneously choose initial price positions $\bar{p}_i \in [0, v]$; and then
(2) the firms' agents simultaneously choose their firms' retail prices $p_i \in [0, \bar{p}_i]$.

Agents' payoffs are expected profits; they are assumed to be risk neutral as usual. The manager of firm i, however, has payoff $u_i(\pi_i)$, a smoothly increasing and concave function of profit. For one result we will additionally assume that managers are only "a little" risk averse, in the sense that u_i is close to linear. Specifically, if manager i is *"ϵ-risk-averse"* then u_i is such that there exists an affine transformation of profits, $A(.)$, such that $|A_i(x) - u_i(x)| < \epsilon$ for all $x \in [0, v]$.

The manager has the ability and incentive to constrain the agent in the second stage: doing so can induce the play of a preferred equilibrium (from the perspective of the manager).

Equilibrium play in any $t = 2$ subgame is unaffected by this "$2n$ player" scenario. If managers choose the initial prices reported in Proposition 5 then they obtain payoffs $u_i(\pi_i)$ where π_i is the profit of firm i. Any upward deviation by firm n to $\bar{p}_n > p^{\dagger}_{n-1}$ leads to a subgame with the same expected profit, but strictly lower expected utility (because agent n would employ a mixed strategy, making profit uncertain). This means that manager n's choice of $\bar{p}_n = p^{\dagger}_{n-1}$ is the unique best reply to the initial price choices $\bar{p}_i = v$ for all $i \neq n$. Our (subgame perfect) equilibrium (with the on-path play of pure strategies) is now strict.

This argument rules out an effective replication of single-stage equilibrium play (with $\bar{p}_i = v$ for all i). In fact, it rules out all other (pure strategy) price profiles at the initial stage.

[26]Our two-stage pricing game has a subgame perfect equilibrium in which all firms choose $\bar{p}_i = v$, which effectively puts them in the single-stage game (but at $t = 2$) so that Propositions 2 and 3 apply.

Proposition 8 (Two-Stage Pricing with Risk-Averse Managers). *Suppose risk-averse managers choose initial prices and risk-neutral pricing agents choose final prices and consider subgame perfect equilibria with the play of pure strategies at the initial stage.*

In any such equilibrium, $\bar{p}_i = p_i$ for all i; one firm, $j \in \{1, \ldots, n\}$, prices at $\bar{p}_j = \min_{k \neq j}\{p_k^\dagger\}$; and $\bar{p}_i = v$ for all $i \neq j$. In addition, there is such an equilibrium in which:

$$\bar{p}_n = p_{n-1}^\dagger \quad \textit{and} \quad \bar{p}_{j<n} = v, \tag{13}$$

which is the prediction of Proposition 5. Now suppose managers are ϵ-risk-averse. For ϵ sufficiently small, the price profile in (13) *is the unique such subgame perfect equilibrium prediction.*

3. Sequential-Move Models of Sales

Two-stage pricing offers a first-stage commitment opportunity that is related to Stackelberg-style settings. Here we briefly discuss an exogenous-move-order game, before building one in which firms choose if and when to commit to advertised price positions.

Exogenously Sequential Play. We retain our supply-side and demand-side specifications and, to ease exposition, assume that $p_1^\dagger > \cdots > p_n^\dagger$ and break ties in favor of more aggressive firms. Now suppose that the firms each choose a price in some set sequence.

An easy case is when firms n and $i < n$ are the last two firms to move, and when all other firms $j \notin \{i, n\}$ have already chosen $p_j = v$. If firm i moves before n then it recognizes that any price $p_i \in [p_i^\dagger, v]$ will be undercut by firm n, and so it focuses on captives by setting $p_i = v$. Firm n (moving last) then serves the shoppers at price v. If instead firm n moves before firm i then it chooses $p_n = p_i^\dagger$ to deter any final-period undercut by firm i.

More generally, firm n (no matter when it moves in the sequence) sells to all the shoppers. Given this, a firm $i < n$ moving before firm n recognizes that it will only sell to captives and so sets $p_i = v$. Firm n needs to deter any undercut by a later-moving firm, and so sets p_i equal to the lowest $p_j^\dagger$ amongst all firms j that follow it.

Proposition 9 (Equilibrium with Exogenous Sequencing). *In any subgame perfect equilibrium, each firm $i < n$ chooses $p_i = v$ when called upon to move. If firm n moves in the last period then it chooses $p_n = v$. If it moves earlier then it chooses $p_n = p_j^\dagger$ where $j < n$ is the most aggressive firm that has yet to move at that time.*

The allocation of shoppers here is unambiguous. However, the price that they (and the captives of firm n) pay (and so the division of surplus between customers and the most aggresive firm) depends on the exogenous move order. Clearly, firm n would rather move later in that order.

However, when sales are made over time (rather than after all firms have moved) exogenous move orders are sensitive to manipulation. Suppose, for example, that we see a firm's decision to set a price as the act of advertising to make that price accessible to price-comparing shoppers.

Any firm $i < n$ has no reason to rush to advertise; ultimately, it sells only to captive customers. On the other hand, firm n wants to move early to bring in those shoppers rather than waiting and foregoing sales to them. We now build a model to incorporate these considerations.

Endogenously Sequenced Play. We now study firms who decide when and at what price to advertise to shoppers. Firms are committed to any price they advertise. However, shoppers can only be reached via advertised prices.

Formally, this is a multi-stage game that takes place over T discrete periods. Each period, firms choose price positions and whether to advertise those positions, where the decision to advertise locks in a firm's price for all future periods. Each time $t \in \{1, 2, \dots, T\}$ proceeds as follows.

(1) All firms observe the entire history of the game so far, and then
   (a) a firm that advertised in period $t-1$ advertises the same price in period t; but
   (b) other firms simultaneously choose a price and whether to advertise it.

(2) Shoppers buy from (one of) the cheapest advertised prices. Captives buy as usual.

Profits accrue without discounting across all T periods where the mass of sales at time t is scaled by $1/T$.[27] Amongst sales to shoppers, we retain the freedom to choose appropriate tie-break rules in the event of tied prices. Firms are strictly ordered so that $p_1^\dagger > \cdots > p_n^\dagger$. These assumptions help us to cope with unimportant technicalities and to facilitate exposition.

Firm n can guarantee a profit of $(p_{n-1}^\dagger - c_n)(\lambda_n + \lambda_S)$, which is equal to its captive-only profit plus $(p_{n-1}^\dagger - p_n^\dagger)(\lambda_n + \lambda_S)$. It can do this by advertising immediately (at time $t = 1$) at a price $p_n = p_{n-1}^\dagger$. If it does this, then other firms are willing to sit back, charge v to their captives, and refrain from advertising. The crucial deviation here is upward by firm n to a higher advertised price at time $t = 1$. Doing so, however, firm n will be undercut and loses shopper sales for the remaining $T - 1$ periods. If T is sufficiently large, then firm n prefers not to do this. And as we confirm in Proposition 10, this is the only subgame-perfect prediction.

Proposition 10 (Equilibrium with Endogenous Sequencing). *If T is sufficiently large, then in any subgame-perfect equilibrium, the most aggressive firm n advertises a shopper-serving low-price position, $p_n = p_{n-1}^\dagger$, at the first opportunity, $t = 1$, and all other firms set the monopoly price in every period, selling only to their captive buyers.*

This outcome replicates the outcome from two-stage play that we characterized in Section 2 in which one firm (firm n) stands apart from the others and uniquely sells to the shoppers. In essence, a desire to avoid delay gives the same outcome here; under two-stage play we saw the desire to avoid risk had a similar effect. These two familiar forces (impatience and risk aversion) push us towards a stable price profile in which an "on-sale" price is offered by a single firm. Nevertheless, the profits here also match those from a single-stage pricing game (Proposition 1).

[27] We can interpret the demand as either a flow of new customers, or as repeated sales. We can also handle several other specifications, including an infinite time horizon with discounting or play in continuous time.

4. Process Innovations and Endogenous Asymmetry

Our comparative-static results, equation (10) or (12), reveal the different incentives that firms face to engage in cost reduction. Here we consider situations in which firms' costs (or customers' valuations for their product) arise endogenously from their innovative activities.

A Model of Process Innovation. Prior to the pricing stage (or stages) firms choose their production technologies via costly innovations. We study the following game.

(1) Firms simultaneously choose and observe production technologies, denoted by z_i.
(2) Firms proceed either to (a) single-stage or (b) two-stage pricing where, respectively:
   (a) firms play a Nash equilibrium of the single-stage pricing game (of Section 1); or
   (b) firms play a subgame-perfect equilibrium of the two-stage pricing game (of Section 2) that supports the on-path play of pure strategies.

We interpret a firm's technology choice, $z_i \in [0, \bar{z}_i]$, as a fixed cost expenditure which lowers its marginal cost of production: it is a (costly) process innovation. It would (as usual) be equivalent to think of a product innovation that raises customers' willingness to pay for that firm's product. What really matters is the net surplus, $v - c_i$, created. We assume that

$$v - c_i = V_i(z_i), \tag{14}$$

where $V_i(z_i)$ is positive, smoothly increasing, concave, and $\lambda_i V_i'(0) > 1 > (\lambda_i + \lambda_S)V_i'(\bar{z}_i)$.

For either version (a) or (b) of pricing, the gross equilibrium expected profit of a firm is taken from (3) of Proposition 1. As such, innovation choices do not depend on which pricing game is adopted. However, we have (so far) chosen labels so that firms n and $n-1$ are most aggressive. Here that status is endogenous because firms choose their technologies. For now, then, we do not label firms in this way. The general expression for a firm's net expected profit is

$$\pi_i = \lambda_i V_i(z_i) + (\lambda_i + \lambda_S) \max\left\{0, \min_{j\neq i}\{p_j^\dagger\} - p_i^\dagger\right\} - z_i \quad \text{where} \quad p_j^\dagger = v - \frac{\lambda_S V_j(z_j)}{\lambda_j + \lambda_S}. \tag{15}$$

Using these n expected profit expressions as the outcomes from stage 2 described above, we specify a simultaneous-move innovation game. We look for its pure-strategy Nash equilibria.

Asymmetric Equilibrium Innovation. Consider the response of firm i's profit to a local increase in z_i. Firm i is the most aggressive firm if and only if $p_i^\dagger < p_j^\dagger$ for all $j \neq i$, which, from (15), is equivalent to $V_i(z_i)/(\lambda_i + \lambda_S) > V_j(z_j)/(\lambda_j + \lambda_S)$. Combining with (12),

$$\frac{\partial \pi_i}{\partial z_i} = -1 + V_i'(z_i)\begin{cases} \lambda_i + \lambda_S & \text{if } \frac{V_i(z_i)}{\lambda_i+\lambda_S} > \max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j+\lambda_S}\right\} \\ \lambda_i & \text{if } \frac{V_i(z_i)}{\lambda_i+\lambda_S} < \max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j+\lambda_S}\right\} \end{cases} \tag{16}$$

so that the profit of firm i has a (convex) kink when i becomes the most aggressive firm.

Equation (16) implies that firm i will never optimally choose z_i such that $p_i^\dagger = \min_{j\neq i}\{p_j^\dagger\}$ and so, in equilibrium, the most aggressive firm is distinct. We can, therefore, characterize two possible solutions for firm i's innovation: z_i^H for when it is the most aggressive firm, and z_i^L for when it is not. These solutions are uniquely determined by the two respective conditions

$$1 = \lambda_i V_i'(z_i^L) \quad \text{and} \quad 1 = (\lambda_i + \lambda_S) V_i'(z_i^H), \tag{17}$$

they satisfy $z_i^H > z_i^L$, and they do not depend on the innovation choices of any other firms.

To find an equilibrium of our innovation game we identify a firm $i \in \{1, \ldots, n\}$ to be the aggressive firm. We then set $z_i = z_i^H$ for that firm and $z_j = z_j^L$ for all other firms $j \neq i$. This is a candidate for an equilibrium. Several checks are needed. Firstly, this works only if firm i is the most aggressive firm. This requires $p_i^\dagger < \min_{j\neq i}\{p_j^\dagger\}$ or

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\} \quad \Leftrightarrow \quad \frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j\in\{1,\ldots,n\}}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}. \tag{18}$$

Secondly, we need to check that firm i does not wish to deviate back to z_i^L, and that no $j \neq i$ wishes to deviate from z_j^L to z_j^H. For example, suppose a firm i uniquely maximizes (across the set of firms) both $V_j(z_j^L)/(\lambda_j + \lambda_S)$ and also $V_j(z_j^H)/(\lambda_j + \lambda_S)$. Such a firm can always take the "most aggressive" role in the innovation game. The proof of Proposition 11 establishes more generally that we can always find at least one firm to take this role.

Proposition 11 (Asymmetric Innovation Equilibria). *Consider the innovation game.*

(i) There is at least one pure-strategy Nash equilibrium, and there are at most n such equilibria.

(ii) In any equilibrium there is a firm that, at the pricing stage, is the uniquely most aggressive.

(iii) If $\arg\max_j\{V_j(z_j^L)/\lambda_j\}$ is unique, there is a unique equilibrium if λ_S is sufficiently small.

(iv) If firms are symmetric then there are exactly n pure-strategy equilibria.

The final claim of the proposition (finding n equilibria, with any one of the symmetric firms eventually taking the aggressive sell-to-the-shoppers position) also holds when firms are sufficiently similar. It is of interest because it implies that exogenously symmetric firms become endogenously asymmetric. One of the firms (it could be any) behaves distinctly differently.

Corollary 2 (Ex Ante Symmetry and Asymmetric Outcomes). *Suppose firms have the same sized captive audiences and technological opportunities. In any pure-strategy equilibrium, innovation and pricing choices are asymmetric. One firm chooses strictly higher innovation, giving it a strictly lower marginal cost; the $n-1$ others choose the same (and lower) innovation.*

We emphasize this as a corollary because it provides a distinct rationale for opening up models of sales to cost asymmetry. Asymmetry is especially relevant because (with endogenous innovation) it arises ex post even when firms are symmetric ex ante.

From Demand-Side to Supply-Side Asymmetry. Proposition 11 establishes circumstances (when λ_S is small) in which there are fewer than n equilibria. However, it does not identify which firms are able to claim the shopper-supplying position. To do this here we specify technological opportunities that are the same for all firms: $v - c_i = V(z_i)$ for all i, where

$$V(z) = \beta z^{\gamma} \quad \text{where} \quad \gamma \in (0, 1). \tag{19}$$

Firms remain asymmetric owing to their different captive-audience sizes. This allows us to ask how the central demand-side terms of models of sales determine their central supply-side terms.

