EVOLUTION IN TEAMS

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Abstract. Team formation will often involve a coordination problem. If no-one else is contributing to a team, there is little point in an agent exerting any effort. Similarly, once a team is formed, an agent within the team will not leave, as to do so would result in team collapse; non-contributing agents would not join, as they currently receive the benefits of the team's efforts whilst paying none of the costs. The methods of the stochastic adjustment dynamics literature can help select between these equilibria. Team and population size, and cost and benefit parameters all play a role in determining the chances of successful team formation. Increasing the pool of agents from which to choose team members seems at first glance to have a positive impact upon team formation. However, just one "bad apple" within the extended pool can have a disproportionate effect on the outcome. Although an agent with high participation costs would never contribute to a successful team, their mere presence alone can result in the failure of an otherwise successful team.

1. Introduction

1.1. **Teamwork, Collective Action, and Public Good Provision.** Many economic and social activities require a number of agents (a "team") to participate in order to ensure the success of a given project or task. As a result, these kinds of activities frequently experience the well known problems associated with collective

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actions.[1] Focus here is directed at two such problems. First, such activities can involve a positive externality: Participation in a team leads to a private cost borne by the individual, but the benefits of the team's efforts accrue to all. Second, successful team formation requires coordination: Absent the participation of a sufficient number of individuals, the project as a whole would fail.[2]

Sporting activities provide one example of the need for coordination. Many sports are team-based, and without the presence of all participants, the game cannot take place. In May of each year Oxford colleges compete in a rowing tournament known as "Eights Week". A team's success in this tournament critically depends upon the ability of its eight members (hence the name) to train on a regular basis. Training sessions are typically conducted in the early morning and require the presence of the entire team — the absence of a single member results in cancellation. A rower will find it optimal to wake early and participate if and only if all the other team members do so. Of course, if seven of the rowers are expected to attend the training session, the eighth team member faces a strong incentive to attend — their attendance will enable the training (and potential success) of the entire team rather than merely the element contributed by an individual.

Another example of teamwork, this time involving a positive externality, is provided by local environmental projects. These projects benefit everyone, however, only those who give up their time to contribute to such projects bear any of the costs. The Oxford Conservation Volunteers (OCV) has been "carrying out practical work conserving the wildlife and traditional landscape of the Oxford area since 1977".[3] Their activities range from hedge-laying to the conservation of wildlife habitats via scrub clearance. The volunteers contribute time and energy (a private cost) to an activity which generates environmental benefits for all (a positive externality).

Both of these examples involve the provision of a public good.[4] According to its classic definition a pure public good is both non-rival and non-excludable. A good is non-rival if its consumption by one individual does not decrease the amount available

[1]The classic analysis of collective action problems appears in Olson (1965). The literature which followed is vast and continues to grow in both applied and theoretical directions, see for example, Bardhan, Ghatak, and Karaivanov (2002) and Marx and Matthews (2000) respectively.

[2]Both these problems have attracted a great deal of interest, and not only from theorists. For a summary of the experimental approach to these problems, see Ledyard (1995).

[3]A description of its activities is available from `http://www.ocv.org.uk/`.

[4]The now standard economic approach to the problem of public good provision is presented in the central contributions of Bergstrom, Blume, and Varian (1986,1992), and a useful diagrammatic exposition can be found in Ley (1996).

for the consumption of others. For instance, everyone is able to enjoy the pleasures of an improved environment. A good is non-excludable if all individuals are free to consume it once it has been provided. All members of an Oxford college are able to revel in the glory of coming in at the "Head of the River".[5]

Here, the focus is on the conditions under which a team might form and successfully provide public goods of this sort. In particular, how do team size, benefit and cost parameters, and the population from which the team is drawn affect the chances of success? In attempting to answer these questions, a game theoretic approach is taken.[6] However, in so doing, a fundamental problem of game theory arises.

1.2. **The Equilibrium Selection Problem.** A simple game is presented here, where each agent in a population must choose whether or not to contribute to a team. To do so is costly, but if sufficiently many other players also choose to contribute, the team is successful in its endeavours and a benefit (larger than private cost) is generated for the population as a whole. This benefit accrues to every member of the population, regardless of whether or not they are part of the team — this is the positive externality element of the game.

Furthermore, there is a coordination problem. If no-one else is currently contributing to the team, there is no incentive for an agent to contribute themselves, as they will bear a cost, but to no benefit. On the other hand, if sufficiently many players are contributing to form a successful team there will be no incentive for any of the agents to change their behaviour. Any agent currently involved in the team does not wish to stop contributing, as to do so would result in the team's collapse. Although they would then bear no cost, nor would they receive the benefit. Non-contributing agents have no incentive to join the team, as the private cost they would face generates no additional benefit.

