A Model of Costly Fixed-Sample Search with Stable Prices

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Abstract. We study markets in which buyers engage in fixed-sample search for a homogeneous good sold by any number of firms. Firms set prices over two stages in which initial pricing positions can be lowered but not raised prior to purchases. Equilibrium features searchers gathering one or two price quotations and firms setting a distinct profile of prices played as pure strategies on the equilibrium path. The analysis and predictions differ markedly from those in the literature where a single-stage of pricing produces mixed strategies in equilibrium. With homogeneous and linear search costs, a stable equilibrium with search exists (for a given range of search cost levels) if and only if the number of firms is greater than five. If however search costs are sufficiently heterogeneous (and convex), then for any number of firms a stable equilibrium with search exists, and if the CDF of costs is concave it is the unique one.

Keywords: consumer search, stable prices, undercut-proofness, price dispersion, price competition, pure strategies, fixed-sample search, consideration sets.

1. Introduction

In many settings prices are not immediately accessible and buyers must engage in costly search to reveal them. For example, where there is a lag between requesting and receiving a price quotation, potential buyers' main decision is how many suppliers to request quotations from. This situation arises in many industries and contexts, including construction and large-project tendering.[2] Even in settings where prices are immediately visible, researchers have documented consumer behavior that appears observationally equivalent to such fixed-sample search.[3]

Once a buyer has chosen how many prices to request (fixed their quotation sample size), they choose among that many prices when they arrive. Requesting each quotation is costly, and so

[1]This paper replaces the earlier version titled "Two-Stage Pricing with Costly Buyer Search." It includes and develops some of the analysis within an earlier working paper (Myatt and Ronayne, 2019, Section 5).
[2]Morgan and Manning (1985) characterize conditions under which fixed-sample search is optimal.
[3]See, for example, De los Santos, Hortaçsu, and Wildenbeest (2012); Honka and Chintagunta (2016) who study data from markets for books and auto insurance, respectively. Both studies fail to find a relationship between the prices consumers see and their propensity to search on. This is consistent with a procedure of buyers deciding in advance how many prices they will recover, and sticking to that no matter the prices they uncover along the way. This "gather then evaluate" protocol is exactly what fixed-sample search models capture.

the problem is non-trivial. Buyers solve this problem taking expected supplier prices into consideration, and firms set their prices taking expected buyer search intensity into consideration.

This problem of fixed-sample or "simultaneous" search has been modeled by many, following the seminal homogeneous-goods setting of Burdett and Judd (1983). Their work has seen numerous extensions such as allowing for differentiated products, by Anderson, De Palma, and Thisse (1992); Moraga-González, Sándor, and Wildenbeest (2021), and allowing for both endogenous and exogenous search, by Janssen and Moraga-González (2004).[4] Burdett and Judd (1983) also inspired work in other areas where search is important, as evidenced by the labor market studies of Acemoglu and Shimer (2000); Burdett and Mortensen (1998).

We examine the classic setting in which buyers engage in costly fixed-sample search for homogeneous goods. On the supply side, the common modeling approach in the literature is to have firms play a single-stage pricing game. This leads to equilibrium pricing in mixed strategies. In this paper we take a different approach and employ the two-stage framework we proposed (Myatt and Ronayne, 2023b) in which firms each first select an initial pricing position, which they can subsequently lower but not raise when setting their final price for buyers. In the classic fixed-sample search setting, we derive a unique profile of such prices, one for each firm, which are stable in the sense that no firm wishes to undercut any other firm's price.

Our analysis and predictions differ markedly from those with the typical single stage of pricing. Firstly, we show the prices we predict can arise from collective or non-cooperative firm interaction. In fact, we show that a price profile is played on the equilibrium path (necessarily in pure strategies) if and only if it is the (unique) profile of undercut-proof and Pareto efficient (from the industry's perspective) prices.

In terms of the prices themselves, in any equilibrium with search we predict firms choose a distinct sequence of prices for which closed-form solutions are readily available.[5] For given buyer search behavior (but within the class that appears in any equilibrium with search), as the number of firms, $n \geq 2$, grows, firms' prices distribute themselves between a fixed (invariant to n) highest and lowest price. Similarly, the level of various prices at other rank-positions (e.g., the median when n is odd) is invariant to n. As the number of firms grows, the average price charged falls (at least for small oligopolies). These points contrast to analysis of the same setting with a single-stage of pricing, the symmetric mixed-strategy of which is invariant to n.

On the demand side, equilibrium search features some buyers getting one or two quotations, as in Burdett and Judd (1983). Whether firms price use mixed or pure strategies does not affect that; what matters is that buyers believe the distribution of prices they face is non-degenerate.[6]

[4] For a more general survey of firm pricing with consumer search, see Anderson and Renault (2018).

[5] In contrast, closed-form solutions (for the pricing distributions of firms' equilibrium mixed-strategies) are typically unavailable in the single-stage framework, as pointed out by many. See, for example, Janssen and Moraga-González (2004, p.1103) or Johnen and Ronayne (2021, p.603).

[6] Burdett and Judd (1983) study a continuum of firms, which permits a pure-strategy interpretation, but solving their model for any finite n would result in mixed-strategy pricing. We allow for any number of firms.

We highlight that equilibria in this setting can be unstable in the sense that they would not survive even small perturbations in buyer behavior. In fact, we show that under the standard assumption that each search is equally costly then for the smallest oligopolies, specifically, $n \in \{2, 3, 4\}$, the only equilibrium with search is unstable (leaving only an equilibrium in which there is no search and hence no sales, so that the market breaks down). We refine this insight by showing that there is an open set of search cost values for which a stable equilibrium with search exists if and only if there are sufficiently many, $n \geq 5$, firms. And, if it exists, it is the only stable equilibrium with search. Any stable equilibrium with search has the comparative static that buyers' search intensity and surplus are higher for a lower search cost.

One reason there may not exist a stable equilibrium with search is that search costs are homogeneous. We relax this assumption and instead look for equilibria when search costs are heterogeneous across buyers and convex with respect to the number of quotation any given buyer gathers. For simplicity and to maintain comparability with earlier sections we assume that the cost of the third search is prohibitive. This leaves buyers collecting one or two quotations in any equilibrium with search, as in the case with homogeneous and linear search costs. This allows us to focus purely on the shape of the distribution of costs for buyers' second search when characterizing the set of equilibria. We show that if there is sufficient richness in the distribution of second-search costs (that there are both buyers with sufficiently low costs and others with sufficiently high costs), then there is a stable equilibrium with search (for any n). In addition, if that distribution is concave, then that is the unique equilibrium with search.

We consider the case when n grows arbitrarily large. There we show that the prices we predict converge to exactly the same distribution as reported by Burdett and Judd (1983). This means that, in the limit, we find the same results as them and thereby reveal a connection between mixed-strategy and stable-price predictions.[7] However, our predictions differ for any finite n. This can lead to differences in qualitative predictions. For example, adopting single-stage pricing within the setting we study implies firm price strategies are independent of n.

We can also straightforwardly accommodate "shoppers" (buyers assumed to retrieve every price) into our analysis, which are a popular addition to models. The only difference for our stable prices is an adjustment to the price of the cheapest firm (the firm that serves the shoppers), which depends on how many shoppers there are. In contrast, the presence of shoppers in a single-stage model results in multiple firms (all firms, in the symmetric equilibrium) having their equilibrium mixed strategies depend on how many shoppers there are. The corresponding analysis with a single-stage of pricing by Janssen and Moraga-González (2004) finds equilibria in which searchers (those who choose how many quotations to gather) retrieve no more than one quotation. In contrast, under two-stage pricing such equilibria exist only for a narrow range of parameter values, and involve no search at all if there are no shoppers.

[7]As Kalai (2004) shows, many mixed-strategy equilibria become ex-post Nash with arbitrarily many players.

2. Model

There are $n \geq 2$ firms with zero costs. Firms $i \in \{1, \ldots, n\}$ decide on their initial pricing positions $\bar{p}_i$ (either collectively or non-cooperatively). Having observed those, firms then have the opportunity to cut but not raise their price, i.e., they set $p_i \leq \bar{p}_i$, prior to purchases.

A unit mass of buyers, or "searchers", are each willing to pay $v > 0$ for one unit. They use a fixed-sample technology to obtain price quotations. Moving simultaneously, each searcher pays κq to obtain $q \in \{0, 1, \ldots, n\}$ quotations, where $0 < \kappa < v$. We extend the analysis to search costs that are heterogeneous across searchers, and (at least weakly) convex for each searcher, in Section 9.[8] The buyer then receives q random draws (without replacement) out of the n price offers. The buyer selects the cheapest offer, with ties broken in any interior way.[9,10]

We envisage a situation in which buyers search at the same time as firms engage in pricing. To do this formally, we assume buyers move first and simultaneously choose their search policies. Those policies are observed by firms, before they choose their initial prices. Any individual buyer has no measurable influence on future play and so acts as though moving simultaneously.[11] We seek solutions in which buyers' search protocols are best-responses to firms' prices, and firms either set efficient undercut-proof prices given buyers' search protocols, or, best-respond to buyers' search protocols (two possibilities that we show to be one and the same in Proposition 1).

3. Stable and Efficient Prices with Costly Fixed-Sample Search

For now, we take buyer search protocols as fixed and examine firms' responses to them. Let $\mu_q \in [0, 1]$ be the proportion of searchers who pay for $q \geq 0$ quotations, so that $\sum_{q=0}^{n} \mu_q = 1$. Given $\{\mu_q\}_{q=0}^{n}$, we seek prices that are stable in the sense that no firm wishes to undercut any other. Within that set, we focus on those which are (Pareto) efficient for industry.[12]

Suppose no searchers retrieve exactly one quotation, so that $\mu_1 = 0$. Undercut-proof prices are forced down so that the Bertrand (zero-profit) outcome realizes (undercut-proofness guarantees that each consumer has a zero price in their consideration set). We set those cases aside and continue with $\mu_1 > 0$.[13] This implies that efficient undercut-proof prices are all strictly positive.

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[8] Relatedly, Moraga-González, Sándor, and Wildenbeest (2017) extend the setting of Burdett and Judd (1983) to one with heterogeneous search costs (and a finite number of firms).