The parameter $\gamma = zV'(z)/V(z)$ is the elasticity of a firm's per-customer surplus with respect to its fixed-cost innovation outlay. It represents, therefore, a measure of the shape of technological opportunity for a firm, whereas the parameter β scales that opportunity.[28]

The specification of (19) generates closed-form solutions for (17). Innovation choices are

$$z_i^L = (\gamma\beta\lambda_i)^{1/(1-\gamma)} \quad \text{and} \quad z_i^H = (\gamma\beta(\lambda_i + \lambda_S))^{1/(1-\gamma)}. \tag{20}$$

The innovative expenditure of a firm increases with the size of its captive audience, and the elasticity of that relationship is determined by the shape (via its elasticity γ) of the technological opportunity. We are more interested in how firms' captive audiences (the exogenous demand-side asymmetry) influence the surplus $v - c_i$ generated by their products (the endogenous supply-side asymmetry). This is (depending on whether a firm expects to serve captives)

$$V(z_i^L) = \beta\left(\gamma\beta\lambda_i\right)^{\gamma/(1-\gamma)} \quad \text{or} \quad V(z_i^H) = \beta\left(\gamma\beta(\lambda_i + \lambda_S)\right)^{\gamma/(1-\gamma)}. \tag{21}$$

The elasticity of per-consumer surplus $v - c_i = V(z_i^L)$ with respect to the captive-audience size λ_i (for a firm that expects to serve only its captives) is, by inspection, $\gamma/(1-\gamma)$. This strictly exceeds one, and so is an elastic relationship, if and only if $\gamma > \frac{1}{2}$.

To pin down the equilibrium of the innovation game we need to identify a firm that ultimately becomes the most aggressive competitor and so serves the shoppers. A key criterion for whether this is possible for a firm i is the inequality reported in (18). The left-hand side is

$$\frac{V(z_i^H)}{\lambda_i + \lambda_S} = \beta^{1/(1-\gamma)}\gamma^{\gamma/(1-\gamma)}\left(\lambda_i + \lambda_S\right)^{(2\gamma-1)/(1-\gamma)}. \tag{22}$$

If $\gamma > \frac{1}{2}$ then this is strictly increasing in λ_i. This means that firm i is able to take the (endogenous) position of the most aggressive firm only if its captive audience is sufficiently large. For this case (that is, when $\gamma > \frac{1}{2}$) the per-customer surplus is relatively responsive to the innovation expenditure. This means that a firm with a larger captive audience endogenously chooses surplus that is disproportionately larger. The implicit cost advantage (which makes it more aggressive) more than offsets its larger captive audience (which makes it less aggressive).

Similarly, if $\gamma < \frac{1}{2}$, so that innovation opportunities are weaker, then it is firms with smaller captive audiences that are able to take the position as the (endogenously) most aggressive firm.

[28] This specification is reminiscent (albeit different from) the classic constant-elasticity relationship between production cost and research-and-development expenditure in Dasgupta and Stiglitz (1980, p. 273). Here the constant elasticity is between the per-unit surplus and any fixed-cost expenditure.

In both cases, there can be more than one possible firm to take on the role of the most aggressive, but the list of possible candidates shrinks with the population of shoppers. In fact, we can exploit claim (iii) of Proposition 11, which identifies a unique equilibrium (and so a unique identity for the most aggressive firm) if λ_S is sufficiently small. To do this, we note that

$$\frac{V(z_i^L)}{\lambda_i + \lambda_S} = \frac{\beta^{1/(1-\gamma)}\gamma^{\gamma/(1-\gamma)}\lambda_i^{\gamma/(1-\gamma)}}{\lambda_i + \lambda_S} \quad \Rightarrow \quad \lim_{\lambda_S \downarrow 0} \frac{V(z_i^L)}{\lambda_i + \lambda_S} = \beta^{1/(1-\gamma)}\gamma^{\gamma/(1-\gamma)}\lambda_i^{(2\gamma-1)/(1-\gamma)}. \tag{23}$$

We use this and claim (iii) of Proposition 11 to obtain our final formal proposition.

Proposition 12 (Who Becomes the Most Aggressive?). *Consider the innovation game in which firms share the same technological opportunity, $v - c_i = V(z_i) = \beta z_i^\gamma$, and set $\gamma \neq \frac{1}{2}$.*

If λ_S is sufficiently small, then in the unique equilibrium of the innovation game:

(i) the firm with the largest captive audience becomes the most aggressive firm if $\gamma > \frac{1}{2}$, but

(ii) the firm with the smallest captive audience becomes the most aggressive firm if $\gamma < \frac{1}{2}$.

Our study of two-stage pricing identified a unique firm that serves shoppers, but did so based on exogenously specified supply-side and demand-side asymmetries. Proposition 12 reveals how those demand-side asymmetries (the configuration of captive-customer populations) can interact with the shape of technological opportunities to determine both supply-side asymmetries and which firm serves the shoppers. Of course, endogenous actions can also influence the demand side and so we comment briefly on this next before offering concluding discussions.

Endogenous Captive Audiences. Asymmetric innovation decisions emerge because the profit of the most aggressive firm reacts in a distinctly different way when we change its costs. Chioveanu (2008) identified a related effect for firms where costly advertising influences the sizes of their captive audiences. The advertising stage is followed by a conventional single-stage pricing game.[29] Chioveanu (2008, Proposition 3) identified an equilibrium in which one firm advertises strictly less than others, so that equilibrium advertising outlays are asymmetric.

Chioveanu (2008) worked with symmetric marginal costs but we can confirm that her important insight also holds when costs are asymmetric. If firm i is not the most aggressive, then

$$\frac{\partial \pi_i}{\partial \lambda_i} = v - c_i \quad \text{and} \quad \frac{\partial \pi_i}{\partial \lambda_j} = 0 \quad \text{for} \quad j \neq i. \tag{24}$$

On the other hand, if firm i is the most aggressive, so that (18) holds, then

$$\frac{\partial \pi_i}{\partial \lambda_i} = v - c_i - \lambda_S \max_{j \neq i}\left\{\frac{v - c_j}{\lambda_j + \lambda_S}\right\} \quad \text{and} \quad \frac{\partial \pi_i}{\partial \lambda_k} > 0 \quad \text{if} \quad k = \arg\max_{j \neq i}\left\{\frac{v - c_j}{\lambda_j + \lambda_S}\right\}. \tag{25}$$

Again there is a kink in the response of a firm's profit, which (if we were to view the acquisition of captives as costly) pushes towards an asymmetric equilibrium.

[29]A similar result, but in a setting with a comparison site that advertises alongside sellers for its captive base can be found in an earlier version of Ronayne and Taylor (2022): Ronayne and Taylor (2020, Appendix W.3).

The result of Chioveanu (2008) also resonates with early work by Ireland (1993) and McAfee (1994), which considered firms that independently advertise to customers à la Butters (1977) and Grossman and Shapiro (1984) before choosing prices.[30] In those papers one firm advertises distinctly more than others, even if they are symmetric ex ante. In our related work we also obtain that result, but using two-stage pricing (Myatt and Ronayne, 2023a, Section 7).

A key difference between the cost-reducing process innovations here and the captive-audience-enhancing advertising of Chioveanu (2008) is that here the distinct firm (that is, the aggressive firm that expects to serve shoppers) faces a stronger incentive to innovate, whereas the standalone firm faces a weaker incentive to advertise. However, the underlying force is the same: that firm faces a stronger incentive than others to become more aggressive. The difference is that aggression is achieved by over-investment in one case, and under-investment in the other.

The forces which push toward asymmetry here come from the decisions of firms. This contrasts the extended model of Baye, Kovenock, and De Vries (1992, Section V) in which captive consumers are able to switch between firms. They identified an incentive for customers to shift toward (in our language) the most aggressive firm, which corresponds (in their setting) to the firm with fewest captives. This underpinned their argument (and their Theorem 3) that the play of a symmetric equilibrium by symmetric firms is most reasonable. Here we join Chioveanu (2008) and others in suggesting that asymmetric outcomes may be more likely.

Shelegia and Wilson (2021) also considered the incentives of firms to engage in costly activities that influence the marketplace. They studied (in their Section III.B, pp. 214–216) a game in which costly effort expands a firm's captive audience (and possibly shrinks the captive audiences of competitors). This is the approach of Chioveanu (2008). However, one result (their Proposition 3) refers to the properties of a symmetric equilibrium (in which all firms make the same effort choices) when such an equilibrium exists. Here, however, we align with Chioveanu (2008) in suggesting that any equilibrium will involve asymmetric choices. Similarly, Shelegia and Wilson's study of the comparative-static effects of firm profitability also begins from a symmetric situation, whereas we suggest that asymmetric outcomes may be more likely.

5. Concluding Discussion

We analysed the classic captive-and-shopper model of sales while allowing for fully asymmetric firms. We know only of one paper (Shelegia and Wilson, 2021) that deals extensively with asymmetric costs in such a setting, albeit modeled via utility offers and with the addition of costly advertising. Our equilibrium characterization is also accompanied by welfare and comparative-static results, including one (the effect of costs on price) that builds upon an insight that we credit to Golding and Slutsky (2000) and Inderst (2002). Our two-stage (and multi-stage) settings extend and complement our own related work (Myatt and Ronayne, 2023a,b).

[30] A customer sees the price of firm i with probability α_i. Given a unit mass of potential customers, the mass of shoppers is $\lambda_S = \prod_{i=1}^{n} \alpha_i$, the captive audience of firm i has size $\lambda_i = \alpha_i \prod_{j \neq i}(1-\alpha_j)$, and (for example) there are $\lambda_{ij} = \alpha_i \alpha_j \prod_{k \notin \{i,j\}}(1-\alpha_k)$ customers who compare the pair of i and j (and no other firm).

Our finding that asymmetric costs endogenously arise (even if firms are symmetric ex ante) first and foremost highlights the importance of dealing with asymmetric models of sales. The endogenous supply-side asymmetry resonates with papers that find endogenously asymmetric demand structures (also despite symmetric setups) via, e.g., product awareness (Ireland, 1993; McAfee, 1994) or captive customer bases (Chioveanu, 2008). A common feature to all this work is that there is one firm with distinct incentives (to innovate, or to advertise, respectively).

Asymmetries on the supply or demand side naturally suggest asymmetric strategies in any pricing game one might apply, and indeed that is what we found in the various pricing games we studied. Interestingly, in the (popular) symmetric (in terms of costs and captives) setting there are multiple equilibria, including asymmetric ones, as explicitly addressed by, e.g., Baye, Kovenock, and De Vries (1992) and Johnen and Ronayne (2021). Those papers each point to (different) rationales to select the symmetric equilibrium, where the former does so in a meta-game with asymmetric captive shares, while the latter does so by considering richer (but still symmetric) demand structures. Those arguments were made holding costs equal across firms. Our work shows how cost-symmetry is not robust to modeling process innovation choices.

A particularly robust prediction from our work is that equilibrium profits in captive-shopper settings are the same across a wide variety of pricing games, including not only the classic single-stage game, but also two-stage and sequential-move games (Propositions 1, 5 and 10). An implication is that researchers who are interested in (expected) profit predictions, for example if modeling a model of sales in a subgame, can have a certain level of confidence in that regard. A recent example is provided by the model of Hagiu and Wright (2023) in which a platform offers a per-transaction benefit to sellers but also chooses a per-unit sales fee to charge them. Both elements determine sellers' marginal costs. The platform also determines the number of captives and shoppers via its design decisions. After the platform acts, sellers set prices. Equilibrium analysis therefore requires the platform to anticipate profits in the pricing subgames. Our work says that a variety of pricing games would yield the same profits and hence do not affect the platform's incentives, ultimately serving to increase the robustness of the equilibria they report.

Appendix A. Omitted Proofs

Proof of Proposition 1. An equilibrium exists because (given the use of our tie-break rule in which ties are broken in favor of a lowest-cost firm) the conditions of Theorem 5 of Dasgupta and Maskin (1986, p.14) are satisfied. Specifically, that theorem asks for the sum of players' payoffs (the total industry profit here) to be upper semi-continuous in actions (here, this is the profile of prices) and this holds if ties are broken in favor of lower-cost firms. (In fact, our work later on shows the set of equilibria will be the same for any tie-break rule.)

Turning to payoffs, if $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ then Lemma B6 in Appendix B shows that any Nash equilibrium gives the payoffs specified in the proposition. In Appendix B we provide an algorithm to construct an equilibrium with the required payoffs for any parameter values. $\square$

Proof of Proposition 2. Basic and standard properties of any equilibrium (see Appendix B) are that all firms mix up to v (Lemma B2) with no gaps (Lemma B3) and any atoms only at v (Lemma B1). Firm n earns more than its captive-only profit (Proposition 1 and Lemma B6) which requires all others to play an atom at v (Lemma B6) while firm n does not (Lemma B1).

With those basic observations in hand, we note that if the lower bound of all prices were to strictly exceed $p^\dagger_{n-1}$ then (at least) firms n and $n-1$ could (by pricing just above $p^\dagger_{n-1}$) sell to all shoppers and achieve a profit strictly exceeding their equilibrium profit. We conclude that the joint support of firms' mixed strategies extends down to $\min_i \underline{p}_i = p^\dagger_{n-1}$. Prices below $\min_{j\in\{1,\ldots,n-2\}} p^\dagger_j$ are strictly dominated for firms $i \in \{1,\ldots,n-2\}$, and so (given the absence of gaps; Lemma B3) firms $n-1$ and n must mix continuously over $[p^\dagger_{n-1}, \min_{j\in\{1,\ldots,n-2\}} p^\dagger_j)$. Given that they both price below $\min_{j\in\{1,\ldots,n-2\}} p^\dagger_j$ with strictly positive probability, a price at or just above $\min_{j\in\{1,\ldots,n-2\}} p^\dagger_j$ will not be chosen by any firm $i \in \{1,\ldots,n-2\}$, and so there is some $p^\ddagger > \min_{j\in\{1,\ldots,n-2\}} p^\dagger_j$ such that firms $n-1$ and n mix on the interval $[p^\dagger_{n-1}, p^\ddagger)$.

The expected profit earned by firm $n-1$ from charging a price $p \in [p^\dagger_{n-1}, p^\ddagger)$ is

$$\pi_{n-1}(p) = (p - c_{n-1})\left(\lambda_{n-1} + \lambda_S\left(1 - F_n(p)\right)\right) = \lambda_{n-1}(v - c_{n-1}), \tag{A1}$$

where the final term is its captive-only profit. Similarly, the expected profit earned by firm n from charging a price $p \in [p^\dagger_{n-1}, p^\ddagger)$ is

$$\pi_n(p) = (p - c_n)\left(\lambda_n + \lambda_S\left(1 - F_{n-1}(p)\right)\right) = \lambda_n(v - c_n) + (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n) \tag{A2}$$

where the final expression is the profit of firm n from Lemma B6. These equations solve:

$$F_n(v) = 1 - \frac{\lambda_{n-1}(v-p)}{\lambda_S(p - c_{n-1})} \quad \text{and} \quad F_{n-1}(v) = 1 - \frac{\lambda_n(v-p)}{\lambda_S(p-c_n)} - \frac{(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)}{\lambda_S(p - c_n)}. \tag{A3}$$

These are valid cumulative distribution functions that strictly and continuously increase from $F_{n-1}(p^\dagger_{n-1}) = F_{n-1}(p^\dagger_{n-1}) = 0$, and they can be re-written to obtain (4).