Thus the game has multiple Nash equilibria. One where no agent contributes, and many where just enough agents participate to form a successful team. Notice that these latter equilibria involve higher payoffs to *all* the agents in the population. That does not mean this is the equilibrium that ought to be predicted, however. It is quite possible that, despite the greater payoffs available from successful team formation, the

[5]In a rowing regatta the overall winner is said to come in at the head of the river.
[6]The use of game theoretic concepts has become standard in the public goods literature, text books commonly employ such language to introduce the problem, for example Cornes and Sandler (1996).

population coordinates on the (perfectly rational and plausible) equilibrium where no team is formed.

Equilibrium selection between strict Nash equilibria of this sort has long been recognised as a problem in game theory.[7] Recent attempts to provide solutions to this problem include the global games literature and the evolutionary dynamics or stochastic adjustment literature.[8] The latter approach is the one taken here.[9]

1.3. **Evolution and Teamwork.** The stochastic adjustment dynamics literature provides a way to solve the equilibrium selection problem.[10] In such models a population of agents is repeatedly matched to play a normal form stage game. Every period agents are given the opportunity to revise their current strategy in the light of previous play. In particular, they might choose a best response to an observation of the history of play.

This leads to a dynamic process that is dependent upon initial conditions. To see this, consider an updating agent. With the team formation game outlined above in mind, a revising player will choose to contribute if and only if their contribution would just result in team formation. Otherwise the best response is non-contribution. Each updating agent faces a similar choice. The best response process will lead to one of the equilibria, but which one will depend on where the process began. Once at an equilibrium, no updating player will alter their current strategy and the process becomes "locked in". To circumvent this problem, noise is added to the process.[11] That is, some small probability of the process moving against the best response is introduced. Now the process is "ergodic" — history independent. As the noise is reduced to zero, the process spends almost all the time in one or other of the equilibria, and selection takes place.

[7] An important early approach is contained in Harsanyi and Selten (1988), their tracing procedure picks out the "risk dominant" equilibrium in simple games (roughly speaking, the equilibrium from which unilateral deviation is more costly). This need not be the Pareto superior equilibrium.
[8] The former began with the seminal work of Carlsson and van Damme (1993). For a summary of the literature and its results, see Morris and Shin (2003) or Myatt, Shin, and Wallace (2002).
[9] Myatt and Wallace (2002) provide an equilibrium selection argument based on the global games literature for collective action and public good provision problems similar to the one discussed here, although the specific application to teams is beyond the scope of that paper.
[10] The key contributions in this field are Kandori, Mailath, and Rob (1993) and Young (1993). Both provide further arguments for selection of the risk dominant equilibrium in simple 2×2 games.
[11] Broadly speaking, Kandori, Mailath, and Rob (1993) and Young (1993) do this by allowing agents to play a strategy which is not a best response with some small probability. This might be interpreted as experimentation on the part of the agents, or simply as mistakes.

One way of introducing such noise is to allow players to differ.[12] Individual agents may well have different costs of contributing to, and different valuations from, the public good that the team might produce. This is modelled in the following way: Each period an agent is chosen at random to update their strategy. Their payoffs are drawn from some distribution, the mean of which corresponds to the payoffs of the original unperturbed game. Given their idiosyncratic payoffs, the agents play a best response to the previous period's strategy profile. An ergodic distribution can be characterised for this stochastic process. As the variances of the distributions are allowed to collapse to zero, the payoffs tend to those of the underlying unperturbed game and the ergodic distribution places all weight on a particular equilibrium. Thus, selection takes place. A precise condition is presented in Proposition 1 which determines which of the equilibria will be played, and hence whether a team will successfully form.

Section 2 presents the argument formally. Then, in Section 3, the model is extended to the case of asymmetry. Prior to this, all agents drew their payoffs from identical distributions, implying symmetry for the underlying unperturbed game. However, players may have systematically different costs of contributions. When this is the case, a single "bad apple" (a player with a higher average cost of contribution) may destabilise an otherwise successful team. This will occur even though that player would rarely be found contributing in a successful team — it is merely their presence in the pool of *potential* contributors that matters. Ergodic distributions are calculated and presented that show how a team's success is critically dependent upon population size, noise, and the introduction of an asymmetry of this form.

2. The Basic Model

In this section, the basic model is developed. The simple teamwork game is described in Section 2.1 and the stochastic evolution dynamic employed in Section 2.2. The main analysis takes place in Section 2.3. Finally a short discussion of the main results of the section is contained in Section 2.4.