[9] In Section 8 we extend the model to include "shoppers", who (for exogenous reasons) gather all n prices.

[10] We assume throughout that whenever every buyer sees at least two prices, which forces zero-profits for firms in any equilibrium, that all prices are zero. This rules out the bizarre equilibria that typically exist in Bertrand games with a common marginal cost in which some prices are equal to cost (sufficiently many to make all firms earn zero profit), while others are unrestricted.

[11] This formulation will allow for firms' initial pricing stage to be the start of a proper subgame, facilitating subgame perfection as a solution concept when we study stable prices that emerge from non-cooperative firm interaction. For the derivation of stable prices we could alternatively assume that buyers do not observe firms' initial price positions and choose their search protocol simultaneously with firms choice of final retail prices.

[12] The set of undercut-proof prices is typically large. Putting aside trivial cases in which some firm charges a price equal to marginal cost, the set is defined by Myatt and Ronayne (2023b, claim (ii) of Lemma 1).

[13] In fact this is without loss, because, as we show later, any equilibrium with search features $\mu_1 > 0$.

If also $\mu_2 > 0$, then undercut-proof prices are entirely distinct. If instead $\mu_2 = 0$, then there are ties in the efficient undercut-proof prices of at least two firms at the monopoly price, v.[14] There are no ties in efficient undercut-proof prices below the monopoly level. For either efficiency or equilibrium we must have $\bar{p}_i = p_i$ for all i and so without loss we state only $\bar{p}_i$ terms from now.

Without loss of generality, label firms such that $\bar{p}_1 \geq \cdots \geq \bar{p}_n > 0$. Because buyers' search technology returns a random subset of price quotations from the pool of prices, buyers' consideration sets of any given size, q, comprise firm indices drawn randomly and symmetrically from $\{1, \ldots, n\}$. It is helpful to denote by X_i, the mass of searchers buying from i:

$$X_i \equiv \sum_{q=1}^{i} \mu_q \left[\binom{i-1}{q-1} \Big/ \binom{n}{q} \right] \tag{1}$$

The term X_i sums over the relevant consideration-set sizes (no sale is made if $q > i$ because either q is cheaper than i, or $\bar{p}_i = \bar{p}_q = v$ in which case $\mu_q = 0$). For each q, there are $\binom{n}{q}$ equally-sized consideration sets. Firm q makes a sale only if compared to $q-1$ others from the $i-1$ firms with higher prices. There are $\binom{i-1}{q-1}$ such sets.

For the case of $\mu_2 > 0$, such that equilibrium prices are entirely distinct, firm i earns $\bar{p}_i X_i$. For firm i not to wish to undercut $j > i$ we must have

$$\bar{p}_i X_i \geq \bar{p}_j X_j. \tag{2}$$

More generally the set of no-undercutting constraints may be complex, but here buyers' consideration is spread uniformly across them, which makes the constraints particularly parsimonious. To be efficient for the industry, the prices $\bar{p}_j$ in (2) must be raised as much as possible.

Firm 1 faces no potential undercutters and so charges $\bar{p}_1 = v$ in any efficient profile of prices. It does then not undercut firm 2 so long as $vX_1 \geq \bar{p}_2 X_2$, which must bind for efficiency and so determines $\bar{p}_2$ and says that firm 2's profit is also vX_1 from any undercut proof efficient profile. Firm 1 and 2's no-undercutting-firm-3 constraints can be written as $vX_1 \geq \bar{p}_3 X_3$, which gives $\bar{p}_3$ when it binds. Continuing iteratively yields a unique profile of efficient undercut-proof prices.[15]

Lemma 1 (Stable Prices given Buyers' Protocol). *For a given buyer search protocol $\{\mu_q\}_{q=0}^{n}$ with $\mu_1 > 0$, there is a unique undercut-proof Pareto efficient profile of prices in which:*

$$\bar{p}_i = v \frac{X_1}{X_i} \quad \textit{for } i \in \{1, \ldots, n\}. \tag{3}$$

When firms set these prices, each firm earns the profit vX_1.

[14] The precise and more general statement is: if $\mu_h = 0$ for all $h \in \{2, \ldots, m\}$ with $2 \leq m \leq n$, then an efficient undercut-proof profile has exactly m firms setting the monopoly price, v.

[15] This constellation of consideration is the *full exchangeability* setting of Myatt and Ronayne (2023b, Section 6) where our Proposition 4 there encompasses Lemma 1 here. The differences between that paper's setting and the present one are that here: (i) firms each have an equal share of consumers with singleton consideration sets (i.e., "captives"); and (ii) we allow for zero masses of buyers with consideration sets with two or more elements.

Non-Cooperative Two-Stage Pricing. Lemma 1 gives the (uniquely) efficient and stable prices for any search protocol from buyers. We now show that those same prices can also be established non-cooperatively.

By construction, no firm wishes to unilaterally deviate downwards from the initial prices in (3), and upwards deviations from them are ruled out by assumption. It remains to ask whether firms would choose to establish those initial pricing positions as part of an equilibrium.

Formally, following searchers' choice of protocol, we employ a two-stage perfect-information pricing game. Together, this forms a three-stage game in which

($t = 1$) searchers simultaneously choose a search protocol, $\{\mu_q\}_{q=0}^n$; then
($t = 2$) firms simultaneously choose their initial price positions, $\bar{p}_i \in [0, v]$; and lastly
($t = 3$) firms simultaneously choose their final retail prices, $p_i \in [0, \bar{p}_i]$.

We look for subgame perfect equilibria. That is, those in which firms choose pure strategies on the (subgame perfect) equilibrium path.

Definition. *If a profile of prices are the on-path pure-stratgies of a subgame perfect equilibrium of a game, we say that the profile is* supported by the equilibrium play of pure strategies.

Consider a profile of prices, $\bar{p}_1 \geq \cdots \geq \bar{p}_n$. If it is supported by the equilibrium play of pure strategies, then $p_i = \bar{p}_i > 0$ for all i.[16] It must also be undercut-proof (else there is a profitable deviation available at the final-stage). Firm 1 must set the monopoly price so that $\bar{p}_1 = v$ (if not, it could profitably raise initial price to v). Similarly, firm 2 can then safely raise its price to either the point at which 1's undercutting constraint binds, or v (in the case of $\mu_2 = 0$).

Continuing iteratively through the (undercut-proof) prices, raising $\bar{p}_i$ until $i-1$ is indifferent to undercutting it, we retrieve a candidate profile for the support of the equilibrium play of pure strategies. The set of prices resulting from the process of iteratively making no-undercutting constraints bind therefore determines a unique candidate profile. This was exactly the process that led us to the efficient undercut proof profile of Lemma 1 and so the unique candidate profile of prices supportable by the equilibrium play of pure strategies is that specified by (3).

We have shown that if a price profile is supported by the equilibrium play of pure strategies, then it satisfies (3). We now ask the converse: are those prices supported by the equilibrium play of pure strategies?

Consider the prices in (3). There is no profitable unilateral deviation downwards at the final stage (by undercut-proofness) or at the penultimate stage (any downwards deviation there can also be made at the final stage). No upwards deviation is available at the final stage by assumption. This leaves us to consider upward deviations at the penultimate stage. Such a

[16]That $p_i = \bar{p}_i$ follows because a final-stage pure-strategy $p_i < \bar{p}_i$ can either be profitably raised, or, in the case of a tie for some buyers' lowest price, undercut; $\bar{p}_i > 0$ follows because $\mu_1 > 0$ means a firm can always set $p_i = \bar{p}_i = v$ and make positive profit from the sales to those who only consider i.

deviation marks the start of a subgame in which we seek a Nash equilibrium that yields a payoff for the deviator no greater than the payoff that results from the candidate on-path prices. A modified version of Lemma 6 of Myatt and Ronayne (2023b) applies, which we state next.[17]

Lemma 2 (Final-Stage Subgames). *Consider a buyer search protocol* $\{\mu_q\}_{q=0}^n$ *with* $\mu_1 > 0$ *and the prices in* (3)*, with the exception of firm* k: $\bar{p}_k < v$*, which deviates to an initial price* $\hat{p}_k > \bar{p}_k$*. There is a Nash equilibrium in that subgame in which each firm earns the profit* vX_1*.*

Suppose that $\{\mu_q\}_{q=0}^n$ is a best-response for buyers to the prices in (3). In combination with the lack of other profitable unilateral firm deviations, Lemma 2 shows that the prices in (3) are the on-path play of a subgame perfect equilibrium. (To complete the specification of a subgame perfect equilibrium, we allow any equilibrium to follow in subgames that are further off-path.)

Of course, we have not said anything about buyers' optimal protocol in $t = 1$, which will be our next focus, but we can now summarize firms' equilibrium behavior.

Proposition 1 (Stable Prices in a Two-Stage Pricing Game). *Fix a given buyer search protocol* $\{\mu_q\}_{q=0}^n$ *with* $\mu_1 > 0$ *and consider the two-stage pricing game starting at* $t = 2$*. A profile of prices is supported by the equilibrium play of pure strategies in the two-stage pricing game if and only if it is the efficient undercut-proof profile of prices given in* (3)*.*

When consumers consideration sets are formed by the protocols considered in this paper, the equilibrium pricing strategies of a single-stage pricing game (in which firms simultaneously set prices), which are necessarily in mixed strategies, typically do not have analytic solutions (see, for example,). In contrast, with Proposition 1 we provide a unique and concise pure-strategy prediction for prices. Each price is equal the ratio of the monopoly profit to the number of buyers the corresponding firm serves in equilibrium.

4. Search protocol in any equilibrium with search

Searchers decide how many draws to take from the pool of firm prices. This generates a single (possibly degenerate) distribution (regardless of firms' strategies), from which searchers draw. Here, we fix firms' behavior and examine buyers' optimal response to it.