Any equilibrium involves mixing by the dance partners $n-1$ and n up to some price $p^\ddagger$. One possibility is that $p^\ddagger = v$, so that only these two firms mix and all others choose $p_i = v$. By inspection, $\lim_{p\uparrow v} F_n(p) = 1$ and $\lim_{p\uparrow v} F_{n-1}(p) \leq 1$ (the latter inequality is strict if $p^\dagger_{n-1} > p^\dagger_n$) which means that we have valid distributions over the entire interval. Firms $n-1$ and n cannot improve by deviating. If there is an equilibrium in which only these two firms mix then (by construction) it is unique. We must check to see if some other firm i might wish to deviate to $p_i < v$. Firm $i \in \{1,\ldots,n-2\}$ earns its captive-only monopoly profit $\lambda_i(v - c_i)$. By deviating to $p \in [p^\dagger_{n-1}, v)$, and assuming that $\lambda_{n-1} \leq \lambda_i$ (the condition in the proposition) it earns

$$\begin{aligned}
\pi_i(p) &= (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_{n-1}(p)\right)\left(1 - F_n(p)\right)\right) && \text{(A4)}\\
&< (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_n(p)\right)\right) && \text{(A5)}\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_{n-1}(v-p)}{(p - c_{n-1})}\right) && \text{(A6)}\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_{n-1}(v-p)}{\lambda_{n-1}(v - p^\dagger_{n-1}) + \lambda_S(p - p^\dagger_{n-1})}\right) && \text{(A7)}\\
&\leq (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_i(v-p)}{\lambda_i(v - p^\dagger_i) + \lambda_S(p - p^\dagger_i)}\right) && \text{(A8)}\\
&= (p - c_i)\left(\lambda_i + \frac{\lambda_i(v-p)}{(p - c_i)}\right) = \lambda_i(v - c_i). && \text{(A9)}
\end{aligned}$$

The third line is obtained by substituting in the expression for $F_n(p)$. The fourth line is obtained by writing c_{n-1} in terms of λ_{n-1} and $p^\dagger_{n-1}$ and then re-arranging. The fifth line holds because $\lambda_{n-1} \leq \lambda_i$ and $p^\dagger_{n-1} \leq p^\dagger_i$. The final line is obtained by substituting back in for $p^\dagger_i$ and then re-arranging. This means that firm i performs strictly worse by deviating.

We have established the existence of a (unique, within this class) "two to tango" equilibrium. It remains to check whether than can be other equilibria. This would require another firm to begin mixing (to "step on to the dance floor") at some price $p^\ddagger < v$. However, the argument above demonstrates (given the lack of atoms below v, and continuity properties) that it would be strictly sub-optimal for such a firm. $\square$

Proof of Proposition 3. This follows from the preceding argument in the main text. To see a specific example of this, consider a "two to tango" equilibrium in which (for simplicity of exposition) firms n and $n-1$ satisfy $c_n = c_{n-1} = 0$ and $\lambda_n = \lambda_{n-1} = \lambda$. This means that

$$p^\dagger_n = p^\dagger_{n-1} = \frac{\lambda v}{\lambda + \lambda_S} \quad \text{and} \quad F_n(p) = F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda + \lambda_S)}{p\lambda_S} = 1 - \frac{(v-p)\lambda}{p\lambda_S}. \quad \text{(A10)}$$

The condition required for this to be an equilibrium is that there is no price at which another lower-indexed firm wishes to join. Equation (6) from the main text here requires

$$\begin{aligned}
(v-p)\lambda_i \geq (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)) = (p - c_i)\lambda_S\left(\frac{(v-p)\lambda}{p\lambda_S}\right)^2 \\
\Leftrightarrow \quad \lambda_i \geq \frac{(p-c_i)(v-p)\lambda^2}{p^2\lambda_S} \quad \forall\, p \in \left(\frac{\lambda v}{\lambda + \lambda_S}, v\right). \quad \text{(A11)}
\end{aligned}$$

By inspection, if $p > c_i$ then this fails if λ_i is small. Pushing further, suppose that this holds for all $i \in \{1, \ldots, n-3\}$, but let us choose λ_{n-2} so that this fails for some p. For firm $n-2$ set

$$c_{n-2} = p^\dagger_{n-1} = \frac{\lambda v}{\lambda + \lambda_S} \quad \text{(A12)}$$

which guarantees that $p^\dagger_{n-2} > p^\dagger_{n-1}$ for any $\lambda_{n-2} > 0$, no matter how small. Firm $n-2$ will wish to step on to the dance floor at the lowest price $p^\ddagger$ which satisfies

$$\lambda_{n-2} = \frac{(p^\ddagger - c_{n-2})(v-p^\ddagger)\lambda^2}{(p^\ddagger)^2\lambda_S} = \frac{\left(p^\ddagger(\lambda+\lambda_S) - \lambda v\right)(v-p^\ddagger)\lambda^2}{(p^\ddagger)^2\lambda_S(\lambda+\lambda_S)}. \tag{A13}$$

Explicitly, this is the lower solution to

$$(\lambda+\lambda_S)\left(1+\frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)(p^\ddagger)^2 - (2\lambda+\lambda_S)vp^\ddagger + \lambda v^2 = 0$$

$$\Rightarrow \quad p^\ddagger = \frac{v(2\lambda+\lambda_S) - v\sqrt{\lambda_S^2 - \frac{4\lambda_{n-2}\lambda_S(\lambda+\lambda_S)}{\lambda}}}{2(\lambda+\lambda_S)\left(1+\frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)}, \tag{A14}$$

where this solution satisfies $p^\ddagger \downarrow \lambda v/(\lambda+\lambda_S)$ as $\lambda_{n-2} \downarrow 0$.

The argument given is that there can be parameters under which a third firm must participate. In Appendix B we provide a characterization of equilibria. The same approach taken here applies: if there are k firms that engaged in randomized sales then we can add an additional firm with relatively high marginal cost but few captives that wishes to participate in sales. □

Proof of Corollary 1. This follows from the preceding argument in the main text. □

Proof of Proposition 4. Claim (i) holds because firms $i \in \{1,\ldots,n-2\}$ play $p_i = v$ as pure strategies, and their costs do not enter the solutions reported in (4). For claim (ii), $F_{n-1}(p)$ is decreasing in $p^\dagger_{n-1}$ which itself is increasing in c_{n-1}. For $F_n(p)$,

$$\frac{\partial F_n(p)}{\partial c_{n-1}} = \frac{\lambda_{n-1}+\lambda_S}{\lambda_S}\left(\frac{(p-p^\dagger_{n-1})}{(p-c_{n-1})^2} - \frac{1}{p-c_{n-1}}\frac{\partial p^\dagger_{n-1}}{\partial c_{n-1}}\right) \tag{A15}$$

$$= \frac{\lambda_{n-1}+\lambda_S}{\lambda_S(p-c_{n-1})}\left(\frac{(p-p^\dagger_{n-1})}{(p-c_{n-1})} - \frac{\lambda_S}{\lambda_{n-1}+\lambda_S}\right) = -\frac{(v-p)\lambda_{n-1}}{\lambda_S(p-c_{n-1})^2} < 0 \tag{A16}$$

The CDFs are both decreasing in c_{n-1}, which means an increase in c_{n-1} pushes up the distributions of prices. Claim (iii) follows an inspection of $F_{n-1}(p)$. Finally for claim (iv), we use the properties $c_n \le c_{n-1}$ and $p^\dagger_n < p^\dagger_{n-1}$. From re-arrangement,

$$\lambda_i + \lambda_S = \frac{\lambda_S(v-c_i)}{v-p^\dagger_i}, \tag{A17}$$

and so substituting for $\lambda_{n-1}+\lambda_S$ and $\lambda_n+\lambda_S$ in $F_n(p)$ and $F_{n-1}(p)$ respectively,

$$F_{n-1}(p) < F_n(p) \quad \Leftrightarrow \quad \frac{v-c_n}{(v-p^\dagger_n)(p-c_n)} < \frac{v-c_{n-1}}{(v-p^\dagger_{n-1})(p-c_{n-1})}, \tag{A18}$$

which holds if $p^\dagger_n < p^\dagger_{n-1}$ and (given that $v \ge p$) $c_n \le c_{n-1}$. □

Proof of Proposition 5. The argument in the text before the proposition establishes that the profile is the only one that can be supported by the on-path play of pure strategies. It remains to show that we can construct a subgame perfect equilibrium that supports such play. The argument in the text also explains that firms maintain their initial prices in the subgame reached on the equilibrium path, and that there is no incentive to deviate downward at the first stage. The only remaining deviation to consider is an upward deviation by firm n at the first stage to an initial price $\bar{p}_n > p^\dagger_{n-1}$. If $\bar{p}_n = v$ then the subgame is equivalent to a single-stage game, and we specify that firms play the equilibrium from that game; it exists (Proposition 2) and gives the same expected profit for firm n as on the equilibrium path. If $\bar{p}_n \in (p^\dagger_{n-1}, v)$, then in the subgame we use the same strategies as for an equilibrium of the single-stage game, but we truncate those strategies. Specifically, firm n follows its single-stage equilibrium strategy for $p < \bar{p}_n$ and then places any remaining mass at $\bar{p}_n$. Any other firm shifts any mass from $\bar{p}_n$ and above up to v. This generates the on-path equilibrium profits for all firms. □

Proof of Proposition 6. As noted in the text, this follows from re-arranging (2) □

Proof of Proposition 7. Straightforward from expressions for equilibrium profits. □

Proof of Proposition 8. We seek subgame perfect equilibria of the two-stage game defined in the text with pure strategy choices at $t = 1$. We now use the definition of $p^\dagger_i$ updated to be in terms of $\bar{p}_i \in (c_i, v]$ as given by (B1) in Appendix B. For our consideration of Nash equilibria in subgames at $t = 2$, we also use several lemmas derived there. We proceed via three cases.

(i) Suppose that $\bar{p}_m \equiv \min_{j\in\{1,\dots,n\}}\{\bar{p}_j\} < p^\dagger_i$ for all $i \neq m$. By Lemma B4 firm m (which must be firm n) earns expected profit $(\lambda_n + \lambda_S)(\bar{p}_n - c_n)$ with certainty. Manager n could safely deviate to a slightly higher initial price and prompt a slightly higher but still certain profit. Hence $\bar{p}_m \geq p^\dagger_i$ for some $i \neq m$ in any equilibrium.

(ii) Suppose $\bar{p}_m > p^\dagger_i$ for some $i \neq m$. Any $t = 2$ subgame's equilibrium is in mixed strategies.

If $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$, then firm n earns expected profit equal to $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$ while any $i < n$ earns captive-only expected profit (Lemma B6). This can only be achieved with agents n and $n-1$ mixing down to $p^\dagger_{n-1}$. Manager n sees this, and can deviate to $\bar{p}_n = p^\dagger_{n-1}$, which results in $p_i = \bar{p}_i$ in $t = 2$ (by Lemma B4). This gives firm n a profit of $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$, with certainty. Given manager n is risk averse, they strictly prefer this outcome.

We now consider the two remaining special (degenerate) sub-cases. First, if there are multiple most aggressive firms, then at least two agents that mix down to $p^\dagger_n$ in $t = 2$ (else multiple firms would earn more than captive only profits, contradicting Lemma B5) and those firms earn their captive-only profit. A manager of one of those firms could set $p^\dagger_n$ and generate the same expected profit, but with certainty, which they strictly prefer because they are risk averse.

The second special sub-case is that with a single most aggressive firm and multiple second-most-aggressive firms. In this sub-case, there are no prices below $p^\dagger_{n-1}$ in the support of any agent. At least two agents mix down to $p^\dagger_{n-1}$ (else multiple firms would earn more than captive only profits, contradicting Lemma B5) and those firms earn their captive-only profit. In this sub-case, there exists an equilibrium in the subgame in which n earns an expected profit of $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n) > \lambda_n(\bar{p}_n - c_n)$. (This is implied the continuity of equilibria with generic parameters; formally, we can use Corollary 2 and its proof in Siegel (2009, p.83). See our Appendix C for the mapping between his setting and ours. These payoffs are "non-pathological" in our two-stage game, in the same sense as those in the single-stage game (as discussed in the main text following Proposition 1). We proceed similarly here and impose that these are the payoffs that result in the subgame in this special sub-case.) In addition, in any such equilibrium, firm n's profit is uncertain: it could only be certain if $p_n = p^\dagger_{n-1}$, which cannot be an equilibrium strategy for n (because any atom must be at $\bar{p}_n > p^\dagger_{n-1}$; Lemma B1). Understanding this, the manager of n can deviate to $\bar{p}_n = p^\dagger_{n-1}$ and ensure the same profit, but with certainty. Given manager n is risk averse, they strictly prefer this outcome.

We have shown that there is no equilibrium in which $\bar{p}_m > p^\dagger_i$ for some $i \neq m$.

(iii) The remaining possibility is that $\bar{p}_m = p^\dagger_i$ for some $i \neq m$. Necessarily, $m = n$ and no firm has any prices strictly below $p^\dagger_{n-1} = \bar{p}_n$ in the support of their equilibrium strategies in the $t = 2$ subgame. Ties cannot occur at $p^\dagger_{n-1} = \bar{p}_n$ (else one of those firms would shift mass slightly downward) and so n plays $p_n = \bar{p}_n = p^\dagger_{n-1}$ and so $p_i = \bar{p}_i$ for $i < n$ because agent i only sells to captives. This cannot be an equilibrium at $t = 1$ if any $\bar{p}_i < v$ because i would strictly prefer to deviate to $\bar{p}_i = v$. The only candidates for equilibrium at $t = 1$ thus involve $\bar{p}_i = v$ for $n - 1$ managers and $\bar{p}_m < v$ for one, which proves the first part of the proposition.