2.1. **The Game.**

There are n players, each with two pure strategies available to them — either "contribute" or "don't contribute". If $m \in \{2, \ldots, n\}$ players contribute then a successful *team* is formed and each of the n players receives a benefit equal to

[12] This is the approach taken in Myatt and Wallace (2003a) and extended to larger games in Myatt and Wallace (2003c). The methods of the latter paper can be brought to bear on the current problem, see Myatt and Wallace (2003b).

$v > 0$. Each contributing player pays a cost of $c \in (0, v)$. Denote player i's action as $z_i \in \{0, 1\}$ where 0 represents "don't contribute", and 1 represents "contribute".

An element z in the state space $Z = \{0, 1\}^n$ represents the strategy profile played. It is helpful to partition the state space into subsets $Z_k = \{z \in Z : \sum_{i=1}^{n} z_i = k\}$. These are the states where exactly k players contribute.

There is a Nash equilibrium where all players don't contribute and receive a payoff of zero. This is represented by the singleton set Z_0. No player has an incentive to contribute as they would pay a cost c to no benefit. There are also multiple Nash equilibria where exactly m players contribute and a successful team is formed. These are represented by all the states $z \in Z_m$, of which there are $\binom{n}{m}$. No non-contributing player has an incentive to contribute as this would only result in them paying a cost to no extra benefit. No contributing player has an incentive to stop contributing since by doing so the team would fail and they would receive a payoff of zero — less than their equilibrium payoff of $v - c$.

For any state z, define $z^{i+} = \{z' \in Z : z'_i = 1, z'_j = z_j \quad \forall j \neq i\}$, and $z^{i-} = \{z' \in Z : z'_i = 0, z'_j = z_j \quad \forall j \neq i\}$. These two states are (potentially) only different from z insofar as player i is now contributing or not contributing respectively. The payoff to player i may then be written as

$$u_i(z) = v\mathcal{I}\left\{\sum_{j=1}^{n} z_j \geq m\right\} - cz_i,$$

and the payoff difference ("contribution incentive"), as

$$\Delta u_i(z) = u_i(z^{i+}) - u_i(z^{i-}) = v\mathcal{I}\left\{z^{i+} \in Z_m\right\} - c,$$

where $\mathcal{I}(\cdot)$ is the indicator function. The latter expression summarises the incentives the player faces. When $\Delta u_i(z) > 0$ player i will contribute, and when $\Delta u_i(z) < 0$ they will not. Using this notation, the set of Nash equilibria becomes

$$Z^* = \{z \in Z : (2z_i - 1)\Delta u_i(z) \geq 0 \quad \forall i\} = Z_0 \cup Z_m.$$

2.2. **The Dynamic.** In order to select between the multiple Nash equilibria present in this game, a simple one-step-at-a-time best-response dynamic is proposed. The idea is that at any time t, an individual agent is chosen randomly from the population (with probability $1/n$) and allowed to revise their current strategy choice. They observe the current population state, z_t, and choose a best response to the strategies being played. Having done so they are returned to the population along with their new strategy choice and the process repeats itself. Hence, $z_{t+1} \in \{z_t^{i+}, z_t^{i-}\}$. In fact,

$$z_{t+1} = \begin{cases} z_t^{i+} & \text{if} \quad \Delta u_i(z_t) > 0, \\ z_t^{i-} & \text{if} \quad \Delta u_i(z_t) < 0. \end{cases}$$

Notice that $\Delta u_i(z) \neq 0$ for all z. The dynamic, as it stands, is path dependent. If the initial state happens to be $z \in Z_k$ where $k \geq m$, a successful team will continue to operate and a Nash equilibrium in Z_m will be reached. If, on the other hand, $k < m - 1$, all contributors will eventually stop contributing and no team will form — the Nash equilibrium in Z_0 arises. Finally, when $k = m - 1$, which equilibrium is reached depends critically upon the first player chosen to revise their strategy. If it is a contributor, they will stop contributing and the team will collapse. If a non-contributor is selected to revise, they will choose to contribute, forming a successful team which then continues. Either way, the set of rest points of the dynamic coincide with the set of Nash equilibria and the equilibrium selection problem remains.

Ergodicity is required to solve this problem. Here this is achieved by the addition of a randomly drawn idiosyncratic element in the players' payoffs. In particular, rather than assuming that payoffs are fixed and known, it is assumed that the individual costs and benefits from participating in a team are to some extent different across players. To model this, suppose that a revising player i has a cost parameter $\tilde{c}_i$ and a benefit parameter $\tilde{v}_i$ such that:

$$\tilde{c}_i \sim N(c, \varepsilon\xi^2), \quad \text{and} \quad \tilde{v}_i - \tilde{c}_i \sim N(v - c, \varepsilon\kappa^2). \tag{1}$$

This suggests a new payoff difference of $\Delta\tilde{u}_i(z) = \tilde{v}_i \mathcal{I}\{z^{i+} \in Z_m\} - \tilde{c}_i$. Now, a revising player would contribute if $\Delta\tilde{u}_i(z) > 0$ and would not whenever $\Delta\tilde{u}_i(z) < 0$. ε here is a scaling factor. When $\varepsilon = 0$, the model reduces to the underlying stage game described in Section 2.1.