If all searchers gather two quotations or more, so that $\mu_0 = \mu_1 = 0$, then all buyers observe at least two prices, and Bertrand competition pushes prices to zero, so consumers strictly prefer to search less. As such, at least one of μ_0 and μ_1 is strictly positive in any equilibrium. If $\mu_1 = 0$ then $\mu_0 = 1$, else some buyers observe at least two prices while no buyer sees exactly one, and so once again prices tend to zero, and those searching twice or more would prefer to search

[17] Myatt and Ronayne (2023b) assumed the analog of $\mu_2 > 0$, which guarantees that any equilibrium of the sort we seek has entirely distinct on-path prices. Because here we allow $\mu_2 = 0$, we need to cover cases in which there may be ties in on-path prices at the monopoly price, v. The relevant proof of Myatt and Ronayne (2023b) is (trivially) extended to cover such subgames by noting that having multiple firms at the top pricing position does not alter the analysis (symmetry implies the undercutting conditions are the same for all such firms) and so in any off-path subgame, one can let them (continue to) price at v so that they do not disturb Lemma 2.

less. And indeed there are equilibria with $\mu_0 = 1$, but these are trivial no search equilibria with firms charging prices in any compatible way (most obviously $\bar{p}_i = v$ for all i).

The reasoning above tells us that in any equilibrium with search, some buyers must gather one price, so that $\mu_1 > 0$. We cannot have all searchers getting one quotation, or firms would respond by all charging the monopoly price, and searchers would rather not search. And so $\mu_1 \in (0, 1)$ and μ_q for some $q \geq 2$, i.e., in any equilibrium with search, some (but not all) searchers gather exactly one quotation while others gather more than one.

From the properties of order statistics, the expected minimum draw is (at least weakly) decreasing in the number of draws, and at a (at least weakly) decreasing rate. This means there are (at least weakly) decreasing returns to search. On the cost side, searchers have the same constant marginal cost of search, $\kappa > 0$. Together these facts imply that no two searchers can obtain two (or more) quotations *more* than any other searcher.[18] Together, these points tell us that in any equilibrium with search, $\mu_1 + \mu_2 = 1$ and $\mu_1 > 0$. Lemma 3 collects these points.

Lemma 3 (Number of Quotations in any Equilibrium). *In any equilibrium, search behavior falls into one of two categories.*

1. *There is no search (i.e., $\mu_0 = 1$) in which case there exist trivial equilibria in which firms price sufficiently high such that no searcher wishes to search even once.*

2. *There is search: some searchers gather exactly one quotation, while others gather two, i.e.,*

$$\mu_1 + \mu_2 = 1 \quad \textit{and} \quad \mu_1 \in (0, 1). \tag{4}$$

The second point of the lemma is exactly Claim 1 of Burdett and Judd (1983, Section 3.2). Neither of our model's differences from theirs (two-stage pricing and a possibly finite number of firms, n) were relevant for the arguments deriving equilibrium searcher behavior above, and so we reach the same conclusion as them.

For fixed firm behavior, our predictions for equilibrium buyer behavior coincide with those found with a single-stage pricing game because in either case buyers choose how many draws to take at random from a distribution of prices and take expectations accordingly. Whether that distribution is a result of firms employing mixed or pure strategies is not relevant.

However, and as we will see next, the stark differences in the nature of prices from our two-stage approach have numerous implications.

[18]A special case is found with $n = 2$, where we can construct an equilibrium in which $\mu_0, \mu_1, \mu_2 > 0$. A slight change to our model eliminates this multiplicity. If we assume that the second quotation is slightly more expensive than the first then once again any equilibrium can only involve two adjacent search strategies.

5. Prices in any equilibrium with search

We have shown that in any equilibrium with search, some searchers gather two price quotations so that every firm goes head-to-head with every other firm.[19] As per the prices we derived in Section 3, given by (3), this ensures that any prices supported by the equilibrium play of pure strategies must involve n distinct prices (prices are positive (because $\mu_1 > 0$) and so a tie between i and j would leave both with an incentive to undercut the other).

Because $\mu_2 > 0$, the quantity of sales to searchers made by each successively lower-priced firm is increasing: $X_1 < \cdots < X_n$. Efficient undercut-proof prices are such that each step down the ladder of prices is exactly compensated for by a corresponding increase in sales. Bringing together equilibrium search behavior and optimal stable prices produces Proposition 2.

Proposition 2 (Prices in any Equilibrium with Search). *In any equilibrium with search, some buyers search once and some twice (case 2 of Lemma 3), and there is a unique profile of prices supported by the equilibrium play of pure strategies, such that for all $i \in \{1, \ldots, n\}$*

$$\bar{p}_i = \frac{v\mu_1(n-1)}{\mu_1(n-1) + 2(1-\mu_1)(i-1)}. \tag{5}$$

As $n \to \infty$ these prices converge to a continuous distribution with support on $[v\mu_1/(2-\mu_1), v]$

$$F(p) = 1 - \frac{v-p}{2p}\frac{\mu_1}{1-\mu_1}. \tag{6}$$

Analyzing the prices reported in Proposition 2 yields the observations in Corollary 1.

Corollary 1. *Prices are decreasing in the proportion, μ_2, of buyers who gather two quotations.*

The highest price is $\bar{p}_1 = v$ and the lowest is $\bar{p}_n = v\mu_1/(2-\mu_1)$. If n is odd, then the median price is equal to $v\mu_1$. Similarly, if $n+3$ is a multiple of 4 then the $\frac{n+3}{4}^{th}$ and $\frac{3n+1}{4}^{th}$ highest prices are $\bar{p}_{\frac{n+3}{4}} = v\mu_1\frac{2}{3-\mu_1}$ and $\bar{p}_{\frac{3n+1}{4}} = v\mu_1\frac{2}{1+\mu_1}$, respectively.

The average price charged is lower in an industry with $n > 2$ competitors than in a duopoly.

Each firm earns profit of $v\mu_1/n$, and so the average paid by a buyer is equal to $v\mu_1$.

The final claims concern profitability and so (equivalently) the average price paid. A fraction of μ_1/n of customers are effectively captive to each firm. A fraction $1-\mu_1$ make pairwise comparisons. The most expensive firm loses any comparisons, and so earns the profit from exploitation of captive customers. The price construction means that all firms earn equal expected profit. The average price paid does not depend directly on the number of competitors. Instead, it depends buyers' actions: more intensive search lowers the distribution of prices posted (in fact it lowers every price except the most expensive, which is constant at $\bar{p}_1 = v$).

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[19]In Myatt and Ronayne (2023b, claim (i) of Lemma 1), we showed this ("twoness") condition, in addition to captive consumers for each firm, is sufficient for positive undercut-proof prices to be entirely distinct.

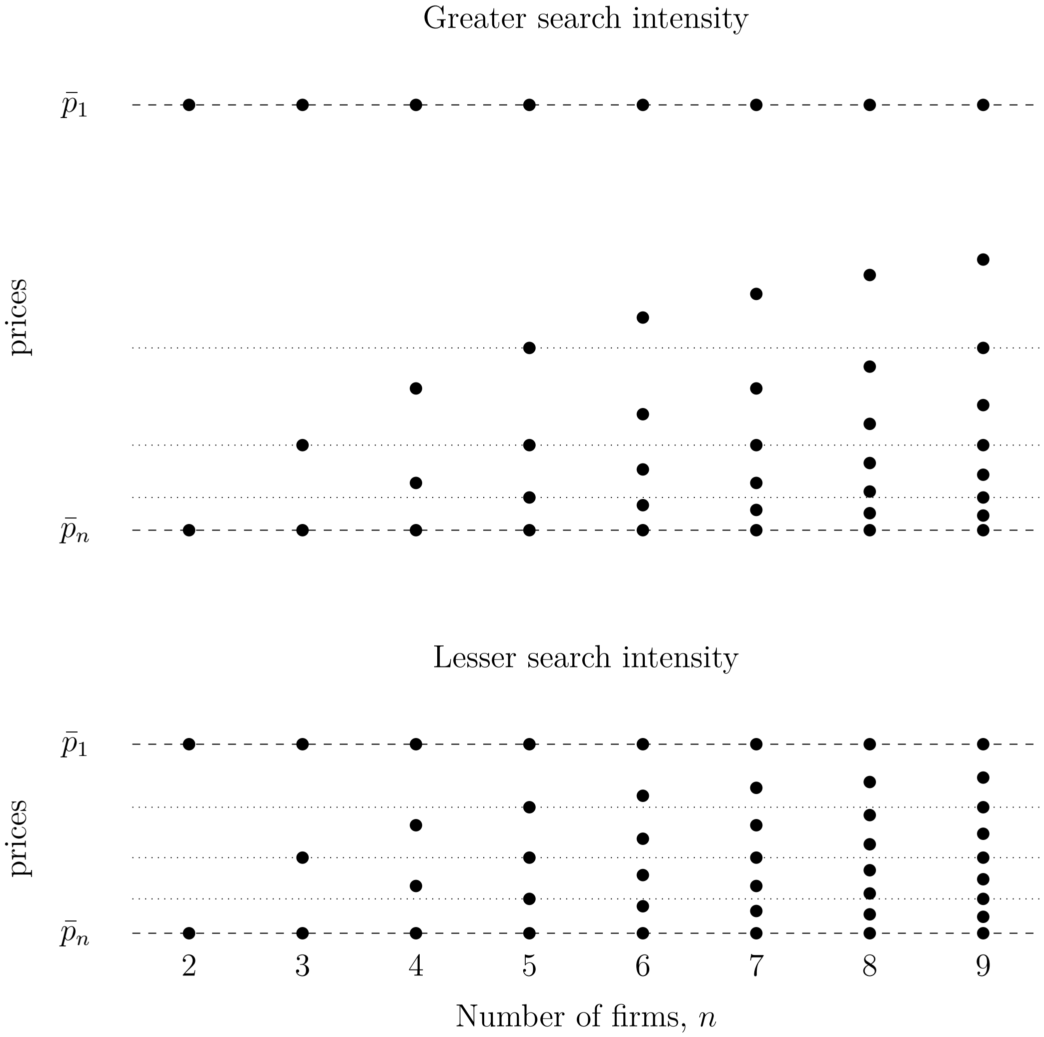

FIGURE 1. The distribution of prices (given in Proposition 2) for $n \in \{2, \ldots, 9\}$. The highest ($\bar{p}_1 = v$) and lowest ($\bar{p}_n = v\mu_1/(2-\mu_1)$) prices do not vary with n. Upper panel: $\mu_2 = 3/5$; a majority of buyers gather two prices (a minority, one). Lower panel: $\mu_2 = 1/5$; a minority of buyers gather two prices (a majority, one). Dotted lines: the median, $\frac{n+3}{4}^{\text{th}}$ and $\frac{3n+1}{4}^{\text{th}}$ highest prices (when well-defined).