We now refine the set of equilibrium predictions to a unique candidate with ϵ-risk-averse managers and ϵ sufficiently small. Because $p^\dagger_i$ is defined relative to $\bar{p}_i$ the identity of firm m is not yet pinned down. Now consider $p^\dagger_i$ when $\bar{p}_i = v$ for all i, which for the remainder of this proof we denote $p^\dagger_i(v)$, and index firms as usual such that $p^\dagger_n(v) < \cdots < p^\dagger_1(v)$. We seek the identity of the low-priced firm, m. If $m < n$, then by case (iii) above, $\bar{p}_m = p^\dagger_n(v)$ in any equilibrium. This gives m a profit of $(\lambda_m + \lambda_S)(p^\dagger_n(v) - c_m)$. If $p^\dagger_n(v) < p^\dagger_m(v)$, this is less than m's captive only expected profits. For sufficiently small risk aversion (ϵ sufficiently small), m would strictly benefit by deviating to $\bar{p}_m = v$ such that an equilibrium of the single-stage game ($\bar{p}_i = v$ for all i) is played in $t = 2$. In any, m makes their captive-only profit in expectation and agent m which may involve mixed strategies by agent m in that $t = 2$ Nash equilibrium, and so for sufficiently small risk aversion manager m strictly prefers that outcome. If instead $p^\dagger_n(v) = p^\dagger_m(v)$ (such that m is one of the most aggressive firms), then m makes exactly its captive-only profit with certainty. Any deviation by manager m to a $\bar{p}_m > p^\dagger_n(v)$ yields at best the same expected profit and may not do so with certainty, so there is no strictly profitable deviation for m. We conclude that the profile $p_i = \bar{p}_i = v$ for $n-1$ firms and $p_i = \bar{p}_i = \min_j\{p^\dagger_j(v) \setminus p^\dagger_n(v)\}$ for one of the most aggressive firms, is the unique on-path equilibrium prediction. When $p^\dagger_n(v) < p^\dagger_{n-1}(v)$ such that there is a uniquely most aggressive firm, it is the unique prediction. □

Proof of Proposition 9. We consider what must be true in a subgame perfect equilibrium for any move order. First, no firm sets a dominated price, and so for all i, $p_i \in [p^\dagger_i, v]$.

Now consider the period, $t \in \{1, \ldots, T\}$, in which it is n's turn to act. Denote by $\underline{p}_{t-1}$ the lowest price set in the first $t-1$ periods (and let $\underline{p}_0 = v$). Denote by j the most aggressive firm yet to move (and let $p_j^\dagger = v$ if $t = T$).

Firm n is the most aggressive and no firm sets a strictly dominated price, so $\min\{\underline{p}_{t-1}, p_j^\dagger\} > p_n^\dagger$.

When n is called upon to play, n strictly prefers to undercut (match, given ties are broken in its favor) any price set so far ($\underline{p}_{t-1}$) than set v and sell only to their captives. Firm n also foresees that any price it sets, $p_n > p_j^\dagger$, will be undercut. As such, firm n sets $p_n = \min\{\underline{p}_{t-1}, p_j^\dagger\}$.

Foreseeing firm n's strategy, any firm that acts before n understands they will be undercut and will only sell to captives, so they set v. In turn, this reduces firm n's strategy to $p_n = p_j^\dagger$, and so any firms acting after n set a price equal to v. □

Proof of Proposition 10. We consider what must be true in any subgame perfect equilibrium. First, if a firm does not advertise so that it only sells to its captives, it sets v, and if firm i advertises, it sets $p_i \in [p_i^\dagger, v]$. (Other prices are strictly dominated.)

Some firm advertises at some point. If not, then any firm could advertise v at any point, sell to shoppers, and strictly increase profits. And the lowest advertised price is strictly below v, because otherwise there would be a profitable undercutting opportunity.

Take the last period, t, in which some firms advertise (let i be the most aggressive) and suppose $t > 1$. The lowest price advertised at $t-1$ must be higher than $p_i^\dagger$ (else i would not advertise in period t), and as firms are strictly asymmetric in aggression, strictly higher. If i is the only firm to advertise in period t, then i could profitably deviate by advertising a price in period $t-1$ above $p_i^\dagger$ but below the lowest price advertised then. Thus, it cannot be that only i advertises in period t. Next consider multiple firms advertising in period t. The subgame starting at t is exactly the single-stage game analysed in Section 1 with payoffs scaled appropriately for the number of remaining periods. By Proposition 1 we know that in any Nash equilibrium (of the subgame starting at t) i makes more per period than their captive-only profit (which they make when they do not advertise). They could strictly improve by advertising $p_i(p_i^\dagger, p_j^\dagger)$ in period $t-1$, where j is the second-most aggressive firm advertising at t. Therefore $t = 1$.

Suppose multiple firms are last to advertise (in period $t = 1$). The game is exactly the same as that analysed in Section 1 and so (from Proposition 1) $n-1$ earns its captive-only profit and that there is (continuous) mixing down to $p_{n-1}^\dagger$. Firm $n-1$ has a profitable deviation to not advertise in $t = 1$, wait until $t = 2$ and then slightly undercut the lowest price advertised in $t = 1$ (because the other firms mix continuously that lowest advertised price is guaranteed to be strictly greater than $p_{n-1}^\dagger$), which gets it its captive-only profit in period $t = 1$ followed by a strictly greater profit in every later period.

We are left with the case that a single firm advertises in period $t = 1$ and no firm advertises in any period $t > 1$. If firm $i < n$ advertised, they would set $p_i \geq p_i^\dagger > p_n^\dagger$, which n would profitably undercut. And so it must be that (only) firm n advertises in period $t = 1$. To prevent profitable undercuts from others, it must be that n advertises $p_n = p^\dagger_{n-1}$. This means firms $i < n$ set $p_i = v$ every period, and so it is immaterial whether they advertise.

It remains to consider whether n could deviate to advertise some higher price, in $(p^\dagger_{n-1}, v]$, in period $t = 1$. It would be undercut in period $t = 2$ and not sell to shoppers again. Because no other firm advertises at $t = 1$, the most profitable such deviation is to v, yielding a profit of

$$\hat{\pi}_n = (v - c_n)(\lambda_n + \lambda_S/T). \tag{A19}$$

Firm n's profit from the candidate equilibrium strategies is

$$\pi^*_n = (v - c_n)\lambda_n + (p^\dagger_{n-1} - p^\dagger_n)(\lambda_n + \lambda_S). \tag{A20}$$

And comparing the two we see:

$$\pi^*_n \geq \hat{\pi}_n \Leftrightarrow T \geq \frac{(v - c_n)\lambda_S}{(p^\dagger_{n-1} - p^\dagger_n)(\lambda_S + \lambda_n)}. \tag{A21}$$

Therefore if there are sufficiently many periods, any subgame perfect equilibrium features the strategies stated in the proposition. □

Proof of Proposition 11. We begin by re-writing the profit of firm i from eq. (15) as

$$\pi_i = \lambda_i V_i(z_i) + \lambda_S \max\left\{0, V_i(z_i) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right\}\right\} - z_i. \tag{A22}$$

As noted in the text, this is maximized by either high or low innovation choices $z_i \in \{z^L_i, z^H_i\}$ which satisfy the first-order conditions from eq. (17) and where $z^H_i > z^L_i$. In essence, the innovation game is a binary-action game where each firm chooses either high or low innovation.

Some firms will always choose low innovation. For example, any firm i where

$$\frac{V_i(z^H_i)}{\lambda_i + \lambda_S} \leq \max_{j\neq i}\left\{\frac{V_j(z^L_j)}{\lambda_j + \lambda_S}\right\} \tag{A23}$$

will not choose z^H_i because it would not be the most aggressive firm (even if every other firm $j \neq i$ choses low innovation) and so it would earn $\lambda_i V_i(z^H_i) - z^H_i < \lambda_i V_i(z^L_i) - z^L_i$. Hence we restrict attention to firms that do not satisfy the inequality of eq. (A23). Amongst this set we further restrict attention to those firms who would choose high innovation if all of their competitors were expected to choose low innovation. These are firms that satisfy

$$\lambda_i V_i(z^H_i) + \lambda_S \max\left\{0, V_i(z^H_i) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z^L_j)}{\lambda_j + \lambda_S}\right\}\right\} - z^H_i \geq$$
$$\lambda_i V_i(z^L_i) + \lambda_S \max\left\{0, V_i(z^L_i) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z^L_j)}{\lambda_j + \lambda_S}\right\}\right\} - z^L_i. \tag{A24}$$

For firms where the inequality of eq. (A23) fails (these a firms that are able to take the most aggressive position when by choosing z_i^H while $j \neq i$ choose z_j^L) the inequality eq. (A24) is

$$(\lambda_i + \lambda_S)\left[V_i(z_i^H) - \max_{j\neq i}\left\{\frac{\lambda_S V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right] - z_i^H \geq \lambda_i V_i(z_i^L) + \lambda_S \max\left\{0, V_i(z_i^L) - (\lambda_i + \lambda_S)\max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j + \lambda_S}\right\}\right\} - z_i^L. \quad \text{(A25)}$$

Consider the set of firms for which eq. (A24) holds. This set is non-empty. To see why, consider a firm $i \in \arg\max_{j\in\{1,\ldots,n\}} V_j(z_j^L)/(\lambda_j + \lambda_S)$. (This is a firm that would be the most aggressive (or jointly most aggressive) firm if all firms chose z_j^L.) For this firm eq. (A24) reduces to

$$(\lambda_i + \lambda_S)V_i(z_i^H) - z_i^H \geq (\lambda_i + \lambda_S)V_i(z_i^L) + \lambda_S V_i(z_i^L) - z_i^L, \quad \text{(A26)}$$

which holds strictly because z_i^H is the unique maximizer of $(\lambda_i + \lambda_S)V_i(z_i) - z_i$. Amongst the non-empty set of firms that satisfy eq. (A24), find a firm that maximizes $V_i(z_i^H)/(\lambda_i + \lambda_S)$.

We now label this firm as firm n. There is a pure-strategy Nash equilibrium of the innovation game in which $z_n = z_n^H$ and $z_j = z_j^L$ for $j \in \{1, \ldots, n-1\}$. Firm n satisfies the inequality of eq. (A24) and so does indeed wish to choose $z_n = z_n^H$. Any other firm $i \neq n$ that satisfies this inequality also satisfies $V_i(z_i^H)/(\lambda_i + \lambda_H) \leq V_n(z_n^H)/(\lambda_n + \lambda_H)$ and so does to achieve the position of the most aggressive firm by deviating to $z_i = z_i^L$. This proves claim (i).

Claim (ii) follows from the argument after eq. (16): the profit of firm i has an upward kink as the firm ties to be the most aggressive, and so cannot be the optimal choice of z_i.

Turning to claim (iii), we note that $\lim_{\lambda_S\downarrow 0} z_i^H = z_i^L$. Suppose that $n = \arg\max_{j\in\{1,\ldots,n\}} V_j(z_j^L)/\lambda_j$. Then for $i \neq n$ and λ_S sufficiently small we can guarantee that

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} < \frac{V_n(z_n^L)}{\lambda_n + \lambda_S}, \quad \text{(A27)}$$

and so firm n is the most aggressive firm.

Claim (iv) is straightforward: any firm can be firm n when they are symmetric. □

Proof of Proposition 12. By assumption firms have different audience sizes. Applying eq. (23), if λ_S is sufficiently small and if $\gamma > \frac{1}{2}$ then the limit on the right-hand side of eq. (23) is increasing in λ_i and so the firm that maximizes $V_i(z_i^L)/(\lambda_i + \lambda_S)$ is the firm with the largest captive audience. Applying claim (iii) of Proposition 11 yields claim (i) of Proposition 12. The same argument generates claim (ii) when $\gamma < \frac{1}{2}$. □

Appendix B. Extended Results

In this appendix we derive fully the properties of equilibria when firms face arbitrary initial-price limitations; we fully construct equilibria (which can involve more than two firms "dancing") for models of sales, and finally we illustrate pathological equilibria in knife-edge cases.

Equilibrium Properties of Simultaneous Pricing. We consider both the standard (and classic) single-stage model, and also the subgames of our two-stage model in which each firm i chooses $p_i \in [0, \bar{p}_i]$ for some $\bar{p}_i \in (c_i, v]$. (The single-stage model is obtained by setting $\bar{p}_i = v$ for all i.) We now document several (relatively standard) properties that must hold for any (Nash) equilibrium. We write $F_i(p) : [0, \bar{p}_i] \mapsto [0, 1]$ for the mixed strategy of firm i. As usual, by an atom we mean a price at which $F_i(p)$ discontinuously increases. For this appendix, we redefine firm i's lowest undominated price to be relative to i's initial price, $\bar{p}_i$, so that:

$$p_i^\dagger = \frac{\lambda_i \bar{p}_i + \lambda_S c_i}{\lambda_i + \lambda_S}. \tag{B1}$$

Equation (1) is recovered with $\bar{p}_i = v$. The firms are indexed by (B1) such that $p_n^\dagger \leq \cdots \leq p_1^\dagger$. We also let $\underline{p}_i$ denote the infimum of the support of prices played by firm i in equilbrium.