As long as $\varepsilon > 0$, the dynamic process described above is ergodic. To see this, notice that a revising player i will find it optimal to choose $z_i = 0$ or $z_i = 1$ with positive probability from any state $z \in Z$. This is guaranteed by the full support of the Normal distribution. Hence the process can move "up" or "down" one state (or stay in the same place). Therefore any state can be reached from any other in finite time and with positive probability. Moreover, since there is positive probability of remaining in the same state, no cycles can become established. These facts ensure that the process is ergodic — see, for example, Grimmett and Stirzaker (2001).

Ergodicity guarantees a unique long-run distribution. Section 2.3 calculates the transition probabilities between states for this process and thus the ergodic distribution. Of course, with $\varepsilon > 0$, the distribution will place positive probability on each of the states in Z. However, as $\varepsilon \to 0$, and the "trembled" game approaches the underlying game of interest, more and more weight in the ergodic distribution will be concentrated on the states corresponding to Nash equilibria. As a result, a condition arises which allows selection between these multiple equilibria. The condition yields an insight into the sorts of factors that allow a successful team to arise and persist.

2.3. **Analysis.** The first task is to calculate the transition probabilities. In other words, the probability of moving from one state z at time t to another state z' at time $t+1$ is required.

Suppose that the process is currently at state $z \in Z_k$ (the process is in the "kth layer") and that $k \leq n-1$. The process can rise a layer to a state in Z_{k+1} only if a non-contributor is selected to revise their strategy *and* that player chooses to contribute as a result. The former occurs with probability $(n-k)/n$ since there are currently $n-k$ non-contributors in the population. The latter occurs with a probability dependent upon whether $k = m-1$ or not.

First, suppose $k = m-1$. In this instance, it only takes one more contributor to form a successful team. Therefore, if there were no noise ($\varepsilon = 0$), a revising player i would find it a best response to set $z_i = 1$ (to contribute). However, when $\varepsilon > 0$, this will only happen with high probability. If the revising agent chooses to continue with their current strategy ($z_i = 0$), they will receive a zero payoff. If they choose to contribute they will receive $\tilde{v}_i - \tilde{c}_i$, since a successful team forms. Now this payoff is drawn from a Normal distribution as described in Equation (1). Hence they will choose to contribute with probability

$$\Pr[\tilde{v}_i - \tilde{c}_i > 0] = \Phi[(v-c)/\kappa\sqrt{\varepsilon}],$$

where $\Phi[\cdot]$ is the cumulative density function of the standard Normal distribution.

In a similar way, the probability of a non-contributor choosing to contribute when $k \neq m-1$ can be calculated. In this case, the agent must act "against the flow of play", choosing an action which would *not* be a best response in the underlying stage game. When $\varepsilon > 0$, however, there is some small probability that such an action would be a best response — given by $1 - \Phi[c/\xi\sqrt{\varepsilon}]$. Hence,

$$\Pr[z_{t+1} \in Z_{k+1} \,|\, z_t \in Z_k] = \frac{n-k}{n} \begin{cases} 1 - \Phi[c/\xi\sqrt{\varepsilon}] & \text{if } k \neq m-1, \\ \Phi[(v-c)/\kappa\sqrt{\varepsilon}] & \text{if } k = m-1. \end{cases} \tag{2}$$

Similarly, the probability of dropping a layer when $k \geq 1$ can be calculated. In this instance, a *contributor* must be randomly selected (which occurs with probability k/n) and they must choose to stop contributing. This will be a best response with one minus the probabilities shown above. Therefore,

$$\Pr[z_{t+1} \in Z_{k-1} \,|\, z_t \in Z_k] = \frac{k}{n} \begin{cases} \Phi[c/\xi\sqrt{\varepsilon}] & \text{if } k \neq m, \\ 1 - \Phi[(v-c)/\kappa\sqrt{\varepsilon}] & \text{if } k = m. \end{cases} \tag{3}$$

Of course, a revising player may also decide to keep their action unchanged. In this case, the state remains within the same layer. These probabilities can be calculated in a way analogous to above, but they are not required for the analysis. Note that the players' identities do not enter the transition probabilities — only the aggregate number contributing and not contributing play a role. A more general framework would allow for such asymmetries across players, and is required for the analysis of Section 3.