Although the average price paid is independent of n, the distribution of prices charged is not. Notably the average price is not increasing in n. Beyond the statement in Proposition 2 we can also show that this average is decreasing for $n \in \{2, \ldots, 10\}$. This stands in contrast to the symmetric mixed-strategy predictions from single-stage pricing, in which firms' equilibrium pricing distribution depends only on the number of quotations buyers retrieve, not the number of firms.[20] The key difference here is the nature of pricing. Instead of each firm continuously mixing in an i.i.d. fashion over an interval of prices, we predict that each firm charges one price. As n increases, prices file in between the highest and lowest prices, $\bar{p}_1$ and $\bar{p}_n$, which are invariant to n. We provide two illustrations (for two different levels of μ_2) of this in Figure 1.

[20]See, for example, Burdett and Judd (1983, Section 3.1), Moraga-González, Sándor, and Wildenbeest (2017, equation 3), or Ronayne (2021, Section 3.3). In the current setting with $\mu_2 > 0$ and single-stage pricing, Johnen and Ronayne (2021) show the symmetric equilibrium is the unique equilibrium. For a more general treatment of single-stage pricing with symmetric consideration sets, see Armstrong and Vickers (2022, Section 3).

We now make a connection between the equilibria with stable prices that we report, and those in the literature. The mixed equilibria of single-stage models are not ex post Nash for finite n. But as $n \to \infty$, the equilibria of a large class of games become ex post Nash (Kalai, 2004, Theorem 1). The mixed-strategy equilibria of the single-stage pricing models we have examined are no exception. In fact, as $n \to \infty$ in our model, prices organize themselves in such way as to asymptotically produce the exact same distribution, F of (6), as reported in those studies.[21] Hence both paradigms are robust to ex post deviations in the limit, but only there. The difference is that our equilibrium is ex post Nash for all n, not only when $n \to \infty$.

6. The benefit of a second search

Now we consider optimal searcher behavior. Using the prices of Proposition 2 we need to find the marginal gain to a buyer from obtaining their second quotation (weakly lower than the gain from the first quotation), which we term B_n. An equilibrium with search obtains when $B_n = \kappa$.

Consider, for example, the duopoly case ($n = 2$) for which the two prices are $\bar{p}_1 = v$ and $\bar{p}_2 = v\mu_1/(2-\mu_1)$. A single quotation finds the cheaper firm with probability 1/2 and generates benefit $v - \bar{p}_2$. Two quotations give this benefit with certainty. Therefore, $B_2 = (v - \bar{p}_2)/2$.[22]

Now consider $n \geq 2$ firms. With probability $1/n$ the first quotation is $\bar{p}_1$. The gain from the second search is $\bar{p}_1 - \bar{p}_i$ cheaper price (of a remaining firm $i > 1$) is found, which happens with probability $1/(n-1)$. With probability $1/n$ the first quotation is $\bar{p}_2$. The gain from the second search is zero if the more expensive $\bar{p}_1$ is drawn, but $\bar{p}_2 - \bar{p}_i$ when a cheaper price (of a firm $i > 2$), which happens with probability $1/(n-1)$. Continuing iteratively,

$$B_n = \frac{1}{n}\left(\frac{1}{n-1}\left[(\bar{p}_1 - \bar{p}_2) + \cdots + (\bar{p}_1 - \bar{p}_n)\right]\right) + \frac{1}{n}\left(\frac{1}{n-1}\left[(\bar{p}_2 - \bar{p}_3) + \cdots + (\bar{p}_2 - \bar{p}_n)\right]\right)$$
$$+ \cdots + \frac{1}{n}\left(\frac{1}{n-1}(\bar{p}_{n-1} - \bar{p}_n)\right), \tag{7}$$

where prices, $\bar{p}_i$, are those specified in (3). Reorganizing terms, we obtain (9) in Proposition 3. For large markets (when $n \to \infty$) we can compute the expected benefit of a second search as

$$B_\infty = \mathrm{E}_F[p] - \mathrm{E}_F[\min\{p', p''\}], \tag{8}$$

where F is the distribution of prices given in (6) and $\min\{p', p''\}$ denotes the minimum of two (randomly drawn) prices. We solve this and arrive at Proposition 3.

[21] See, for example, Burdett and Judd (1983, equation 2). There, n is large throughout, but with single-stage pricing, the exact same CDF in symmetric mixed-strategies is recovered for any finite $n \geq 2$. And so when n is assumed to be finite, strategies are necessarily not ex post Nash.

[22] For this special case of $n = 2$, the marginal benefit of both the first and second quotations is equal to $(v - \bar{p}_2)/2$.

Proposition 3 (Buyers' Benefit from a Second Search). *Given firms set prices as given in Proposition 2, the expected benefit to a buyer from a second search is*

$$B_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} (n - 2i + 1)\bar{p}_i. \tag{9}$$

As the number of firms grows arbitrarily large, the expected benefit from a second search is

$$B_\infty = \frac{v}{2} \frac{\mu_1}{1-\mu_1} \left(\frac{1}{1-\mu_1} \log\left(\frac{2-\mu_1}{\mu_1}\right) - 2 \right). \tag{10}$$

For any n and $\mu_1 \in (0,1)$, B_n is strictly positive; is strictly concave in μ_1; and converges to v/n as $\mu_1 \downarrow 0$ and 0 as $\mu_1 \uparrow 1$.

Proposition 3 reports some properties of the expected benefit of the second search. As $\mu_1 \uparrow 1$, all prices converge to the monopoly price, and so $B_n \downarrow 0$. One can also confirm that B_n converges to v/n as $\mu_1 \downarrow 0$. This is because $\bar{p}_i \downarrow 0$ for $i > 1$ while $\bar{p}_1 = v$, and so a second search is only helpful in the case that a buyer draws $\bar{p}_1 = v$ with their first search (so that a second search guarantees a benefit of $v - 0$), which happens with probability $1/n$.

7. Equilibrium

We now bring together the work in the preceding sections. An equilibrium with search specifies a value of $\mu_2 \in (0,1)$ (or $1 - \mu_2 = \mu_1 \in (0,1)$) that sets the expected benefit of the second search equal to the cost of the second search:

$$B_n = \kappa. \tag{11}$$

From Proposition 3, there is at least one equilibrium with search, and at most two, so long as $\kappa < v/n$. Of course, that condition is onerous as the number of firms grows. However, we will show that this is only a relevant parameter constraint for small oligopolies. Before exploring the set of equilibria more, we examine the stability of equilibria with search.

Stable Equilibria with Search. Suppose $\mu_2^* \in (0,1)$ solves $B_n = \kappa$ and so gives an equilibrium with search. Consider a perturbation in which we take some (sufficiently small) number of buyers searching once, make them search twice instead, and let firms recompute prices with the new profile of buyer behavior. If B_n is strictly upward sloping in μ_2 at μ_2^*, then the perturbation makes the second search more attractive such that all buyers would strictly prefer to search twice, and so the equilibrium at μ_2^* unravels. Instead, if B_n is strictly downward sloping in μ_2 at μ_2^*, then the perturbation makes the second search less attractive and so those buyers would strictly prefer to switch back to searching once, reinstating the equilibrium at μ_2^*. In this sense we refer to any equilibrium with former property as unstable and any with the latter as stable.

The case above considered a perturbation increasing search. For an equilibrium to be stable we require stability for perturbations increasing and reducing search. This leads to Lemma 4.

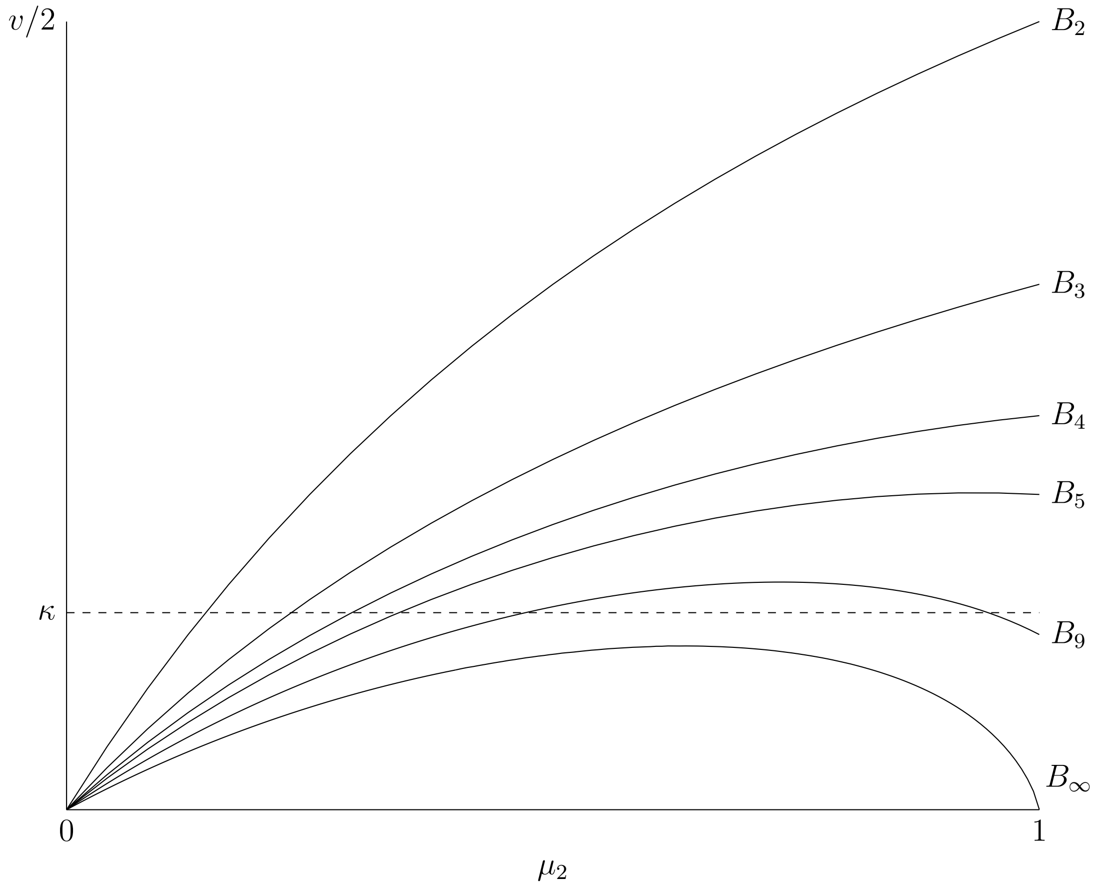

FIGURE 2. Equilibrium with homogeneous and linear search costs. The benefit of buyers' second search, B_n, for $n \in \{2, 3, 4, 5, 9, \infty\}$ is shown with solid lines. An example cost level, $\kappa = v/8$, is shown with a dashed line. With $n \leq 5$ there is one equilibrium with search, which is unstable. With $n = 9$ there are two with search, one unstable (the one with lower μ_2) and one stable (with higher μ_2).