Lemma B1 (Atoms). *Any atom can only ever be placed at a firm's initial price $\bar{p}_i$. In a single-stage model there are no atoms strictly below v and at most $n-1$ atoms at v.*

Proof. A firm will choose a price that is strictly below its initial price only if that price can win shoppers. However, if an atom is placed at such a price then no other firm chooses that price or just above it; it would be better for them to undercut and capture the atom. This means that the atom-playing firm can safely raise its price locally (strictly gaining profit from captives) without losing any sales to shoppers. This contradiction proves the first claim. Turning to the second claim, a direct implication is that in a single stage-model any atoms must be played at v. If there were n atoms then at least one firm would undercut the others and so (by capturing the joint atom of the other $n-1$ firms) strictly increase expected profit; again a contradiction. □

Lemma B2 (Highest Prices). *The upper bound of the support of prices for firm i is either its initial price $\bar{p}_i$ or the lowest of all firms' initial prices $\min_{j\in\{1,\ldots,n\}}\{\bar{p}_j\}$. In a single-stage model the upper bound of the support of prices for all firms is v.*

Proof. We write $\tilde{p}_i$ for the upper bound of the support for a firm i. If $\tilde{p}_i < \min_{j\in\{1,\ldots,n\}}\{\bar{p}_j\}$ then no other $j \neq i$ would choose $p_j \in [\tilde{p}_i, \bar{p}_j)$, because this would sell only to captives and j would strictly prefer $p_j = \bar{p}_j$. This means that firm i can strictly gain from raising its price from $p_i = \tilde{p}_i$. Suppose instead that $\tilde{p}_i \in \left(\min_{j\in\{1,\ldots,n\}}\{\bar{p}_j\}, \bar{p}_i\right)$. Charging $\tilde{p}_i$ or just below it sells only to captives, because at least one competitor as an upper-bound initial price that is strictly lower. When selling only to captives firm i would do strictly better to charge $\bar{p}_i$. The second claim of the lemma is an immediate corollary of the first claim when $\bar{p}_i = v$ for all i. □

Lemma B3 (Gaps). *No firm plays a price $p_i \in (\bar{p}_m, \bar{p}_i)$, where $\bar{p}_m \equiv \min_{j\in\{1,\ldots,n\}}\{\bar{p}_j\}$, and so there are gaps in the joint support of firms' strategies above the lowest initial price. However, there is no gap in the joint support of firms' strategies below $\bar{p}_m$, and so in a single-stage model there is no gap in the joint support below v. Relatedly, if any interval of prices is in the support for some firm i then it is in the support for some other firm $j \neq i$.*

Proof. The first claim follows from the proof of Lemma B2. Turning to the second claim, suppose that there is such a gap. Expand the gap downward to its lower bound to find a price in the support of some firm i. Noting that there are no atoms within this range, firm i can strictly gain (it loses no sales) by shifting that price upward into the gap. Relatedly, suppose that a firm i prices within an interval that is not in the support of any other firm, so that it is in a gap in everyone else's support. Firm i could again raise that price without losing sales. □

Lemma B4 (Equilibrium with a Low Initial Price). *If $\bar{p}_m \equiv \min_{j\in\{1,\ldots,n\}}\{\bar{p}_j\} \leq p_i^\dagger$ for all $i \neq m$ then there is a unique Nash equilibrium in which $p_i = \bar{p}_i$ for all i.*

Proof. Note that $p_j^\dagger < \bar{p}_j$ for all j and so if $\bar{p}_m \leq p_i^\dagger$ then $\bar{p}_m < \bar{p}_i$ so that only firm m has the initial price $\bar{p}_m$. If $\bar{p}_m < p_i^\dagger$ for $i \neq m$, then any $p_i \leq \bar{p}_m$ is strictly dominated for i. Firm i only sells to captives in any equilibrium and so $p_i = \bar{p}_i$. This means $p_m = \bar{p}_m$. Now suppose $\bar{p}_m = p_i^\dagger$ for some i. Any $p_i < \bar{p}_m$ is strictly dominated. There are no prices played just above $\bar{p}_m$, by Lemma B3. The remaining possibility is that i places an atom at $\bar{p}_m = p_i^\dagger < \bar{p}_i$, but that contradicts Lemma B1. It must be that $p_i = \bar{p}_i$ for $i \neq m$, and so $p_m = \bar{p}_m$ in any equilibrium. Last, it is immediate that $p_i = \bar{p}_i$ for all i is an equilibrium. We conclude that if $\bar{p}_m \leq p_i^\dagger$, then the unique Nash equilibrium is $p_i = \bar{p}_i$ for all i. □

Lemma B5 (Captive-Only Profits). *At least $n-1$ of the firms earn their captive-only profit.*

Proof. If $\bar{p}_m \leq p_i^\dagger$ for $i \neq m$, then Lemma B4 shows i earns their captive-only payoff, $\lambda_i(\bar{p}_i - c_i)$.

If $\bar{p}_m > p_i^\dagger$ for $i \neq m$, there is no pure-strategy Nash equilibrium. We consider mixed equilibria. Any firm with $p_i^\dagger \geq \bar{p}_m$ must set $p_i = \bar{p}_i > \bar{p}_m$. It sells only to captives and earns captive-only profits. Any mixing occurs for firms i such that $p_i^\dagger < \bar{p}_m \leq \bar{p}_i$. We focus on these firms.

We know that the maximum of the support of a strategy is $\bar{p}_i > \bar{p}_m$ or $\bar{p}_m$, that no firm chooses a price $p_i \in (\bar{p}_m, \bar{p}_i)$, and that any atom is placed either at $\bar{p}_i$ or $\bar{p}_m$ (Lemmas B1 and B2).

Consider any firm that places at atom at $\bar{p}_i > \bar{p}_m$. This generates a firm's captive-only profit.

We now focus on firms that do not price above $\bar{p}_m$. This set includes firm m, and is so non-empty. Any firms outside this set (as just noted) earn captive-only profits. Hence the set of firms to be considered includes any firm that earns strictly more than its captive-only profit.

The set of firms could consist only of m. In this case, firm m is the only candidate for earning more than its captive-only profit, which in turn implies that $p_n^\dagger = p_m^\dagger$, so that firm m has (or shares) the lowest undominated price. Each $i \in \{1, \ldots, n-1\}$ earns captive-only profits.

Alternatively, there are multiple firms in this set. Suppose all place an atom at $\bar{p}_m$. Necessarily at least one of these firms wins the shoppers at this price. There is, therefore, an incentive for one of the firms to shift the mass of its atom to slightly lower prices. We conclude that at least one such firm does not play an atom at $\bar{p}_m$, and yet is willing to price arbitrarily close to $\bar{p}_m$.

We have a situation in which (at least) one of multiple firms mixes continuously up to $\bar{p}_m$ without playing an atom. Consider another firm i. If that firm plays an atom at $\bar{p}_m$, then that atom will lose to the identified non-atom-playing competitor and so will sell only to captives. This is strictly dominated if $\bar{p}_i > \bar{p}_m$. We conclude that any firm that does play an atom is either firm m or otherwise $\bar{p}_i = \bar{p}_m$. In either case, this firm earns its captive-only profit.

Any other firms mix up to (but do not place an atom at) $\bar{p}_m$. Suppose that more than one firm does so. Each firm is willing to price arbitrarily close to $\bar{p}_m$ and doing so almost always loses all shopper sales. This is strictly dominated if $\bar{p}_i > \bar{p}_m$. We we conclude that all such firms (if there are two or more) satisfy $\bar{p}_i = \bar{p}_m$ and earn captive-only profits.

The remaining situation is when exactly one firm mixes up to (but does not place an atom at) $\bar{p}_m$. All others place atoms at their initial prices, and earn their captive-only profits. $\square$

Lemma B6 (Profits). *Suppose that $\bar{p}_m \equiv \min_{j \in \{1,\ldots,n\}}\{\bar{p}_j\} > p_i^\dagger$ for some firm $i \neq m$, then:*

(i) Firm n earns at least $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n) \geq \lambda_n(\bar{p}_n - c_n)$; the inequality is strict if $p_n^\dagger < p_{n-1}^\dagger$.

(ii) If $p_n^\dagger < p_{n-1}^\dagger$ then firm $i < n$ earns its captive-only profit and places an atom at $\bar{p}_i$.

(iii) If $p_{n-1}^\dagger < p_{n-2}^\dagger$, then firm n earns exactly $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n)$.

Proof. (i) Firm n can guarantee at least the stated profit by setting $p_n = p_{n-1}^\dagger$ and so earning at least $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n) = \lambda_n(\bar{p}_n - c_n) + (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. By inspection, this strictly exceeds the captive-only profit, $\lambda_n(\bar{p}_n - c_n)$, if $p_{n-1}^\dagger > p_n^\dagger$.

(ii) By Lemma B5, $n-1$ firms earn captive-only profits. Given that n earns strictly more, this must apply to all $i \in \{1, \ldots, n-1\}$. If $\bar{p}_n > \bar{p}_m$ then n does not have $\bar{p}_n$ in its support (if it did, it would earn its captive-only profit). By Lemma B2, n has $\bar{p}_m$ at the top of its support (and, if $\bar{p}_n = \bar{p}_m$, then Lemma B2 immediately implies the same). For n to earn strictly more than captive-only profits and yet be willing to price arbitrarily close to $\bar{p}_m$, it must win the shoppers with probability bounded away from zero, which implies that each $i < n$ places an atom at $\bar{p}_i$.

(iii) If firm n were to earn strictly more than $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n)$, then $\underline{p}_n > p_{n-1}^\dagger$. Firm $n-1$ could then set a price $p_{n-1} \in (p_{n-1}^\dagger, p_{n-2}^\dagger)$ which would capture shoppers with certainty and earn strictly more than its captive-only profit; a contradiction of Lemma B5. $\square$

We have established several properties of any equilibrium. (Given that we break ties in favor of a lowest-cost firm, we can use Theorem 5 of Dasgupta and Maskin (1986) to establish existence.) We know the captive-only profits of $n-1$ firms (from Lemma B5) and that if firm n is unique then it earns strictly more (claim (i) of Lemma B6). We also know (if firm $n-1$ is uniquely defined) the exact (strictly positive) profit of firm n (claim (iii) of Lemma B6).

Indeed, firm n earning $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$ is the profit level implied by the characterisation of equilibrium strategies we give below, for any model parameters. That characterization also gives a unique equilibrium profile for generic parameter values.

This leaves open special cases with $p^\dagger_n = p^\dagger_{n-1}$ or $p^\dagger_{n-1} = p^\dagger_{n-2}$ or both. A continuum of other equilibria can exist, in which exactly one of $\{n, n-1, n-2\}$ does strictly better than the profits given by Proposition 1. These "pathological" payoffs are mentioned in the main text, and then set aside. In this appendix we also describe such (pathological) equilibria.

Equilibrium Strategies in the Single-Stage Model. The equilibrium construction we provide next delivers equilibria with non-pathological profits (as in Proposition 1) for any parameters. We focus on the classic single-stage model, which our two-stage game nests in its second stage with $\bar{p}_i = v$ for all i in the first stage. (The approach can be modified to deal with more general second-stage games.) It is helpful to use the following notation and terminology.

Definition (Required and Minimum Win Probabilities). *The* required win probability *$w_i(p)$ is the probability with which firm i must win the business of shoppers for it to earn its equilibrium profit from the price $p \in (0, v)$. Relatedly, the* minimum win probability *$\underline{w}_i(p)$ is the probability that gives the firm its captive-only profit $\lambda_i(v - c_i)$ from charging the price p.*

From Lemma B5, $n-1$ firms earn captive-only profits, and in this construction we look only at cases in which non-pathological payoffs arise in equilibrium, which means $i < n$ earn captive-only profits. As such, $w_i(p) = \underline{w}_i(p)$ for all $i < n$. For the remaining firm, $w_n(p) \geq \underline{w}_n(p)$.

We will express equilibrium mixtures in terms of required win probabilities: if p is in the support of firm i then it must capture the business of shoppers with probability $w_i(p)$. If p is not in the support, then it captures that business with probability weakly less than $w_i(p)$.

We have noted that firm i's required win probability is its minimum win probability which gives it the monopoly profit from exploiting its captive customers. This satisfies

$$\lambda_i(v - c_i) = (p - c_i)\left(\lambda_i + \lambda_S \underline{w}_i(p)\right) \quad \Rightarrow \quad \underline{w}_i(p) = \frac{\lambda_i(v-p)}{\lambda_S(p - c_i)}. \tag{B2}$$

This is decreasing in p and satisfies $\underline{w}_i(\bar{p}_i) = 0$.

We pause briefly to relate minimum win probabilities to the ordering of firms by aggressiveness. A firm's lowest undominated price $p^\dagger_i$ satisfies $\underline{w}_i(p^\dagger_i) = 1$, which solves to give (1) from the main text. We defined firm i to be (strictly) more aggressive than firm j if $p^\dagger_i < p^\dagger_j$, so firm i is willing to choose a lower price to capture shoppers. Equivalently, this holds if

$$\frac{v-c_i}{v-c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}. \tag{B3}$$

By construction $\underline{w}_j(p) > 1 \geq \underline{w}_i(p)$ for $p \in [p_i^\dagger, p_j^\dagger)$. However, a stronger aggression ranking entails $\underline{w}_j(p) > \underline{w}_i(p)$ for all $p \in [p_j^\dagger, v)$. This (partial ordering of firms) requires

$$\frac{\lambda_i}{p-c_i} < \frac{\lambda_j}{p-c_j}, \tag{B4}$$

which holds for all relevant p if and only if

$$\frac{v-c_i}{v-c_j} > \max\left\{\frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \frac{\lambda_i}{\lambda_j}\right\}. \tag{B5}$$

(We can use this to derive a weaker condition than that stated in Proposition 2 to establish the uniqueness of a "two to tango" equilibrium.) If this does not hold, so that

$$\frac{\lambda_i}{\lambda_j} \geq \frac{v-c_i}{v-c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \tag{B6}$$

then i is more aggressive ($p_i^\dagger < p_j^\dagger$) and so is more willing to charge a lower price, but for sufficiently high prices firm j has a lower minimum win probability, which means that j is relatively more enthusiastic about offering a higher price. In this situation there is a unique price $p_{ij}^\dagger$ at which $\underline{w}_i(p_{ij}^\dagger) = \underline{w}_j(p_{ij}^\dagger)$, and as p rises through this point $\bar{w}_j(p)$ crosses $\bar{w}_i(p)$ from above to below. We will use this property (which corresponds to an effective change in "aggression" ranking as price increases) when we fully characterize an equilibrium below.

We now consider the required win probability for firm n. It can earn more than its captive-only profit. We have already discussed conditions under which it earns $\pi_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - c_n) = \lambda_n(v - c_n) + \Delta_n$ where $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Its required win probability is

$$w_n(p) = \frac{\lambda_n(v-p) + \Delta_n}{\lambda_S(p-c_n)} = \underline{w}_n(p) + \frac{\Delta_n}{\lambda_S(p-c_n)}. \tag{B7}$$

We now characterize the distributions used in firms' mixed strategies.

Equilibrium Construction. In the context of any equilibrium, the mixed strategy $F_i(p)$ of firm i is continuously increasing and satisfies $F_i(p) < 1$ for $p < v$ (from Lemmas B1 and B2). We write $I(p) \subseteq \{1, \ldots, n\}$ for the firms that are on the "dance floor" at price $p \in (p_{n-1}^\dagger, v)$. These are firms where $F_i(p)$ is strictly increasing at that price.[31] If a firm is on the dance floor then at that price its expected profit must equal its equilibrium profit, or equivalently its probability of winning the shoppers must equal its required win probability $w_i(p)$. It wins the shoppers only if all other firms $j \neq i$ price above it, which happens with probability $\prod_{j\neq i}(1 - F_j(p))$. That is,

$$w_i(p) = \prod\nolimits_{j\neq i}(1 - F_j(p)) \quad \Leftrightarrow \quad 1 - F_i(p) = \frac{1 - F_S(p)}{w_i(p)}$$
$$\text{where} \quad 1 - F_S(p) \equiv \prod\nolimits_{j=1}^{n}(1 - F_j(p)), \tag{B8}$$

[31]By which we mean that $F_i(p_L) < F_i(p_H)$ for all p_H and p_L satisfying $p_H > p > p_L$.

where we obtained the second equality after multiplying and dividing by $1-F_i(p)$ which we are able to do given that $F_i(p)<1$ and so $1-F_i(p)>0$ for $p<v$. Here $F_S(p)$ is the distribution of the cheapest price and so the distribution of prices paid by the shoppers. Relatedly, if any new firm were to charge a price p then it would win the sales of shoppers with probability $1-F_S(p)$.