For the moment, however, the symmetry allows a simplification of the state space into the layers. This "reduced" state space may be written

$$\mathcal{Z} = \{Z_0, \dots, Z_n\}.$$

Figure 1 illustrates this reduced-form state space for a simple example with $n = 3$. The arrows represent the possible transitions up and down the layers. The probabilities calculated in Equations (2) and (3) can be associated with each of these arrows. In addition it is always possible to remain in any given state. The next step is to use the transition probabilities to deduce the ergodic (long-run) distribution of this stochastic process.

Having concluded above that the process is indeed ergodic, there will exist a set of probabilities $[\pi^z]_{z \in Z}$ such that $\pi^z = \lim_{t\to\infty} \Pr[z_t = z]$, where $\sum_{z\in Z} \pi^z = 1$. Using the reduced state space, define

$$\pi_k = \sum_{z \in Z_k} \pi^z = \lim_{t\to\infty} \Pr[z_t \in Z_k].$$

This is the ergodic (long-run) probability of the process being in the layer where k agents contribute. Again, $\sum_{k=0}^{n} \pi_k = 1$. The reduced Markov process induced by the

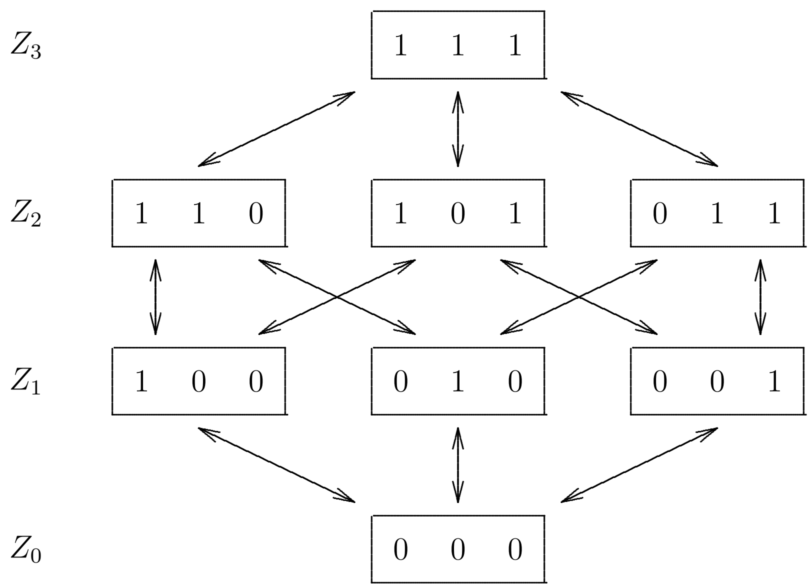

FIGURE 1. Z for $n = 3$ and $m = 2$.

dynamic moves "one-layer-at-a-time". The following "detailed balance conditions" are readily calculated:

$$\begin{aligned}
\pi_0 &= \pi_0 \Pr[z_{t+1} \in Z_0 \,|\, z_t \in Z_0] + \pi_1 \Pr[z_{t+1} \in Z_0 \,|\, z_t \in Z_1], \\
\pi_n &= \pi_n \Pr[z_{t+1} \in Z_n \,|\, z_t \in Z_n] + \pi_{n-1} \Pr[z_{t+1} \in Z_n \,|\, z_t \in Z_{n-1}], \\
\pi_k &= \pi_{k-1} \Pr[z_{t+1} \in Z_k \,|\, z_t \in Z_{k-1}] + \pi_k \Pr[z_{t+1} \in Z_k \,|\, z_t \in Z_k] \\
&\qquad + \pi_{k+1} \Pr[z_{t+1} \in Z_k \,|\, z_t \in Z_{k+1}],
\end{aligned}$$

where $0 < k < n$. These are intuitive conditions. In the long run, the probability of being in state k must be the probability of being in state k and staying there, plus the probabilities of being elsewhere and moving into state k. Straightforward algebra reveals that these equations can be solved to yield values for π_k. In fact,

$$\pi_k = \frac{q_k}{\sum_{j=0}^{n} q_j}, \text{ where } q_k = \prod_{j<k} \Pr[z_{t+1} \in Z_{j+1} \,|\, z_t \in Z_j] \times \prod_{j>k} \Pr[z_{t+1} \in Z_{j-1} \,|\, z_t \in Z_j].$$

Now consider the relative ergodic probabilities with $j < k$. It is easy to see that

$$\frac{\pi_j}{\pi_k} = \frac{q_j}{q_k} = \prod_{i=j}^{k-1} \frac{\Pr[z_{t+1} \in Z_i \,|\, z_t \in Z_{i+1}]}{\Pr[z_{t+1} \in Z_{i+1} \,|\, z_t \in Z_i]}. \tag{4}$$