Lemma 4 (Stability of Equilibria with Search). *An equilibrium with search (with equilibrium search intensity μ_2^*) is stable if and only if B_n cuts κ at μ_2^* from above, i.e.,*

$$\left.\frac{dB_n}{d\mu_2}\right|_{\mu_2=\mu_2^*} < 0. \tag{12}$$

Oligopoly. For small oligopolies with $n = 2, 3$ or 4 firms, it is straightforward to confirm that B_n is strictly decreasing over $\mu_1 \in (0, 1)$. This tells us that there exists an equilibrium with search if and only if $\kappa < v/n$, and that when one exists, it is unique. But by Lemma 4, it is unstable.[23] The only other equilibrium in the model is for there to be no search, i.e., $\mu_0 = 1$ (as per case 1 of Lemma 3), and the market breaks down. Figure 2 provides a visual counterpart to this reasoning while Proposition 4 summarizes the points.

Proposition 4 (Equilibrium in Small Oligopolies). *Consider oligopolies with $n = 2, 3, 4$. An equilibrium with search exists if and only if $\kappa < v/n$. If one exists, it is both unique and unstable. The only other equilibrium, which is stable, is that with no search (case 1 of Lemma 3).*

[23]For $n \in \{2, 3\}$, that equilibrium's quantities are concise and we report them in the appendix's Proposition A1.

However, the lack of a stable equilibria with search is confined to the smallest oligopolies. In fact, and as we prove in the appendix, as we move from $n = 4$ to $n = 5$, the peak of B_n is reached at an interior point, $\mu_2 \in (0, 1)$, which implies there is a range of values that κ can take such that there are exactly two equilibria, one of which is stable.

Proposition 5 (Stable Equilibrium Existence and Uniqueness). *There is an open set of search cost values for which a stable equilibrium with search exists if and only if $n \geq 5$. If a stable equilibrium with search exists, it is the only one.*

Restricting our attention to the stable equilibrium with search we can consider the impact of a change in the search cost, κ, which we state in Corollary 2.

Corollary 2 (Comparative Static with respect to Search Cost). *Suppose $n \geq 5$ and consider two search cost levels, $\kappa' > \kappa''$, for each of which a stable equilibrium with search exists. Equilibrium search is greater, prices are lower, and consumer welfare is higher, with κ' than κ''.*

Large Markets. Given the equivalence of our equilibrium with that in the literature for a large number of firms, we derive optimal search by following Burdett and Judd (1983, Section 3.2). The expression for B_∞ given in (10) is exactly what comes out of their equations (2)-(3). It is single-peaked, rising from zero at $\mu_2 = 0$ and falling back to zero at $\mu_2 = 1$. This implies that if κ is sufficiently small there are (generically) two equilibrium values for μ_2. And by Lemma 4 exactly one (the one with more search, i.e., greater μ_2) is stable.

Proposition 6 (Equilibria with Many Suppliers). *Consider $n \to \infty$. There is some $\bar{\kappa}$ such that if $\kappa < \bar{\kappa}$ then there are two equilibria of which the one with more search is stable.*

As discussed, our predictions depart markedly from those with single-stage pricing for finite n and coincide in the limit as $n \to \infty$. This means that equilibria are distinct in a setting with stable prices vis-á-vis the mixed-strategies arising from single-stage pricing games. In particular, for a given κ, equilibria with search and stable prices have different levels of search and hence prices, profits and consumer welfare. In addition, because B_n is a function of n with stable prices, comparative statics are available with respect to entry.

In particular, replacing our stable prices with the symmetric mixed-strategy equilibrium strategies from a single-stage pricing game (but otherwise leaving the model unchanged), would (as discussed in Section 3) leave equilibrium strategies (both firm's pricing and buyers' search choices) independent of n. Therefore, all equilibrium outcomes are invariant to the number of competing firms. In stark contrast, we find outcomes from equilibria with search depend on n.

One aspect of the classic search setting we studied so far is that search costs are homogeneous across buyers and linear for each buyer. We relax those assumptions in Section 9 and show how richness there can result in stable equilibria that feature search (regardless of n). But first we examine how our analysis (so far with buyers who endogenously choose how many quotations to receive) accommodates the addition of buyers who see all prices.

8. Incorporating Shoppers

We have modeled buyers who (endogenously) choose how many quotes to retrieve. A popular and different specification additionally allows for "shoppers" who (for exogenous reasons) see all prices (as in classic papers such as Rosenthal, 1980; Stahl, 1989; Varian, 1980, and many since). Such an assumption is often convenient, for example, it ensures some price comparisons are made and therefore avoids the Diamond paradox or trivial equilibria with no search (such as that in case 1 of Lemma 3). We now show how our analysis extends to include shoppers.

Specifically, a proportion, $\lambda_S \in [0,1)$, of buyers are now shoppers (who buy at the lowest price among all firms).[24] The model is otherwise unchanged. The remaining $1-\lambda_S \in (0,1]$ buyers are the (endogenous) "searchers" we studied so far.[25] The co-existence of both buyer types in a fixed-sample search setting brings the demand-side assumptions away from Burdett and Judd (1983) and in line with those of Janssen and Moraga-González (2004).

Stable prices in the presence of shoppers. We arrived at the stable prices in (3) by considering (binding) no-undercutting constraints such that each firm charges the highest price they can such that no more-expensive rival wishes to undercut them.

We can apply our earlier analysis replacing each X_i term with $(1-\lambda_S)X_i$. The no-undercutting constraints for firms $i < n-1$ to undercut non-cheapest firms $j \in \{2,\ldots,n-1\}$ are unchanged (firm 1 does not face potential undercuts and so $\bar{p}_1 = v$) and so those $\bar{p}_j$ are exactly as in (3), and their profits are $v(1-\lambda_S)X_1$. Therefore, no firm $i < n$ wishes to undercut $\bar{p}_n$ if

$$v(1-\lambda_S)X_1 \geq \bar{p}_n((1-\lambda_S)X_n + \lambda_S), \tag{13}$$

which binds in any equilibrium. Re-arranging gives the stable price predictions in (14).

$$\bar{p}_i = v\frac{X_1}{X_i} \quad \text{if } i \in \{1,\ldots,n-1\} \quad \text{and} \quad \bar{p}_n = v\frac{(1-\lambda_S)X_1}{(1-\lambda_S)X_n + \lambda_S}. \tag{14}$$

This updates the predictions of (3) to include shoppers. The affect shoppers have on equilibrium prices with our two-stage pricing is simple: the lowest price is adjusted downwards according to how many shoppers there are, while all other prices are independent of shoppers.

In contrast, the (necessarily mixed-strategy) equilibria of the corresponding single-stage pricing models spread the affect of shoppers across multiple firms, and in the case of the (most commonly studied) symmetric equilibrium, all firms. Such (random) equilibrium pricing strategies means every mixing firm thinks about the fact they may be cheapest and so may serve shoppers. This encourages them to place some of their pricing distribution's probability mass at low prices. On the other hand, they do not want to price too low, because then they would rather give up on the shoppers and charge the monopoly price to (captive) buyers who receive a quotation only from them. This trade-off leads to the mixed-strategy pricing in equilibrium.

[24] The assumption that $\lambda_S > 0$ has been justified by appealing to the fact that some people enjoy shopping (Stahl, 1989) or have no opportunity cost of time (Janssen and Moraga-González, 2004).

[25] In the trivial case that all buyers are shoppers, i.e., $\lambda_S = 1$, the Bertrand (zero-profit) outcome follows.

That tradeoff is there in our model too, but our two-stage approach allows for it to be resolved through no-undercutting constraints, leading to the equilibrium play of pure strategies.

Optimal search protocols in the presence of shoppers. The same arguments as without shoppers ensure that at least one of μ_0 and μ_1 is strictly positive in any equilibrium. But now we cannot have $\mu_0 = 1$ because then shoppers are the only active buyers so prices are competed to zero, so searchers would strictly prefer to take one quotation, a contradiction (and so we rule out case 1 of Lemma 3). This implies $\mu_1 > 0$ in any equilibrium.

This means that the set of prices charged by firms, specified in (14) contains more than one price, and so the distribution from which searchers draw is non-degenerate. This means there are typically strictly decreasing gains to search. In combination with the constant and common marginal cost of search, it implies that no two searchers can obtain two (or more) quotations *more* than any other searcher.[26] We collect these observations in Lemma 5.

Lemma 5 (Number of Quotations in Equilibrium with Shoppers). *In equilibrium $\mu_q = 0$ for $q \geq 3$, and $\mu_1 > 0$: searchers gather at most two quotations, and some gather exactly one. In terms of μ_0 and μ_2, there are three possibilities for equilibrium searcher behavior:*

$$(i)\ \mu_0 \in (0,1) \text{ and } \mu_2 = 0; \quad (ii)\ \mu_0 = 0 \text{ and } \mu_2 = 0; \quad (iii)\ \mu_0 = 0 \text{ and } \mu_2 \in (0,1). \tag{15}$$

These potential equilibrium searcher strategies are exactly those found with the single-stage pricing game. Janssen and Moraga-González (2004) term them the “low”, “moderate”, and “high” search intensities, respectively. We also adopt that terminology.