We can substitute the expression for $F_i(p)$ back into the expression for $F_S(p)$, and obtain

$$1-F_S(p)=\prod\nolimits_{j=1}^{n}(1-F_j(p))=\prod\nolimits_{i\in I(p)}\frac{1-F_S(p)}{w_i(p)}\prod\nolimits_{j\notin I(p)}(1-F_j(p))$$

$$\Leftrightarrow\quad 1-F_S(p)=\left(\frac{\prod_{j\in I(p)}w_j(p)}{\prod_{j\notin I(p)}(1-F_j(p))}\right)^{1/(|I(p)|-1)}$$

$$\Rightarrow\quad 1-F_i(p)=\frac{1}{w_i(p)}\left(\frac{\prod_{j\in I(p)}w_j(p)}{\prod_{j\notin I(p)}(1-F_j(p))}\right)^{1/(|I(p)|-1)},\tag{B9}$$

where $|I(p)|$ is the number of firms actively mixing (or "dancing") at price p. The term $\prod_{j\notin I(p)}(1-F_j(p))$ corresponds to firms who do not dance, and so is (locally) constant. For those firms $i\in I(p)$ who dance we need the solutions $F_i(p)$ to be valid distribution functions that are (given that firms actively mix) strictly increasing. The density $f_i(p)$ is

$$f_i(p)=(1-F_i(p))\left(\frac{w_i'(p)}{w_i(p)}-\frac{1}{|I(p)|-1}\sum_{j\in I(p)}\frac{w_j'(p)}{w_j(p)}\right),\tag{B10}$$

and this is strictly positive for all $i\in I(p)$ (as required) if and only if

$$(|I(p)|-1)\max_{i\in I(p)}\left\{\frac{-w_i'(p)}{w_i(p)}\right\}<\sum_{j\in I(p)}\frac{-w_j'(p)}{w_j(p)}.\tag{B11}$$

If $|I(p)|=2$ (so that there is a "tango" between two firms) then this is always satisfied. However, it can fail (and, as we show, it will fail for asymmetric firms) if $|I(p)|>2$.

We now construct an equilibrium. Such an equilibrium partitions the full "dance floor" $[p^\dagger_{n-1},v)$ into at most $n-1$ sub-intervals. In each sub-interval one firm $i\in\{1,\dots,n-1\}$ continuously mixes (or "dances") together with firm n, and then at the top of the sub-interval firm i shifts all remaining mass to v and is replaced by a substitute firm j mixing in the next sub-interval. Firm n mixes over the entire interval $[p^\dagger_{n-1},v)$ but (in essence) swaps dance partners at various points so that the "two to tango" property (Baye, Kovenock, and De Vries, 1992) holds within each sub-interval, but more than two firms can participate in randomized sales overall.

Suppose that the two most aggressive firms are distinct: $p^\dagger_n<p^\dagger_{n-1}<p^\dagger_{n-2}$. We have found exact equilibrium profits in this case (Lemma B6) and firms n and $n-1$ must mix down to $p^\dagger_{n-1}$ (if $p^\dagger_n=p^\dagger_{n-1}$ or $p^\dagger_{n-1}=p^\dagger_{n-2}$ we can also proceed with the profits implied by Lemma B6). We set $I(p)=\{n-1,n\}$ and so $|I(p)|=2$ for all $p\in[p^\dagger_{n-1},p^\dagger_{n-2})$, and use the solutions for the mixing distributions reported in eq. (B9), which simplify to $F_n(p)=1-w_{n-1}(p)$ and $F_{n-1}(p)=1-w_n(p)$ and where $F_S(p)=1-w_{n-1}(p)w_n(p)$. We note that

$$1-F_S(p)=w_{n-1}(p)w_n(p)<w_{n-1}(p)<1\le w_i(p)\tag{B12}$$

for all $p \in (p^\dagger_{n-1}, p^\dagger_{n-2}]$ and all $i \in \{1, \ldots, n-2\}$. This means that no other firm wishes to "join the dance floor" at a price $p^\dagger_{n-2}$ and just above. Thus we continue to apply the solutions here as p increases through $p^\dagger_{n-2}$. One possibility is that $1 - F_S(p) < w_i(p)$ for all $p \in [p^\dagger_{n-2}, v)$ and all $i \in \{1, \ldots, n-2\}$. If so then we have constructed a unique equilibrium in which firms $n-1$ and n "tango" over $[p^\dagger_{n-1}, v)$ while all other firms strictly prefer to maintain $p_i = v$. We note that the solutions reported here satisfy $\lim_{p\uparrow v} F_n(p) = 1$.

The other possibility is that we reach a price at which $1 - F_S(p) = w_i(p)$ for some firm $i \in \{1, \ldots, n-2\}$ so that firm i wishes to "step on to the dance floor" to join the tango. Without loss of generality, we label the firms so that it is firm $n-2$ that wishes to join the dance floor and we write $p^\ddagger_{n-2}$ for the (lowest) price at which this happens. For generic parameter choices, firm $n-2$ is uniquely defined and so our construction will be unique. If there is more than one firm that wishes to "join in" then we pick a firm for which $w_j(p)$ is falling most rapidly, so that $w'_{n-2}(p^\ddagger_{n-2}) \leq w'_j(p^\ddagger_{n-2})$ for any other firm $j \in \{1, \ldots, n-3\}$ where $w_j(p^\ddagger_{n-2}) = w_{n-2}(p^\ddagger_{n-2})$. There can be (non-generic) circumstances in which we have multiple choices available. One such situation is when two firms i and j are symmetric in the sense that $\lambda_i = \lambda_j$ and $c_i = c_j$, and in this circumstance our choice of firm that "steps in" is arbitrary; there are multiple equilibria in this case. For our chosen (generically unique) firm $n-2$,

$$w_{n-2}(p^\ddagger_{n-2}) = 1 - F_S(p^\ddagger_{n-2}) < w_{n-1}(p^\ddagger_{n-2}). \tag{B13}$$

This means that $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below within the interval $(p^\dagger_{n-1}, p^\ddagger_{n-2})$. Given that the minimum win probability functions can cross only once (as established earlier) we can conclude that $w_{n-1}(p) > w_{n-2}(p)$ for all $p \in (p^\ddagger_{n-2}, v)$. This means (as we will confirm) that once firm $n-2$ joins the dance floor, firm $n-1$ will strictly prefer to stay off it.

We continue the construction for prices above $p^\ddagger_{n-2}$. We set $F_{n-1}(p) = F_{n-1}(p^\ddagger_{n-2})$ for all $p \in [p^\ddagger_{n-2}, v)$ so that firm $n-1$ leaves the dance floor and places remaining mass at v. We then set $I(p) = \{n-2, n\}$ (and so we maintain $|I(p)| = 2$) for prices at and (at least locally) above $p^\ddagger_{n-2}$. These firms then mix according to eq. (B9) where these solutions satisfy

$$F_n(p) = 1 - \frac{w_{n-2}(p)}{w_n(p^\ddagger_{n-2})} \quad \text{and} \quad F_{n-2}(p) = 1 - \frac{w_n(p)}{w_n(p^\ddagger_{n-2})} \quad \Rightarrow$$

$$1 - F_S(p) = (1 - F_n(p))(1 - F_{n-1}(p^\ddagger_{n-2}))(1 - F_{n-2}(p)) = \frac{w_n(p) w_{n-2}(p)}{w_n(p^\ddagger_{n-2})}. \tag{B14}$$

We apply these solutions for prices rising above $p^\ddagger_{n-2}$ until a price (discussed below) at which we see another "partner swapping event." Before we do this, however, we perform two checks.

Firstly, we consider whether firm $n-2$ could join the dance floor to form a threesome rather than replacing firm $n-1$, so that $I(p) = \{n-2, n-1, n\}$ at $p^\ddagger_{n-2}$ and just above. Given that $|I(p)| = 3$, then inequality of eq. (B11) required for positive densities is

$$2 \min_{i \in \{n-2, n-1, n\}} \left\{ \frac{w'_i(p)}{w_i(p)} \right\} > \sum_{j \in \{n-2, n-1, n\}} \frac{w'_j(p)}{w_j(p)}. \tag{B15}$$

A necessary condition for this to hold is

$$\frac{w'_{n-2}(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} \geq \sum_{j\in\{n-1,n\}} \frac{w'_j(p^{\ddagger}_{n-2})}{w_j(p^{\ddagger}_{n-2})}. \tag{B16}$$

However, we know that $w_{n-2} > 1 - F_S(p) = w_n(p)w_{n-1}(p)$ for $p < p^{\ddagger}_{n-2}$ but with equality at $p = p^{\ddagger}_{n-2}$, and so $w_{n-2}(p) - w_n(p)w_{n-1}(p)$ is decreasing at $p^{\ddagger}_{n-2}$. That is

$$w'_{n-2}(p^{\ddagger}_{n-2}) < w_n(p^{\ddagger}_{n-2})w'_{n-1}(p^{\ddagger}_{n-2}) + w'_n(p^{\ddagger}_{n-2})w_{n-1}(p^{\ddagger}_{n-2}). \tag{B17}$$

Dividing through by $w_{n-2}(p^{\ddagger}_{n-2}) = w_{n-1}(p^{\ddagger}_{n-2})w_n(p^{\ddagger}_{n-2})$, this inequality is

$$\frac{w'_{n-2}(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} < \frac{w'_{n-1}(p^{\ddagger}_{n-2})}{w_{n-1}(p^{\ddagger}_{n-2})} + \frac{w'_n(p^{\ddagger}_{n-2})}{w_n(p^{\ddagger}_{n-2})}, \tag{B18}$$

a contradiction. This means that we cannot have firm $n-1$ remaining on the dance floor.

Secondly, we need to check that firm $n-1$ does not wish to return to the dance floor:

$$w_{n-1}(p) \geq (1 - F_n(p))(1 - F_{n-2}(p)) = \frac{w_{n-2}(p)w_n(p)}{(w_n(p^{\ddagger}_{n-2}))^2} \tag{B19}$$

This holds as an equality at $p^{\ddagger}_{n-2}$. It holds strictly for all higher p if, taking derivatives,

$$\frac{w'_{n-1}(p)}{w_{n-1}(p)} > \frac{w'_{n-2}(p)}{w_{n-2}(p)} + \frac{w'_n(p)}{w_n(p)}, \tag{B20}$$

and given that $w'_n(p) < 0$ a sufficient condition for this to hold is

$$\frac{w'_{n-1}(p)}{w_{n-1}(p)} \geq \frac{w'_{n-2}(p)}{w_{n-2}(p)} \quad \Leftrightarrow \quad c_{n-1} \leq c_{n-2}, \tag{B21}$$

which follows from differentiation of the expression for the minimum win probability. This holds because the left-hand inequality held at some price $p < p^{\ddagger}_{n-2}$. We know this because $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below as it descended at some point within $(p^{\dagger}_{n-1}, p^{\ddagger}_{n-2})$, and at the point where it crossed $w_{n-2}(p) = w_{n-1}(p)$. As an aside, this also tells us that a firm that joins the dance floor is always a firm with a higher marginal cost, and so (given that it then has a lower minimum win probability) a smaller captive audience.

Having completed these checks, we maintain our new solutions for the mixing of firms n and $n-2$. Just as before, a possibility is that these solutions satisfy $1 - F_S(p) < w_i(p)$ for all $p \in [p^{\ddagger}_{n-2}, v)$ and all $i \in \{1, \ldots, n-3\}$ and if so that we have constructed an equilibrium. Otherwise, we find a price at which another firm for which we choose the label $n-3$ wishes to step in at price $p^{\ddagger}_{n-3} \in (p^{\ddagger}_{n-2}, v)$ where $1 - F_S(p^{\ddagger}_{n-3}) = w_{n-3}(p^{\ddagger}_{n-3})$. We execute another partner swap so that firm $n-3$ replaces firm $n-2$, and firm $n-2$ shifts all remaining mass to $p_{n-2} = v$.

This construction continues iteratively until we reach the upper bound v.

We note that for generic parameter values (by which we mean that no two firms wish to join the dance floor at the same price) this construction is unique. For other knife-edge cases (including, for example, $p^{\dagger}_n = p^{\dagger}_{n-1}$ or $p^{\dagger}_{n-1} = p^{\dagger}_{n-2}$) the construction also works, but there can be multiple equilibria. We summarize the construction and its implications in the following lemma.

Summary of the Equilibrium Construction for Generic Parameters. In summary, there is a Nash equilibrium of the single-stage game in which the profit of firm $i \in \{1, \ldots, n\}$ is

$$\pi_i = \lambda_i(v - c_i) + \begin{cases} (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n) & \text{if } i = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases} \tag{B22}$$

This means $w_i = \underline{w}_i$ as given by (B2) for $i < n$, while w_n is given by (B7). In this equilibrium:

- Firm n plays a mixed strategy with support $[p^\dagger_{n-1}, v]$.
- Firm $n-1$ plays a mixed strategy with support $[p^\dagger_{n-1}, p^\ddagger_j]$ with $p^\ddagger_j \in (p^\dagger_{n-1}, v]$ where $p^\ddagger_j$ is the lowest such price that solves $w_j(p) = 1 - F_S(p) = w_{n-1}(p)w_n(p)$ for any $j \le n-2$.
- No firm $i \le n-2$ has $[p^\dagger_{n-1}, p^\ddagger_j)$ in their support.
- If $p^\ddagger_j = v$ then each $i \le n-2$ plays the pure strategy $p_i = v$.
- If $p^\ddagger_j < v$ then j (effectively takes over the role of $n-1$ and) plays a mixed strategy with support $[p^\ddagger_j, p^\ddagger_k]$ with $p^\ddagger_k \in (p^\ddagger_j, v]$ where $p^\ddagger_k$ is the lowest such price that solves $w_k(p) = 1 - F_S(p) = w_j(p)w_n(p)/(w_n(p^\ddagger_j)^2)$ for any $k \notin \{n, n-1, j\}$.
- If $p^\ddagger_k = v$ then each $i \notin \{n, n-1, j, k\}$ plays the pure strategy $p_i = v$.
- If $p^\ddagger_k < v$ then k (effectively takes over the role of j and) the construction continues exactly as it did above for j with $p^\ddagger_j < v$.

The procedure above ends when either (i) all firms have been assigned mixed strategies, or (ii) we find that for each firm yet to be assigned a mixed strategy, l, there is no $p < v$ such that $w_l(p) = 1 - F_S(p)$. In case (ii), each of those remaining firms, l, plays the pure strategy $p_l = v$.