The next step is to replace these probabilities with the ones in Equations (2) and (3), and then allow $\varepsilon \to 0$. The main result of this section follows from this exercise:

Proposition 1. *As $\varepsilon \to 0$ all weight in the ergodic distribution is concentrated on the pure strategy Nash equilibria in $Z_0 \cup Z_m$, and*

$$\frac{(v-c)^2}{\kappa^2} > (m-1)\frac{c^2}{\xi^2} \quad \Rightarrow \quad \lim_{\varepsilon\to 0}\left[\lim_{t\to\infty} \Pr[z_t \in Z_m]\right] = 1.$$

When the opposite (strict) inequality holds then $\lim_{\varepsilon\to 0}\left[\lim_{t\to\infty} \Pr[z_t \in Z_0]\right] = 1$.

Rather than prove this result formally at this stage, an intuition is offered based on Figure 1. Appendix A contains the formal proof.

Consider the simple four layer example illustrated in the figure. Selection depends upon the relative difficulty of moving between the different pure equilibria states of the game, at Z_2 and Z_0 in the example with $m = 2$. Suppose the process is currently in a state in Z_2. In order to get to $z \in Z_0$, a revising contributor must choose to stop contributing. This happens with probability

$$1 - \Phi\left[\frac{v-c}{\kappa\sqrt{\varepsilon}}\right].$$

For very small (but non-zero) ε this probability will be positive but tiny. The process is now in a state in the layer Z_1. From here it is easy to reach Z_0. All that is required is that a contributor be picked, which occurs with non-negligible probability (a third in this example), and that they then choose to stop contributing. For very small ε, this choice is a best response with a probability close to one.

Thus the "difficult" part of the journey occurs at the outset when a contributor chooses to stop contributing. The expression above naturally suggests that the term $[(v-c)/\kappa]^2$ acts as an index of this difficulty. As $\varepsilon \to 0$, this is the only part of the process that will matter.

An analogous story can be told starting in the Z_0 Nash equilibrium. Here, a non-contributor must revise their action against the flow of play, and choose to contribute. This will be a best response for them with probability

$$1 - \Phi\left[\frac{c}{\xi\sqrt{\varepsilon}}\right],$$

which again is close to zero when ε is small. Once this has happened the process is at a state in Z_1. With non-negligible probability a non-contributor will be chosen (two thirds in the example) and will find it a best response to contribute with high probability for small ε. Once again, the difficult step is the first, and $[c/\xi]^2$ indexes this difficulty.

A comparison of these two numbers reveals which of the two most difficult steps is *more* difficult than the other, and yields a selection result akin to that in Proposition 1. Of course, when m is larger there are many difficult steps to be taken on the way *up* through the layers, although it does not affect the difficulty of moving *down* (since only the first step has low probability). Thus for general m there are $m-1$ low probability steps to be made when moving from Z_0 to Z_m. Hence the index of difficulty becomes $(m-1)[c/\xi]^2$ and the result of the proposition obtains.

2.4. **Discussion.** Proposition 1 states that whenever

$$\frac{v-c}{\kappa} > \sqrt{m-1} \times \frac{c}{\xi}, \tag{5}$$

all weight in the ergodic distribution is concentrated in Z_m when ε gets vanishingly small. That is, a successful team evolves. If the inequality is reversed, a successful team will not evolve. Rather, the Nash equilibrium in Z_0 is selected where no agent contributes. In this section, the selection inequality in Equation (5) is examined.

Notice that n, the population size, does not play a role at all. The other parameters all enter in an intuitive way. The larger $v-c$ (corrected for its variability), the smaller c (again, corrected for variance), and the smaller m, the easier it becomes for a successful team to arise and persist in the long run.

The precise nature of the selection result obtained in the proposition arises when the limit is taken as $\varepsilon \to 0$. Of course, the ergodic distribution can still be calculated numerically for $\varepsilon \gg 0$. When this is done, n *will* play a role in the likely success of a team. Figure 2 illustrates this point. The ergodic distribution is shown in the graph for two different values of n.[13]

In the first graph, where $n = 20$, most weight in the ergodic distribution lies around the $z \in Z_0$ state. That is, nearly all of the time, no successful team is formed. A similar outcome is suggested by Equation (5) — with the parameters used for the figure, the left hand side equals 3 whilst the right hand side is 2. However, although the equation indicates that increasing n should have no impact on selection, once ε is large enough, the limiting result of Proposition 1 ceases to be informative. As can be

[13]The other parameters have values $v = 3, c = \xi = \kappa = 1, m = 10$, and $\varepsilon = 0.3$.