Low and Moderate Intensity: Searching Once or Never. We search for equilibria in which $\mu_0 + \mu_1 = 1$ and $\mu_1 > 0$. In this case, $\mu_q = 0$ for $q > 1$ and we have

$$(1-\lambda_S)X_1 = (1-\lambda_S)X_i = (1-\lambda_S)\mu_1/n \quad \text{for all } i, \tag{16}$$

which, using (14), implies prices are given by

$$\bar{p}_1 = \cdots = \bar{p}_{n-1} = v \quad \text{and} \quad \bar{p}_n = v\frac{(1-\lambda_S)X_1}{(1-\lambda_S)X_1 + \lambda_S}. \tag{17}$$

In this setting, the only buyers in the market (i.e., excluding any getting zero quotations) are “captives” (with singleton consideration sets) and shoppers. In this captive-shopper setting with symmetric firms, the symmetric mixed-strategy best-replies in single-stage pricing games are famously derived by Varian (1980). In addition to the symmetric equilibrium, there is also an uncountable infinity of asymmetric (mixed-strategy) equilibria (Baye, Kovenock, and de Vries, 1992). In contrast, there is a unique profile of stable prices that emerges from two-stage pricing in the (symmetric) captive-shopper setting, as characterized by (17).[27]

[26] A special case is found with $n = 2$, where we can construct an equilibrium in which $\mu_0, \mu_1, \mu_2 > 0$. In that case, searchers collecting two prices are functionally equivalent to shoppers from the persepctive of firms.

[27] In settings with asymmetries in captive shares and single-stage pricing, see Baye, Kovenock, and de Vries (1992, Section V). For a treatment allowing asymmetries in captive shares and marginal costs covering both single-stage and two-stage pricing, see Myatt and Ronayne (2023a).

Turning to optimal search, a quotation costs κ and earns surplus $v - \bar{p}_n$ if firm n supplies the quotation, which happens with probability $1/n$, and earns zero surplus if they do not find firm n's quotation. The fact that a buyer is searching for a single lowest price means that the gains from search (just like the costs) are linearly increasing in the number of quotations q; the probability that q searches without replacement find the cheapest firm is q/n. This means that if it is strictly preferred to search for one quotation rather than none, then it is strictly optimal to search for quotations from all n suppliers. To construct an equilibrium with low or moderate search therefore requires a searcher to be exactly indifferent between searching and not. Solving for μ_1 and checking that $\mu_1 \in (0, 1]$ yields the following result.

Lemma 6 (Equilibrium with Low or Moderate Search). *If $n < v/\kappa \leq n - 1 + (1/\lambda_S)$ then there is an equilibrium in which the following proportion of searchers get a single quotation*

$$\mu_1 = \frac{\lambda_S}{1 - \lambda_S}\left(\frac{v}{\kappa} - n\right), \tag{18}$$

while the $1 - \mu_0$ others do not search, and firms set prices as per (17), which can be written

$$\bar{p}_1 = \cdots = \bar{p}_{n-1} = v \quad \text{and} \quad \bar{p}_n = \frac{v\mu_1(1 - \lambda_S)}{\mu_1(1 - \lambda_S) + \lambda_S n}. \tag{19}$$

Moderate search intensity (with $\mu_1 = 1$) obtains only for a degenerate set of parameters.

Janssen and Moraga-González (2004) defined a "moderate search intensity" equilibrium as one in which every buyer obtains exactly one quotation, so that $\mu_1 = 1$. For generic parameters (the exception is when $n - 1 + (1/\lambda_S) = v/\kappa$) there is no such equilibrium.[28] Therefore, our solution corresponds to their "low search intensity" equilibrium. Interestingly, they (in their Propositions 4–5) established the existence of such an equilibrium when n is large. In contrast, we also require n to be sufficiently small.[29] The equilibrium property that only one firm chooses a price lower than v means that buyers are searching for the proverbial needle in the haystack. When n is large enough, even for low cost-to-value ratios, buyers prefer not to search at all.

More generally, the fraction of those who search is decreasing in n. An increase in n also pushes down the lowest price while making the highest prices (equal to v) more frequent, and so makes the distribution of prices riskier while the average price remains constant.[30]

However, perhaps the most important observations concern the fraction of shoppers who exogenously see all prices. Suppose $n < v/\kappa \Leftrightarrow \kappa < v/n$ (so that a buyer would search once if $\lambda_S \downarrow 0$; note $\lim_{\lambda_S \downarrow 0} \bar{p}_n = 0$ so buyer surplus from finding firm n's price goes to v, so

[28] Our game (with pure-strategy play) rules out Propositions 1–3 of Janssen and Moraga-González (2004) which correspond to their moderate-intensity equilibrium. In their world, the use of single-stage pricing and hence mixed strategies means there are many different possible prices. This implies that there are decreasing returns to search, which contrasts with the linearity here. One way to recover existence for a non-degenerate set of parameters would be to assume that search costs are strictly increasing, rather than constant.

[29] The number of firms cannot be too small (because of the second inequality in Lemma 6): if $v/\kappa > (1+\lambda_S)/\lambda_S$ then that second inequality fails for n small enough. This is because too few firms can make the probability that a searcher finds the low price so high that all searchers wish to search. The fact that the low price rises as n falls tempers this effect, but does not undo it.

[30] These results reinforce those reported in Proposition 5 of Janssen and Moraga-González (2004).

the expected marginal benefit from search goes to v/n). If λ_S is small, then the inequality $v/\kappa < n-1+(1/\lambda_S)$ required for existence holds. However, as this fraction becomes negligible trade collapses: $\mu_1 \downarrow 0$ as $\lambda_S \downarrow 0$ i.e., endogenous search falls to zero as the exogenous search is removed from the model. In essence this says that a low search intensity equilibrium is a no search equilibrium. Equilibria in which searchers gather at most one quotation require there to be buyers who are willing to search for exogenous reasons.

Equilibria with Search and Shoppers. We have shown that lower (at most one quotation) search intensities either are not chosen in equilibrium or involve negligible search. This leaves us with the case (iii) of Lemma 5, i.e., high search intensity. That was the only possibility for buyer behavior in an equilibrium with search in the model without shoppers (Lemma 3).

Expanding the analysis to include shoppers is therefore straightforward because of stable prices. On the buyer side, we can just focus on searchers gathering one or two quotations. And as discussed, (14) says us we only need to adjust $\bar{p}_n$ given that some searchers gather one price and others two, to incorporate shoppers on the firm side.[31] Solving for the equilibrium then proceeds in exactly the same way as described in Section 7. There are differences in the equilibrium quantities, but no qualitative results change. Proposition 7 summarizes.

Proposition 7 (The Effect of Shoppers). *Suppose $\lambda_S \in (0,1)$ of buyers are shoppers, while $1-\lambda_S$ are searchers. For generic parameters, any equilibrium with search features searchers obtaining one or two quotations. Otherwise, the results of Propositions 4 to 6 apply.*[32]

As discussed earlier for the case of $\lambda_S = 0$, stable prices imply a buyer's benefit of a second search is a function of n. The presence of shoppers had a negligible impact on our analysis because the effect of the shoppers is completely absorbed by the firm with the lowest price, as they are the firm that serves them. The corresponding single-stage analysis finds that the (symmetric and mixed-strategy) pricing distribution depends on n, and hence so does the buyers' benefit from a second search (Janssen and Moraga-González, 2004). The reason for this is that the size of shoppers' consideration set is n, and so, as the number of firms increases, so does (shoppers') search behavior. Again, predictions under stable prices and symmetric mixed strategies are distinct. For example, in a duopoly we have two distinct price points while with $n \to \infty$ we have a distribution of prices laid out between those exact two duopoly prices. In contrast, Janssen and Moraga-González (2004) show that the symmetric mixed-strategy resulting from a single-stage game is exactly the same with duopoly and with many firms, leading to results regarding the how expected prices vary with entry (their Proposition 8).

[31]Specifically, the lowest price becomes $\bar{p}_n = \frac{v\mu_1(1-\lambda_S)}{(2-\mu_1)(1-\lambda_S)+\lambda_S n}$, which converges to $\bar{p}_n$ in (5) as $\lambda_S \to 0$. For a depiction of how prices differ, in the appendix we illustrate the distribution of prices in the presence of shoppers in Figure A0, which can be compared to the upper panel of Figure 1.

[32]One quantitative adjustment is required if $\lambda_S > 1/3$. In that case, Proposition 4 instead holds for $n = 2,3$ and Proposition 5 instead holds for $n \geq 4$.

9. Heterogeneous and Convex Search Costs

In the preceding sections we assumed that each buyer's search cost is the same (homogeneity) and that the marginal cost of search is constant (linearity). Here we examine how things change when each buyer's search costs are heterogeneous and convex over the first two searches.

We assume that each buyer b faces costs $\kappa_b(j)$ for their j^{th} search such that $\kappa_b(2) \geq \kappa_b(1) > 0$ and $\kappa_b(j)$ for all $j > 2$ is sufficiently high such that no buyer searches more than two times.[33] Doing so allows us to build naturally on the analysis of the previous sections, which is nested at $\kappa_b(1) = \kappa_b(2) = \kappa = K^{-1}$ for all b. To model the heterogeneity of $\kappa_b(2)$, we assume it is drawn from some continuous distribution K the support of which is some interval $[\underline{\kappa}, \bar{\kappa}] \subseteq (0, \infty)$.

These assumptions do not affect the results that buyers continue to search once or twice in any equilibrium with search, and therefore also do not affect firms' pricing. As such, we can look for equilibria with search as before, by looking for values of μ_2 at which

$$B_n = K^{-1}. \tag{20}$$

Lemma 4 is also unaffected by the search cost heterogeneity and tells us that stable equilibria are found where B_n cuts K^{-1} from above. By considering the properties of these functions, we can arrive at several observations regarding the set of equilibria.

Firstly, if there are equilibria with search, at least one is unstable (because $K^{-1}(0) > 0$). Secondly, conditions which together imply the existence of a stable equilibrium with search are that (i) there is an unstable equilibrium with search (which guarantees B_n cuts K^{-1} at least once); and (ii) some buyers have sufficiently high search costs (to guarantee B_n cuts K^{-1} a second time, necessarily from above).[34] We collect these observations in Proposition 8.