Expressions for the CDFs of prices for firms that play mixed strategies are recovered from (B9).

This equilibrium is unique for (generic) parameters which satisfy: (i) $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$; and (ii) for any two values $p^\ddagger_i, p^\ddagger_j < v$ encountered during the iterative procedure, $p^\ddagger_i \ne p^\ddagger_j$.

Our algorithm constructs an equilibrium (for all parameter values) in which $n-1$ firms earn their captive-only profits, while firm n earns exactly $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$ more than its captive-only profit. Moreover, the equilibrium is unique for generic parameter choices, and the equilibrium profits are uniquely defined when $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.

For the remaining knife-edge cases when either $p^\dagger_n = p^\dagger_{n-1}$ or $p^\dagger_{n-1} = p^\dagger_{n-2}$ or possibly both we have the existence of an equilibrium with the stated profits. However, there could also be an equilibrium in which (for example) firm n earns strictly more than the stated profit. We deemed such equilibria as "pathological" in the main text and excluded them from analysis.

However, we address these cases, along with other matters, in our online material, Appendix C.

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Appendix C. Further Discussions and Extensions

In this appendix we cover the knife-edge cases not covered by our analysis in Appendix B and provide a pathological equilibrium; relate our single-stage model to the work on contests by Siegel (2010); extend our model to a situation in which firms must advertise in order to reach shoppers; and we discuss the recent "advertised sales" model of Shelegia and Wilson (2021).

Knife-Edge Cases. Suppose that $p_n^\dagger < p_{n-1}^\dagger = p_{n-2}^\dagger$ so that the second-most-aggressive firm is not uniquely defined, and further suppose (for the simplicity of exposition) that all other firms $i \in \{1, \ldots, n-3\}$ choose $p_i = v$, leaving an effective three-player game between firms $i \in \{n-2, n-1, n\}$. We know that all firms other than n earn their captive-only profits, and that firm n earns Δ_n more than its captive-only profit where $\Delta_n \geq (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Suppose that this holds as a strict inequality. This implies that $\underline{p}_n > p_{n-1}^\dagger$, and so $w_n(p_{n-1}^\dagger) > 1$. Firms $n-1$ and $n-2$ must mix together down to $p_{n-1}^\dagger$, do so according to distributions $1 - F_{n-1}(p) = w_{n-2}(p)$ and $1 - F_{n-2}(p) = w_{n-1}(p)$, and so $F_S(p) = 1 - w_{n-1}(p)w_{n-2}(p)$. We know that $1 - F_S(p_{n-1}^\dagger) = 1 < w_n(p_{n-1}^\dagger)$, and that $1 - F_S(v) = 0 < w_n(v)$. Moreover, the second (strict) inequality holds even if $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. This means that $w_n(p)$ lies (strictly) above $1 - F_S(p) = w_{n-1}(p)w_{n-2}(p)$ at both the beginning and the end of the interval $[p_{n-1}^\dagger, v]$. We also know that firm n must join the dance floor at some point. This implies that there exists some $p^\ddagger$ where $w_n(p^\ddagger) = 1 - F_S(p^\ddagger) = w_{n-1}(p^\ddagger)w_{n-2}(p^\ddagger)$, which implies that $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below and then subsequently crosses from below to above within the interval $[p_{n-1}^\dagger, v]$. It is also true that $w_n(p)$ crosses $w_i(p)$ in this way for each $i \in \{n-2, n-1\}$. To proceed, we note that $w_n(p)$ is below below $w_i(p)$ whenever

$$\frac{\lambda_n(v-p) + \Delta_n}{\lambda_S(p - c_n)} \leq \frac{\lambda_i(v-p)}{\lambda_S(p - c_i)} \quad \Leftrightarrow \quad \lambda_n + \frac{\Delta_n}{v-p} \leq \lambda_i \frac{p - c_n}{p - c_i}. \tag{C1}$$

The left-hand side is increasing in p. The right-hand side is decreasing in p if $c_n < c_i$. This means that $w_n(p)$ crosses $w_i(p)$ at most once from below to above. This is a contradiction. From this we conclude that if firm n has lower costs than others then there cannot be an equilibrium in which $\Delta_n > (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Equivalently, we have unique equilibrium profits. An equilibrium with $\underline{p}_n > p_{n-1}^\dagger$ must entail $c_n > c_i$, or in this case $c_n > \max\{c_{n-1}, c_{n-2})$. Firm n is the most aggressive firm, and so necessarily this also implies that $\lambda_n < \min\{\lambda_{n-1}, \lambda_{n-2}\}$.

Working with such a configuration (so that firm n has high costs but few captives) let us begin by specifying $w_n(p)$ such that $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. We have already constructed a non-pathological equilibrium for this case. To construct a pathological equilibrium, we need

$$w_n'(p_{n-1}^\dagger) < w_{n-1}'(p_{n-1}^\dagger) + w_{n-2}'(p_{n-1}^\dagger). \tag{C2}$$

Noticing that $w_{n-2}(p_{n-1}^\dagger) = w_{n-2}(p_{n-1}^\dagger) = 1$ for this case, this says that $w_n(p)$ declines more quickly than $w_{n-1}(p)w_{n-2}(p)$ when evaluated at $p_{n-1}^\dagger$. (This inequality also stops the construction of an equilibrium in which all three firms $\{n-2, n-1, n\}$ mix as a threesome.) This means that we can raise Δ_n, so that $w_n(p_{n-1}^\dagger) > 1$, but still guarantee (so long as we don't increase Δ_n too much) that there is some larger $p^\ddagger > p_{n-1}^\dagger$ at which $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below. We then construct an equilibrium by allowing $n-1$ and $n-2$ to "dance" until $p^\ddagger$ when n joins for a "partner swap" at $\underline{p}_n = p^\ddagger$. We next illustrate with a specific example.

Construction of Pathological Equilibria. Consider a triopoly (so setting $n = 3$) in which two pairwise-symmetric firms have low costs but many captives, whereas the third firm has high cost and few captives. Costs satisfy $c_1 = c_2 = 0$ and $c_3 = c > 0$ while the sizes of captive audiences satisfy $\lambda_1 = \lambda_2 = \lambda_H$ and $\lambda_3 = \lambda_L$ where $\lambda_H > \lambda_L$.

We choose parameters so that firms share the same lowest undominated price $p_1^\dagger = p_2^\dagger = p_3^\dagger = p^\dagger$:

$$p^\dagger = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \frac{\lambda_L v + \lambda_S c}{\lambda_L + \lambda_S} = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \quad \Leftrightarrow \quad c = \frac{(\lambda_H - \lambda_L)v}{\lambda_H + \lambda_S}. \tag{C3}$$

Henceforth when we vary the λ parameters we adjust c so that it satisfies this equation.

For this example the non-pathological equilibrium profits are captive-only for all three firms. For such profits the required win probabilities are the minimum win probabilities. They are:

$$w_1(p) = w_2(p) = \frac{\lambda_H(v-p)}{\lambda_S p} \quad \text{and} \quad w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H - \lambda_L)v}{\lambda_H + \lambda_S}\right)} \tag{C4}$$

where of course these satisfy $w_i(p^\dagger) = 1$ for all i. Also

$$w_1'(p^\dagger) = w_2'(p^\dagger) = -\frac{(\lambda_H + \lambda_S)^2}{\lambda_H \lambda_S v} \quad \text{and} \quad w_3'(p^\dagger) = -\frac{(\lambda_H + \lambda_S)(\lambda_L + \lambda_S)}{\lambda_L \lambda_S v}. \tag{C5}$$

The minimum win probability functions intersect (by construction) at $p^\dagger$. However, that function for the third firm (which has higher costs but fewer captives) declines more quickly:

$$w_3'(p^\dagger) < w_i'(p^\dagger) \text{ for } i \in \{1,2\} \ \Leftrightarrow\ -\frac{(\lambda_H + \lambda_S)(\lambda_L + \lambda_S)}{\lambda_L \lambda_S v} < -\frac{(\lambda_H + \lambda_S)^2}{\lambda_H \lambda_S v} \ \Leftrightarrow\ \lambda_L < \lambda_H. \tag{C6}$$

They key requirement to construct an equilibrium with pathological profits is that $w_3(p)$ declines more quickly than $w_1(p)w_2(p)$ when evaluated at $p^\dagger$. In this case,

$$w_3'(p^\dagger) < w_1'(p^\dagger) + w_2'(p^\dagger) \ \Leftrightarrow\ \lambda_L < \frac{\lambda_H \lambda_S}{\lambda_H + 2\lambda_S}. \tag{C7}$$

We construct an equilibrium in which firm 3 earns Δ above its captive-only profit, by setting

$$w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H - \lambda_L)v}{\lambda_H + \lambda_S}\right)} + \frac{\Delta}{\lambda_S(p-c)}, \tag{C8}$$

which for $\Delta > 0$ now satisfies $w_3(p^\dagger) > 1 = w_1(p^\dagger) = w_2(p^\dagger)$. This means firm 3 does not wish "to dance" at $p^\dagger$. Instead, we construct an equilibrium in which firms 1 and 2 mix continuously over $[p^\dagger, p^\ddagger]$ according to $1 - F_1(p) = w_2(p)$ and $1 - F_2(p) = w_1(p)$ which (from pairwise symmetry) reduces to $1 - F_i(p) = w_i(p)$ for $i \in \{1,2\}$ and $1 - F_S(p) = (w_i(p))^2$. By construction $w_3(p) > 1 - F_S(p)$ for prices rising above $p^\dagger$. The key threshold is then the price $p^\ddagger$ which satisfies $w_3(p^\ddagger) = 1 - F_S(p^\ddagger) = w_1(p^\ddagger)w_2(p^\ddagger)$. Explicitly, $p^\ddagger$ (which does exist so long as $\Delta > 0$ is not chosen to be too large) satisfies

$$\frac{\lambda_L(v-p^\ddagger)+\Delta}{\lambda_S\left(p^\ddagger-\frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} = \left(\frac{\lambda_H(v-p^\ddagger)}{\lambda_S p^\ddagger}\right)^2. \tag{C9}$$

At $p^\ddagger$ there is a partner swap. Firm 2 (for example; this could be firm 3) shifts all further mass to v, which is an atom of size $w_1(p^\ddagger)$. Firms 1 and 3 then mix over the interval $[p^\ddagger, v)$ where

$$F_1(p) = 1 - \frac{w_3(p)}{w_1(p^\ddagger)} \quad \text{and} \quad F_3(p) = 1 - \frac{w_1(p)}{w_1(p^\ddagger)}. \tag{C10}$$

These firms earn their claimed equilibrium profits over this interval. The solutions satisfy $F_3(v) = 1$ (so that firm 3 does not play an atom) but $\lim_{p\uparrow v} F_1(p) < 1$ (so that firm 1 does play an atom). We need only check that 2 does not wish to rejoin the dance floor within this interval. We note that the probability that firm 2 wins the shoppers if it were to join is

$$(1-F_1(p))(1-F_3(p)) = \frac{w_3(p)}{w_1(p^\ddagger)}\frac{w_1(p)}{w_1(p^\ddagger)} < \frac{w_3(p^\ddagger)}{(w_1(p^\ddagger)^2}w_1(p) = w_1(p). \tag{C11}$$

Summarizing, we have constructed a (pathological) equilibrium.

Relation to the Siegel (2010) Model of Contests. Siegel (2010) studied a contest in which n players compete for $m \in \{1,\dots,n-1\}$ (homogeneous) prizes in a single-stage game by each simultaneously choosing a "score," $s_i \geq 0$. Each of the m players with the highest scores wins one prize. For a given vector of scores $\mathbf{s} = (s_1,\dots,s_n)$, player i's payoff is:

$$u_i(\mathbf{s}) = P_i(\mathbf{s})v_i(s_i) - (1-P_i(\mathbf{s}))c_i(s_i), \tag{C12}$$

where $v_i(s_i)$ is i's valuation for winning, $c_i(s_i)$ is their cost of losing such that $c_i(0) = 0$, and $P_i(\mathbf{s})$ is their probability of winning a prize (with ties broken arbitrarily).

We now re-write the expected profit of a firm in a single-stage model of sales to show the mapping between the settings. In a model of sales there are $n \geq 2$ firms. The business of shoppers is the $m = 1$ prize, which all firms compete for.[32] Consider some profile of prices $\mathbf{p} = (p_1,\dots,p_n)$. Instead of highest scores winning, lowest prices win. No firm would choose a price $p_i > v$ and so a score of zero is equivalent to a price of v. To construct the analog of (C12), note that firm i's "cost of losing" when it sets v should be zero, i.e., $c_i(v) = 0$. When $p_i = v$ and it loses, it makes its captive-only profit, $(v-c_i)\lambda_i$, and winning and losing are relative to that quantity. When firm i wins, it gets profit $(p_i-c_i)(\lambda_i+\lambda_S)$, which is $\lambda_S(p_i-c_i)-\lambda_i(v-p_i)$ more than its captive-only profit and so constitutes the valuation for winning. When firm i loses it makes $(p_i-c_i)\lambda_i$, which is a loss of $(v-p_i)\lambda_i$ relative to $(v-c_i)\lambda_i$. In sum,

$$\pi_i(\mathbf{p}) = \underbrace{(v-c_i)\lambda_i}_{\text{normalization}} + P_i(\mathbf{p})\underbrace{(\lambda_S(p_i-c_i)-\lambda_i(v-p_i))}_{\text{analog of } v_i(s_i)} - (1-P_i(\mathbf{p}))\underbrace{(v-p_i)\lambda_i}_{\text{analog of } c_i(s_i)}. \tag{C13}$$

It is straightforward to confirm that Assumptions B1-B2 of Siegel (2010) are satisfied. Siegel indexed players by their "reach," which is the score at which the valuation for winning is zero,

[32] Notice that all firms vie for the prize. If there were consumers other than captives and shoppers (i.e., with a consideration set that is not a singleton or the set of all firms), the connection between the models would break.

so that $r_i = v^{-1}(0)$. In our analysis, this corresponds to a firm's aggression, which is determined by $p_i^\dagger$ in (1). Assumption B3 is equivalent to assuming $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ in our analysis.

The model of Siegel (2010) is a special case of that in Siegel (2009).[33] Among other differences, Siegel (2009) assumed that each player i chooses a score $s_i \geq a_i \geq 0$. A player's "initial score" a_i is analogous to an (exogenously chosen) initial price, $\bar{p}_i$, in our two-stage model of Section 2. As we noted in the main text, these papers do not provide a full treatment for a model of sales: our Proposition 1 is covered by Siegel (2009), but our other results are not. Siegel (2010) derived equilibrium strategies for contests with m (homogeneous) prizes and $m+1$ players, and so covers models of sales only in the case of duopoly.