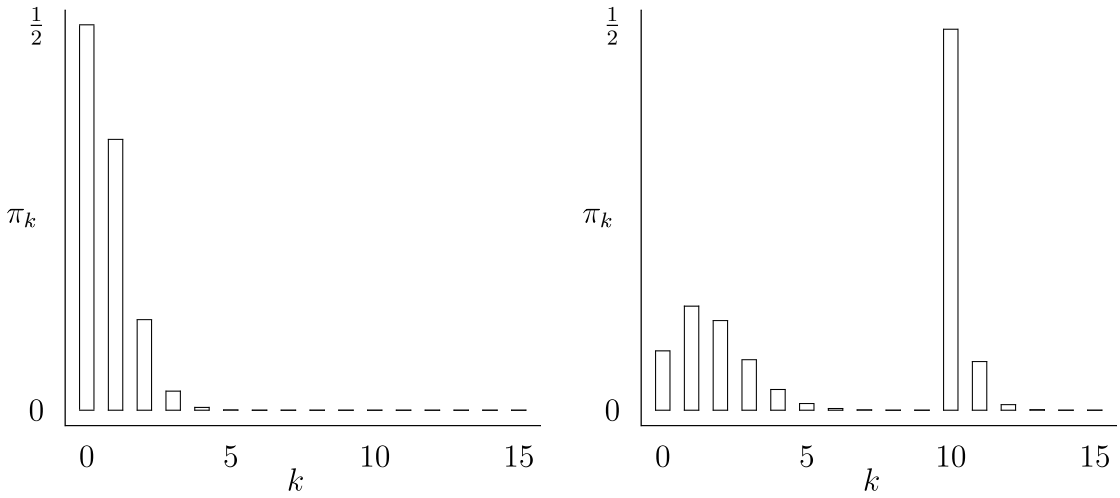

FIGURE 2. Ergodic Distributions for $n = 20$ and $n = 50$.

seen from the second graph in the figure, once $n = 50$, most weight is concentrated around the states $z \in Z_m$, where a successful team is operating.

Why is this? The intuition at the end of Section 2.3 needs modification. Away from the limit the probabilities of choosing non-contributors as opposed to contributors become important. Consider the states in Z_1. Only one agent is currently contributing. The chances of picking this player are much smaller as n gets large, and hence it becomes easier to move away from Z_0 and toward Z_m. A similar change of emphasis occurs at states close to Z_m. Here the team size, m, as well as its relative size to the population at large (m/n) will play a role. As a result, an increase in the population size increases the weight attached to the states in Z_m, making it more likely that a successful team will form. Essentially, there are more non-contributors to pick from and "many hands make light work".

The same point can be made in a slightly different way by fixing the population size and increasing ε. Figure 3 illustrates. Here, n is fixed at 30 and ε rises from 0.25 to 1. The first ergodic distribution is almost identical to that of the first graph in Figure 2. By increasing ε, the population size becomes more important and overturns the selection result that operates in the limit. The intuition is identical. Moving away from the limit, a successful team will operate for more of the time when the population is sufficiently large.

Nor does it take long for this feature to become dominant. Figure 4 plots the probability of the process being in any state $z \in Z_k$ where $k \geq m$ in the long run against

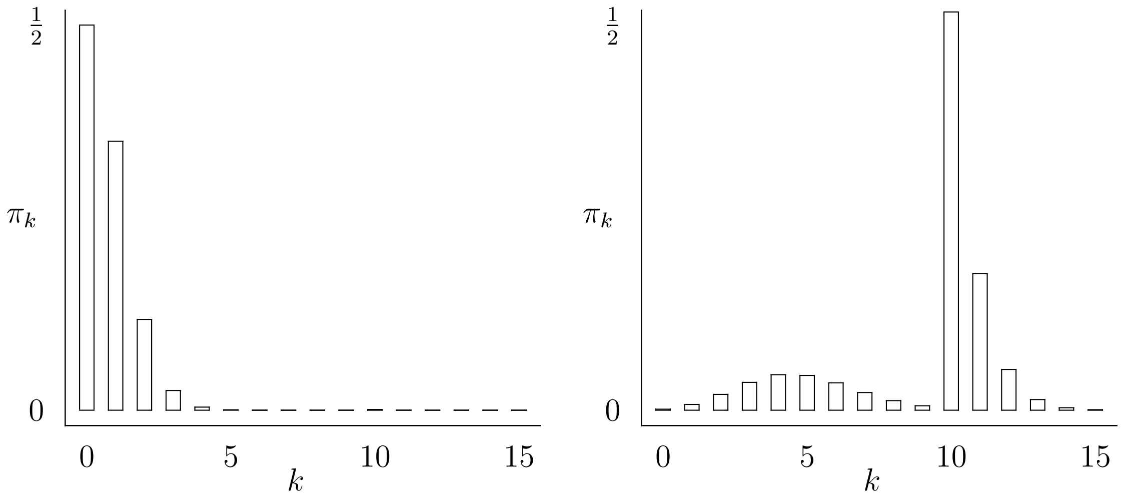

FIGURE 3. Ergodic Distributions for $\varepsilon = 0.25$ and $\varepsilon = 1$.