Proposition 8 (Heterogeneous and Convex Search Costs). *If an equilibrium with search exists and some buyers have sufficiently high second-search costs, then there is at least one equilibrium with search that is stable.*

First Search for Free. A popular assumption for search models is that the first search is free (and all buyers make that search). This removes equiliibria without search (case 1 of Lemma 3).[35] We adopt that too, assuming $\kappa_b(2) \geq \kappa_b(1) = 0$, and that the lower bound of the distribution of costs for the second search is zero, i.e., $\underline{\kappa} = 0$. For convenience we also assume K is continuously differentiable so that it has a continuous density function, k.

Because of the properties of B_n (Proposition 3), we can find properties on the distribution of search costs that dictate equilibrium existence and uniqueness. Specifically, so long as the slope of K^{-1} at zero (denoted by $K^{-1\prime}$) is not too steep (equivalently, the density at zero,

[33] For example, $\kappa_b(j) > v/2$ would be sufficient because B_n is bounded from above by $v/2$.

[34] For example, $\bar{\kappa} > v/n$ would satisfy (ii) because $B_n = v/n$ at $\mu_2 = 1$.

[35] This means that there is search in any equilibrium. The cases with no search are replaced by Diamond equilibria in which all buyers search exactly once.

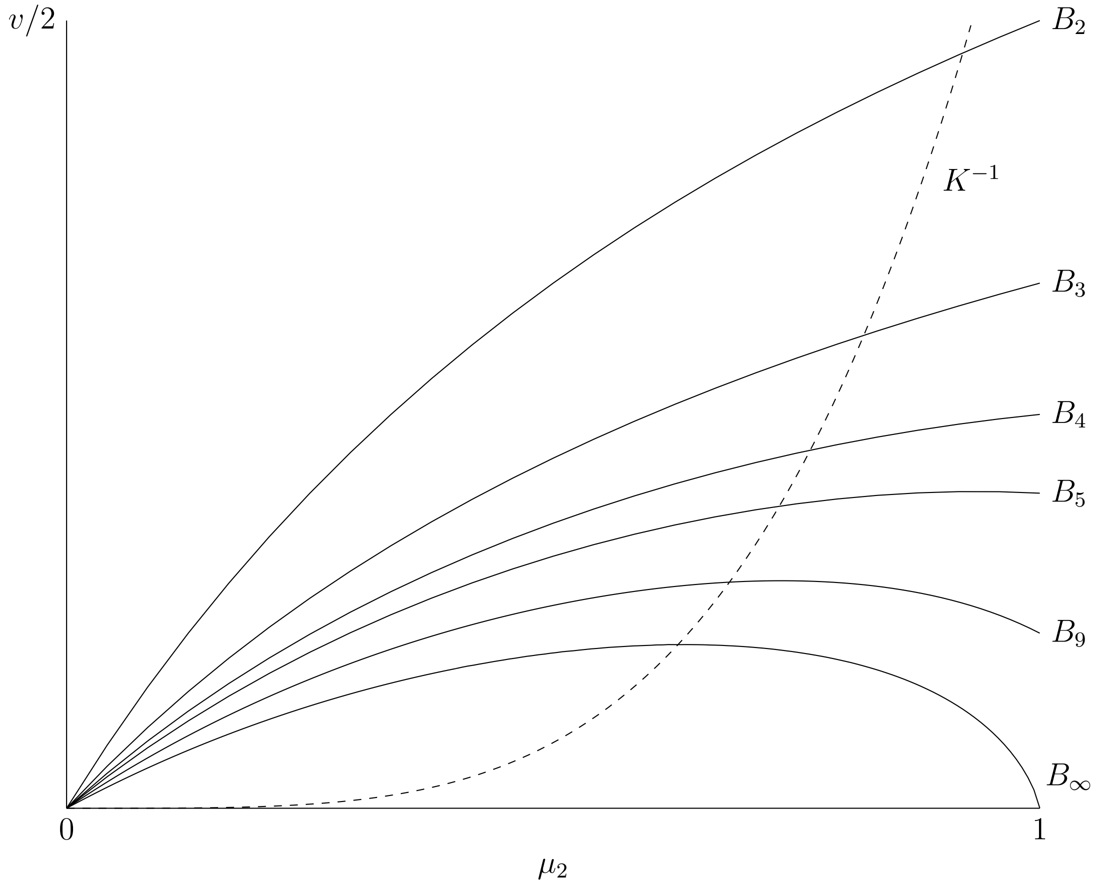

FIGURE 3. Equilibrium with heterogeneous and convex search costs. The second search's benefit, B_n, for $n \in \{2, 3, 4, 5, 9, \infty\}$ is shown by solid lines. An example inverse CDF of second-search costs satisfying the conditions of Proposition 9 is shown with the dashed line: there is a unique and stable equilibrium with search.

$k(0)$, is large enough) and $K^{-1}(1)$ is sufficiently large, then there exists at least one equilibrium with search, where at least one is stable.[36] We loosely interpret these as requiring there to be sufficiently many buyers with low and at least some with high second-search costs, respectively. In other words, they require sufficient heterogeneity in second-search costs.

If, in addition, the distribution of second-search costs, K, is concave (so K^{-1} is convex), then there is a unique equilibrium with search. We collect these observations in Proposition 9. An example distribution satisfying these assumptions and requirements is illustrated in Figure 3.

Proposition 9 (Heterogeneous and Convex Costs with First Search for Free). *Suppose buyers' first search is free and the distribution of second-search costs satisfies* $\underline{\kappa} = 0$.

If ${K^{-1}}'(0) < v/3$ *and* $K^{-1}(1) > v/n$, *then a stable equilibrium with search exists.*

If, in addition, K is concave, then there is a unique and stable equilibrium with search.

[36] We continue to refer to "equilibria with search" and when the first search is for free we mean that to refer to equilibria in which $\mu_2 > 0$.

Appendix: Omitted Proofs

This appendix provides proofs for results not fully proved in the main text followed by supplementary material concerning the extension with shoppers (Figure A0 and Proposition A1).

The proofs below allow for any proportion of "shoppers" $\lambda_S \in [0,1)$, as introduced to the analysis in Section 8, and therefore nest the results of sections where $\lambda_S = 0$.

Proof of Proposition 2. Because $\mu_q = 0$ for $q > 2$, (1), adjusted to include shoppers simplifies:

$$X_i = (1-\lambda_S)\left(\frac{\mu_1}{n} + \frac{2\mu_2(i-1)}{n(n-1)}\right). \tag{A1}$$

Along with $X_1 = (1-\lambda_S)\mu_1/n$, we can substitute terms into (14) to find the prices stated.

Now consider $n \to \infty$. The proportion of shoppers, λ_S, only impacts the lowest price. As n grows large, the probability of any given price being chosen falls to zero and so the impact of shoppers disappears. As such we set $\lambda_S = 0$. Take any price p within the interval bounded by these highest and lowest prices, $\bar{p}_1$ and $\bar{p}_n$, and write $F_n(p)$ for the cumulative distribution function of prices. For finite n,

$$\begin{aligned}
F_n(p) = \frac{n-i}{n-1} \quad \Leftrightarrow \quad \bar{p}_{i-1} > p \geq \bar{p}_i \quad \Leftrightarrow \\
\frac{\mu_1(n-1)v}{\mu_1(n-1)+2(1-\mu_1)(i-2)} > p \geq \frac{\mu_1(n-1)v}{\mu_1(n-1)+2(1-\mu_1)(i-1)} \\
\frac{i-2}{n-1} < \frac{\mu_1}{1-\mu_1}\frac{v-p}{2p} \leq \frac{i-1}{n-1} \quad \Leftrightarrow \quad i = \left\lceil (n-1)\frac{\mu_1}{1-\mu_1}\frac{v-p}{2p} \right\rceil + 1,
\end{aligned} \tag{A2}$$

where "$\lceil\cdot\rceil$" means "the least integer weakly greater than." Hence

$$F_n(p) = 1 - \frac{1}{n-1}\left\lceil (n-1)\frac{\mu_1}{1-\mu_1}\frac{v-p}{2p} \right\rceil, \tag{A3}$$

converges to $F(p)$ as reported, as $n \to \infty$. Let the distribution of the minimum of two random draws from $F(\cdot)$ be $G(p) = 1-(1-F(p))^2$. Taking expectations appropriately, we can compute $B_\infty = \mathrm{E}_F[p] - \mathrm{E}_G[p]$ to give the expression stated. □

Proof of Corollary 1. The claim regarding the effect of μ_1 (or μ_2) on prices holds by inspection.

For the median price, suppose that n is odd. The median firm i satisfies $i-1 = (n-1)/2$. Applying the pricing solution for this firm yields $\bar{p}_i = \mu_1 v$ as claimed. Similarly if $n+3$ is a multiple of 4 then the $i-1 = (n-1)/4$ and $i-1 = 3(n-1)/4$ may be applied accordingly to recover the $\frac{n+3}{4}^{\text{th}}$ and $\frac{3n+1}{4}^{\text{th}}$ highest price, respectively.

The final claims follow because each firm is indifferent between the specified list price and charging v to captives, and because the average price paid equals industry profit. □

Proof of Proposition 3. To see some of the properties of B_n, it is instructive to rearrange it as

$$B_n = \tfrac{1}{n(n-1)}\left[(n-1)(\bar{p}_1 - \bar{p}_n) + (n-3)(\bar{p}_2 - \bar{p}_{n-1}) + \dots\right], \tag{A4}$$

so that we can see B_n as the weighted sum of decreasingly-less-polar pairs of prices. Defining $f(i) = \bar{p}_i - \bar{p}_{n-i+1}$ for $1 \le i \le n/2$, we have

$$\frac{d^2 f(i)}{d\mu_1^2} = 4v(n-1)m\left(\frac{1-i}{(2(i-1)+\mu_1 m)^3} + \frac{i-n}{(2(n-i)+\mu_1 m)^3}\right), \tag{A5}$$

where $m \equiv n - 2i + 1 > 0$. Both terms in parentheses are negative with the second strictly so; f is strictly concave in μ_1. Thus, B_n is a sum of functions that are strictly concave in μ_1.