Costly Advertising. Below we discuss the model of Shelegia and Wilson (2021) in which firms pay to advertise prices to shoppers. We first extend our model to have such a feature.

We assume that each firm i either does not advertise and earns $\lambda_i(v - c_i)$ from its captive customers, or otherwise advertises a price $p_i \in [0, v]$ at a cost $a_i > 0$. Shoppers buy from a firm advertising the lowest price; if no firm advertises then they do not buy at all. We write $F_i(p)$ for the distribution of prices chosen by firm i across $p \in [0, v]$ so that $1 - F_i(v)$ is the probability that firm i does not advertise. The assumption that shoppers are unable to buy from an unadvertised firm contrasts with the model of Shelegia and Wilson (2021): they assumed that shoppers treat firms that do not advertise and those that advertise $p_i = v$ similarly (in effect, not advertising is treated as equivalent to advertising the price $p_i = v$) but also specified a restriction how ties between such "$p_i = v$" firms must be broken.

Clearly, if $a_i > \lambda_S(v-c_i)$ then firm i will never advertise, and so we suppose that $a_i \leq \lambda_S(v-c_i)$ for all i. We can the define for firm i the minimum undominated (advertised) price $p_i^\dagger$ satisfying

$$(p_i^\dagger - c_i)(\lambda_i + \lambda_S) = \lambda_i(v - c_i) + a_i \quad \Leftrightarrow \quad p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i + a_i}{\lambda_i + \lambda_S}, \tag{C14}$$

and we can label firms appropriately, as in our main analysis, in terms of aggression. For the purposes of this extension, we strictly rank the most aggressive firms so that $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. Notice that a firm's "aggression" now depends on its cost a_i of advertising to shoppers as well as its marginal cost of production and the size of its captive audience.

For each price p we define (as in our Appendix B) the critical minimum win probability for i,

$$\underline{w}_i(p) = \frac{\lambda_i(v - p) + a_i}{\lambda_S(p - c_i)}, \tag{C15}$$

where by construction these functions satisfy $\underline{w}_i(p_i^\dagger) = 1$ and $\underline{w}_i(v) > 0$ for all firms.

In this costly-advertising environment, many properties from our main analysis hold true.

[33] The details are described clearly by Siegel (2010, Footnote 8). Other related papers (Siegel, 2012, 2014) differ because the contest prize does not depend on a player's "score" choice (here, equivalent to the advertised price).

Lemma C1 (Equilibrium Properties with Costly Advertising). *(i) Any atom in a firm's mixed strategy is placed at v or at the "no advertising" decision. (ii) The upper bound of the support for a firm that always advertises is v. (iii) There are no gaps in the joint support of firms' strategies below v. (iv) At most one firm places an atom at the advertised price v. (v) At least $n-1$ of the firms earn their captive-only profit obtained from not advertising.*

Proof. Claims (i) to (iii) follow from the arguments used in the proofs of Lemmas B1 to B3. Claim (iv) also follows from standard arguments: firms only advertise a price v if that can win the business of shoppers, and if two or more do so with positive probability that at least one of those firms has the incentive to undercut the atom played by the others.

For claim (v), we note that any firm that earns strictly more than its captive-only profit must always advertise a price. (Choosing not to advertise gives a firm its captive-only profit.) If two (or more) firms earn strictly more than their captive-only profits then, from claim (ii), they use a support extending up to v. At least one of those firms does not play atom at v. This means other such firms know that pricing at or close to v results in arbitrarily few sales to shoppers, and so their profit is arbitrarily close to the captive-only profit. This is a contradiction. □

With the following result, we next recover the key equilibrium profit prediction of Proposition 1.

Proposition C1 (Profits with Costly Advertising). *If $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$, so that the second-most-aggressive firm is identified, then in equilibrium every firm $i \in \{1, \ldots, n-1\}$ earns its captive-only profit while firm n earns $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ more than its captive-only profit.*

Proof. Firm n can guarantee selling to all shoppers and achieving $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ more than its captive-only profit by choosing $p_n = p_{n-1}^\dagger$. From claim (v) of Lemma C1 all other firms earn captive-only profits. If firm n were to earn strictly more than this, then the lower bound of its support would strictly exceed $p_{n-1}^\dagger$. This would give firm $n-1$ an opportunity to earn strictly more than its own captive-only profit; a contradiction. □

It remains for us to construct such an equilibrium. We set the required win probability for each firm $i \in \{1, \ldots, n-1\}$ to equal its minimum win probability: $w_i(p) = \underline{w}_i(p)$. We set the required win probability for firm n to reflect its additional expected profit of $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. These win probabilities satisfy $w_n(p_{n-1}^\dagger) = w_{n-1}(p_{n-1}^\dagger) < w_i(p_{n-1}^\dagger)$ for $i \in \{1, \ldots, n-2\}$.

We now follow the same procedure as in Appendix B: firms n and $n-1$ continuously mix from $p_{n-1}^\dagger$ upwards using distribution functions $1 - F_n(p) = w_{n-1}(p)$ and $1 - F_{n-1}(p) = w_n(p)$, so that the distribution $F_S(p)$ of the cheapest price satisfies $1 - F_S(p) = w_n(p)w_{n-1}(p)$. One possibility (and the leading case) is that this solution satisfies $w_n(p)w_{n-1}(p) < w_i(p)$ for all $i \notin \{n-1, n\}$ and $p \in [p_{n-1}^\dagger, v)$. If so, then we allow firms $n-1$ and n to mix over the whole interval. Firm n then places remaining mass at v, while firm $n-1$ places remaining mass on the act of not advertising. All other firms refrain from advertising a price. This is an equilibrium.

The other possibility (just as in Appendix B) is that there is some price $p^\ddagger < 1$ at which some other firm j satisfies $w_j(p^\ddagger) = w_n(p^\ddagger)w_{n-1}(p^\ddagger)$. If so, then we execute a "partner swap" at this price, just as have done in our earlier constructions. It is straightforward to confirm (again, as we did in Appendix B) that there are situations in which such partner-swapping necessarily occurs, and the equilibria in such circumstances involve active mixing by more than two firms. Generically, we only see two firms dancing within any interval of prices.

Relation to the Shelegia and Wilson (2021) model of Advertised Sales. The specification of Shelegia and Wilson (2021) differs from our core model in three ways: (i) firms make general utility offers; (ii) a firm must pay to advertise a price below v to shoppers; and (iii) there are restrictive tie-break rules. The first two features represent potentially significant and valuable generalizations of the classic model of the sales.

A utility offer (in the sense of Armstrong and Vickers, 2001) by firm i gives consumer surplus u_i to a customer and profit $\pi_i(u_i)$ to the supplying firm. In our unit-demand specification, $u_i = v - p_i$ and so the price is recovered by $p_i = v - u_i$, while the associated profit $\pi_i(u_i) = p_i - c_i = v - u_i - c_i$ is linear in u_i. A utility-offer specification allows more generality than this; for example if a customer has downward-sloping demand then the $\pi_i(u_i)$ becomes non-linear. There is a monopoly utility offer u_i^m that maximizes this, which in our setting corresponds to $u_i^m = 0$, with associated profit $\pi_i(u_i^m) = v - c_i$. Shelegia and Wilson (2021, p. 202) made an "Assumption U" which says that u_i^m is constant across firms. This is true here (for us). However, it fails when there is downward-sloping demand and different marginal costs: the consumer surplus u_i^m received by a customer under the monopoly price typically varies with marginal cost, given that the monopoly price does so. A leading application of the "utility offers" approach is when there is multiple-unit demand; but this is effectively ruled out.

A welcome strength of Shelegia and Wilson (2021) is that they made "Assumption U" explicit. However, for situations with asymmetric marginal cost, and given uniform pricing, the effective restriction is to a unit-demand situation. We do that for the remainder of this discussion, so that the two remaining differences between our core model and that of Shelegia and Wilson (2021) are the positive advertising cost and the specific tie-break rules.

The second key feature (and an important and very welcome generalization) of Shelegia and Wilson (2021) is that a firm i must pay a (fixed) strictly positive advertising cost $a_i > 0$ (this is A_i in their notation) to communicate its price to shoppers. By assumption, captive customers must buy from their captors, and if a shopper chooses to visit a non-advertising firm then that shopper becomes captive to that firm. This means that any non-advertised price will be equal to v. Shoppers seek out the lowest price. If there is a unique lowest advertised price $p_i < v$, then shoppers buy from that firm i. However, this leaves open three questions. Firstly, what happens if there is tie below v for the cheapest advertising firms, so that $p_i = p_j < v$? Secondly, what happens if the cheapest advertising firm charges $p_i = v$, and so is tied with the expected price of a non-advertising firm? Thirdly, what happens if no firm advertises?

The first question is unimportant, simply because the tie-break rule in such cases is never applied: it is straightforward to confirm that no atoms are played below v, and so such ties do not occur with positive probability. However, the other two questions are crucially important.

The answer to the second question follows from "Assumption X" of Shelegia and Wilson (2021). This says that any tie-break rule depends only on the identities of the firms and not on whether one (or more) of the tied prices is advertised. Crucially, this means that there is no advantage to advertising a price $p_i = v$ which means that no firm would do so. (This differs significantly from our "costly advertising" extension described earlier in this appendix.) This has important consequences. It means that a firm cannot optimally undercut non-advertising firms with an advertised price. (If all $n-1$ other firms do not advertise, then any attempt to advertise $p_i < v$ and take all shoppers is inferior to undercutting with a higher price; the set of profit-improving prices is open above, and so a best-reply undercut does not exist.) The usual route (a simple technical convenience) to deal with this would be to allow a firm to advertise $p_i = v$ and then break the tie in favor of the advertising firm. This is explicitly ruled out by the "Assumption X" criterion. This also rules out an equilibrium in which exactly one firm places an atom at an advertised price $p_i = v$. This is important because in our extension described above we construct equilibria with just this feature: firm n places an atom at an advertised price v which successfully undercuts the implicit pricing at v of all firms who choose not to advertise.

The third question above asks: what happens when no firm advertises? One possibility (as used in our own costly-advertising extension) is that shoppers stay unaware of suppliers, and so do not buy at all. Another possibility is that shoppers are distributed amongst firms (and, indeed, amongst a "no purchase" option) according to a profile of exogenous probabilities. Shelegia and Wilson (2021) specified precise tie-break probabilities in this "nobody advertises" case in order to obtain an equilibrium. Their reason for this is that they ruled out, by their Assumption X (as described just above), the ability for an advertising firm to place an atom at v, and this in itself (which has the consequence that all firms engage in no advertising with positive probability) requires the very precise allocation of firms in the no-advertising condition.

Turning to results, Lemma 2 of Shelegia and Wilson (2021) established that at least two firms "use sales" (meaning: set prices below v) by mixing down to some lower-bound price.[34] We share this finding: generically, firms $n-1$ and n mix from $p^\dagger_{n-1}$ upward. However, we also find that (for a non-degenerate set of parameter values) there are "partner swaps" so that a third firm steps on to the dance floor to replace firm $n-1$. We also find (for generic parameter choices) that only two firms "dance" within any interval. These predictions differ from those

[34]Other familiar equilibrium properties hold: there are no atoms below v and there is no gap in the joint support of firms' strategies. A key difference from our model and its extension is that all firms place an atom at (an unadvertised price of) v. The logic is that if one firm always advertises then necessarily any other firm that prices close to v loses all shopper sales. (An advertised atom at v was ruled out by assuming that it is treated the same as not advertising.) Such a firm would prefer not to advertise. This means that the other $n-1$ firms do not price close to v, which means there is gap below v; and this contradicts standard arguments that there can be no gap in the joint support of firms' mixing. W can return to only $n-1$ firms placing an atom at an unadvertised price v (as we do in our own costly advertising extension) if we drop Assumption X.

of Shelegia and Wilson (2021): they did not consider "partner swapping" and they predicted that (with moderate advertising costs) more than two firms mix within the same interval.

The absence of "partner swapping" was by assumption: Shelegia and Wilson (2021, p. 204) recognized that "when $n > 2 \ldots$ there may be multiple forms of sales equilibria with firms using different supports" but went on to "avoid these significant complications" by focusing "only on sales equilibria where all advertising firms use the full convex support." In essence, that means that they assumed away the possibility there could be (or in fact, for some parameters, must be; a variant of our Proposition 3 also holds with positive advertising costs) equilibria in which a firm "joins the dance floor" at some higher price. For example, a condition of their Proposition 2 (Shelegia and Wilson, 2021, p. 209) refers to "when a sales equilibrium exists under our restrictions" and this, implicitly at least, must rule out partner swapping. Their Corollary 1 (Shelegia and Wilson, 2021, p. 210) says that "only two firms use sales when advertising costs are sufficiently small." This does not coincide with our findings: in our zero-advertising-cost core model and in a costly advertising extension (even when those advertising costs are small) we readily find (non-degenerate) situations in which at least three firms "dance" (or "use sales"). However, such "thrango" equilibria involve different firms using different supports. The fact that Shelegia and Wilson (2021) looked only for equilibria in which "all advertising firms use the full convex support" seems to explain our different findings.

A sufficient condition (in our model) for there to be no partner swapping is when firms are strongly ranked, in the sense that more aggressive firms have lower marginal costs, and fewer captive customers, and (in the costly advertising extension) lower advertising costs. In this situation (for generic parameter choices) only firms n and $n-1$ "tango" over the whole range of prices. This contrasts another finding of Shelegia and Wilson (2021) in which (again, from their Corollary 1) "all firms use sales when advertising costs are moderate." In our model this can only happen if $p_i^\dagger = p_{n-1}^\dagger$ for all $i \in \{1, \ldots, n-1\}$; this is of course a non-generic case. The explanation for this is that for Shelegia and Wilson (2021) the "limit price" that a firm is willing to set to win all shoppers depends on the exact tie-break rule, and that tie-break rule is specified as part of their solution. In essence, the tie-break probabilities adjust so that all firms all willing to start dancing at the same price.

Finally, Shelegia and Wilson (2021) provided some results (pp. 214–215) for situations in which "each firm's share of nonshoppers is determined endogenously as a function of the firms' actions prior to sales competition." This is related to the endogenous captive audiences of Chioveanu (2008) and our own consideration (Section 4) of endogenous marginal costs. A characteristic that we share with Chioveanu (2008) is that the equilibrium first-stage choices are asymmetric. A key result (Proposition 3) of Shelegia and Wilson (2021, p. 215) restricts to when "a symmetric SPNE exists," whereas we note that a symmetric equilibrium seems not to exist.[35]

[35] Other results that follow (Propositions 4–6; pp. 216–218) are also based around symmetric equilibria.