The term B_n is strictly positive because $f(i)$ is strictly positive for $\mu_1 \in (0,1)$.

The limit properties stated in the proposition are discussed in the main text. □

Proof of Proposition 5. An equilibrium with search is stable if and only if the benefit of a second search, B_n, cuts κ from below (Lemma 4). That can only happen if B_n is upward sloping for some $\mu_1 \in (0,1)$. Because B_n is strictly positive and concave for $\mu_1 \in (0,1)$ with $\lim_{\mu_1 \uparrow 1} B_n = 0$ (Proposition 3), if B_n is upward sloping anywhere in $(0,1)$, it is upward sloping when $\mu_1 \downarrow 0$:

$$\lim_{\mu_1 \downarrow 0} \frac{dB_n}{d\mu_1} > 0 = \frac{v}{2}\frac{n-1}{n}\left(H_{n-1} - 2\right) > 0, \tag{A6}$$

where $H_{n-1} = \sum_{i=1}^{n-1} i^{-1}$ is the $(n-1)^{\text{th}}$ harmonic number. Evaluating for $n = 3$ and 4 gives $H_3 = 11/6$ and $H_4 = 25/12$, such that $n = 5$ is the first natural number to satisfy $H_{n-1} > 2$ and hence (A6). The function H is strictly increasing in its argument. It follows that for all $n \ge 5$ the (unique) maximizer of B_n is some $\mu_1 \in (0,1)$ and that there is some open set of values for B_n for which B_n is upward sloping in μ_1 (the set is open because (A6) is strict). That set of values is given by $(v/n, \kappa^*)$ where κ^* is the maximum of B_n. In other words, if $\kappa \in (v/n, \kappa^*)$, then there is a unique equilibrium with search where $\mu_1 \in (0,1)$ solves $B_n = \kappa$.

If $n \le 4$, B_n strictly decreases in $\mu_1 \in (0,1)$ and any equilibrium with search is unstable. □

Proof of Lemma 5. This follows from the argument in the main text. Janssen and Moraga-González (2004, p. 1112) presented in a formal argument in the proof of their Lemma 1. □

Proof of Lemma 6. As described in the text, equilibrium μ_1 solves $(v - \bar{p}_n)/n = \kappa$, where $\bar{p}_n$ is from (16) and (17). Checking the resulting $\mu_1 \in (0,1)$ gives the inequalities reported. □

Proof of Proposition 9. If K^{-1} is less steep than B_n at 0 and $K^{-1}(1)$ is sufficiently high, for all n, then we know there is at least some point in $(0,1)$ such that $K^{-1} = B_n$.

To derive the highest relevant upper bound on $K^{-1\prime}(0)$ we find the n with the least steep B_n as $\mu_2 \downarrow 0$. The relevant calculation is

$$\lim_{\mu_2 \downarrow 0} \frac{dB_\infty}{d\mu_2} = \frac{v}{3} \tag{A7}$$

For the lower bound on $K^{-1}(1)$, we find the B_n that is highest as $\mu_2 \uparrow 1$. We calculate

$$\lim_{\mu_2 \uparrow 1} B_n = \frac{v}{n} \tag{A8}$$

and so $n = 2$ corresponds to the lowest relevant lower bound on $K^{-1}(1)$. $\square$

Proof of Proposition A1. The argument follows from the main text. One note here is that if κ is exactly equal to the unique (by the strict concavity reported in Proposition 3) maximum of B_n, then one can construct the argument for stability in the case of a perturbation decreasing search away from an equilibrium, but not increasing search away from an equilibrium. Because stability is not available in both directions, that equilibrium is unstable. $\square$

Proof of Proposition A1. For $n = 2$, set κ equal to the gain from the second quotation:

$$\kappa = B_2 = \frac{v - \bar{p}_2}{2} = v\frac{1 - \mu_1(1 - \lambda_S)}{2 - \mu_1 + \lambda_S \mu_1} \quad \Leftrightarrow \quad \mu_1 = \frac{v - 2\kappa}{(v - \kappa)(1 - \lambda_S)}, \tag{A9}$$

which is in $(0, 1)$ when the stated constraints hold. Regarding the average price charged,

$$\bar{p}_1 = v \quad \text{and} \quad \bar{p}_2 = \frac{v\mu_1(1 - \lambda_S)}{2 - \mu_1 + \lambda_S \mu_1} = v - 2\kappa \quad \Rightarrow \quad \frac{\bar{p}_1 + \bar{p}_1}{2} = v - \kappa. \tag{A10}$$

For equilibrium with $n = 3$,

$$\kappa = B_3 = \frac{\bar{p}_1 - \bar{p}_3}{3} = \frac{v}{3}\left(\frac{2(1 - \mu_1) + \lambda_S(1 + 2\mu_1)}{2 - \mu_1 + 3\lambda_S}\right) \Leftrightarrow \mu_1 = \frac{(v - 3\kappa)(2 + \lambda_S)}{(2v - 3\kappa)(1 - \lambda_S)}. \tag{A11}$$

The solution for μ_1 is in $(0, 1)$ when the stated constraints hold. Substituting back into prices,

$$\begin{aligned} &\bar{p}_1 = v, \quad \bar{p}_2 = \frac{v(v - 3\kappa)(2 + \lambda_S)}{(2v - 3\kappa)(1 - \lambda_S)}, \quad \bar{p}_3 = v - 3\kappa \\ \Rightarrow &\frac{\bar{p}_1 + \bar{p}_2 + \bar{p}_3}{3} = v - \kappa - \frac{v(v\lambda_S + \kappa(1 + 2\lambda_S))}{(2v - 3\kappa)(1 - \lambda_S)} \quad \square \end{aligned} \tag{A12}$$

Supplementary Material for the Model with Shoppers

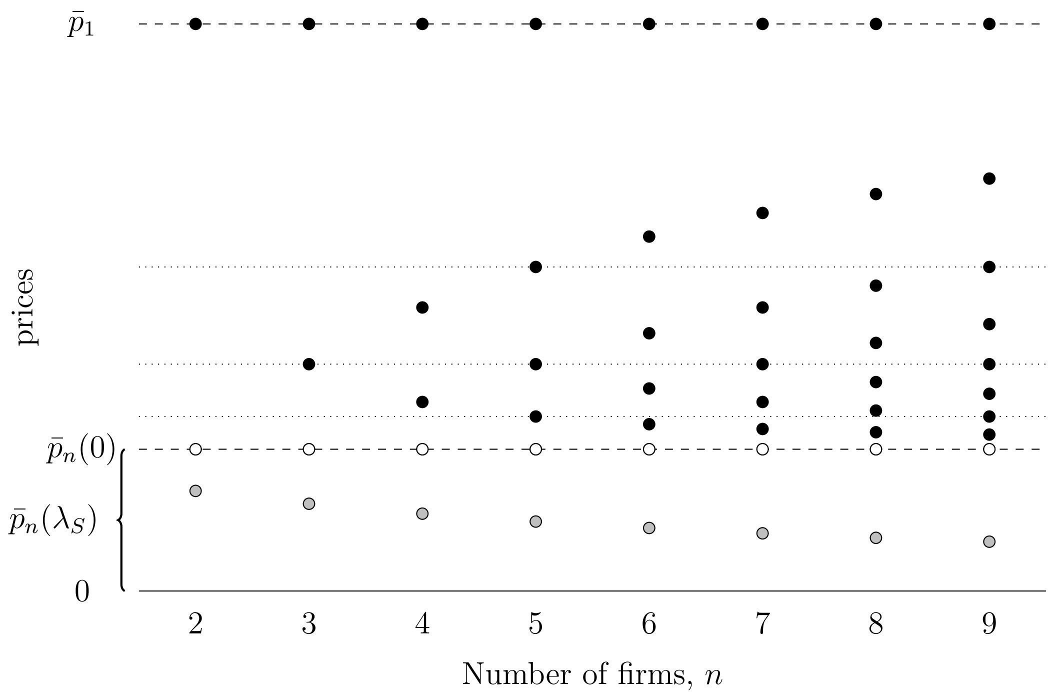

Figure A0. The distribution of prices with shoppers (who exogenously search for all prices), drawn for $\mu_2 = 3/5$ and $\lambda_S = 1/4$. The lowest price (marked as $\bar{p}_n(\lambda_S)$ with points in gray) varies with the proportion of buyers who are shoppers. All other prices ($\bar{p}_i$ for $i < n$) are unaffected by shoppers (for reference these are reproduced here, in black, along with hollow dots for $\bar{p}_n(0)$, the lowest price without shoppers, i.e., $\bar{p}_n$ in Figure 1). Figure 1's caption otherwise applies.

Proposition A1 (Equilibrium Search in a Duopoly and Triopoly). *Suppose a proportion $\lambda_S \in (0,1)$ of buyers are shoppers, while $1-\lambda_S$ are searchers.*[37]

For $n=2$ and $v/\kappa \in (2, 1+1/\lambda_S)$, there is a unique and unstable equilibrium with search where

$$\mu_2 = \frac{\kappa - \lambda_S(v-\kappa)}{(v-\kappa)(1-\lambda_S)}; \quad \sum_{i=1}^{2} \frac{\bar{p}_i}{2} = v - \kappa; \quad \textit{av. price paid} = v\mu_1 = \frac{v(v-2\kappa)}{(v-\kappa)(1-\lambda_S)}. \tag{A13}$$

Buyers are indifferent between 0,1, or 2 searches; they earn zero expected payoff.

For $n=3$ and $v/\kappa \in (3, 2+1/\lambda_S)$, there is a unique and unstable equilibrium with search where

$$\mu_2 = \frac{3(\kappa - \lambda_S(v-2\kappa))}{(2v-3\kappa)(1-\lambda_S)}; \quad \sum_{i=1}^{3} \frac{\bar{p}_i}{3} = v - \kappa - \frac{v\mu_2}{3}; \quad \textit{av. price paid} = \frac{v(v-3\kappa)(2+\lambda_S)}{(2v-3\kappa)(1-\lambda_S)}. \tag{A14}$$

[37]The applies also for $\lambda_S = 0$ except that the upper bound constraint on v/κ is not there.

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