ε for $0 \leq \varepsilon \leq 2$ (the other parameters remain as before). Relatively quickly weight moves to the states where a successful team operates.

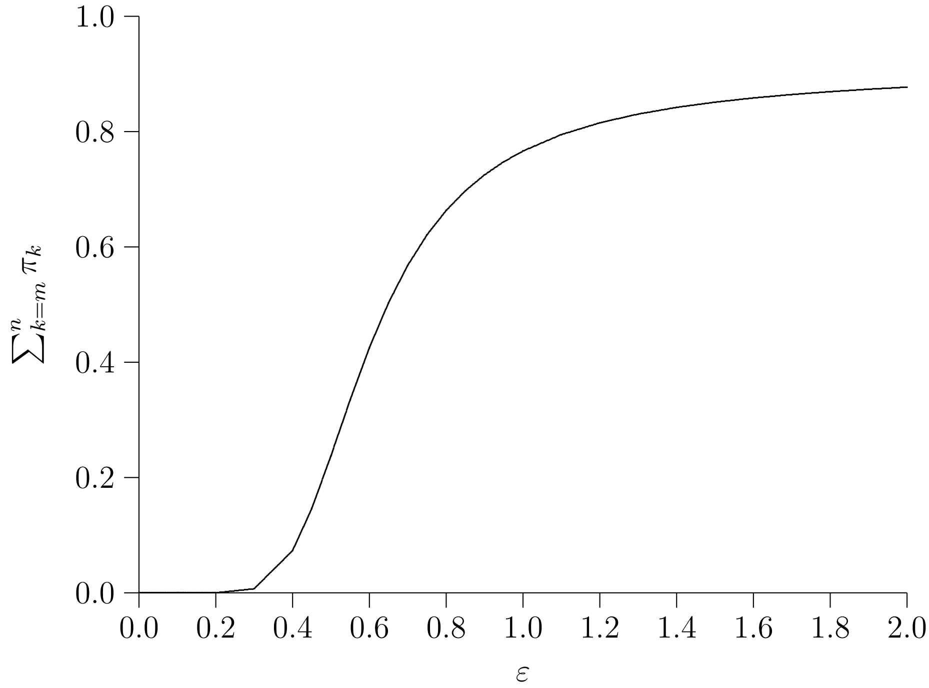

FIGURE 4. Team Success and ε.

Of course, these facts do not overturn the statement of Proposition 1, but they do put limiting results of this sort in a proper context. In the next section, a further exercise of this kind helps provide additional insight. Another critical assumption made at the outset was the symmetry between players. Section 3 examines the case where there is one "bad apple" — a single player with a higher cost of contribution. Although no-one would expect such a player to take part in a successful team in the long run, their existence within the population does have important consequences for team formation — both when $\varepsilon = 0$ *and* away from the limit.

3. Extensions and Simulations

So far, the agents in the model have had symmetric payoffs. Interest also lies in the case where agents differ in this respect. A full model of this kind is beyond the scope of this paper, and is presented in a companion paper — Myatt and Wallace (2003b). Here a simple example of asymmetric payoffs is considered. One player is assumed to be a "bad apple", that is, to have a higher average cost of contribution than the other agents. In the long run this player does not spend much time as part of any successful team (and, as $\varepsilon \to 0$, no time at all), nonetheless, their presence alone can cause a dramatic shift in the fortunes of the population.

The ergodic process is not so readily available in this case. The problem arises because it is no longer possible to simplify the state space into layers. With just one different player, the state space can be simplified to

$$\mathcal{Z} = \left\{ Z_0, Z_1^-, Z_1^+, \ldots, Z_{n-1}^-, Z_{n-1}^+, Z_n \right\},$$

where Z_k^+ represents all the states such that k agents contribute, including the bad apple — and Z_k^- represents the states where k contribute, excluding the bad apple. Thus $Z_k^- \cup Z_k^+ = Z_k$. Formally, if agent n is arbitrarily designated the bad apple,

$$\begin{aligned} Z_k^- &= \left\{ z : \sum_{i=1}^{n} z_i = k \quad \text{and} \quad z_n = 0 \right\}, \text{ and} \\ Z_k^+ &= \left\{ z : \sum_{i=1}^{n} z_i = k \quad \text{and} \quad z_n = 1 \right\}. \end{aligned}$$

The difficulty is that this process can no longer be represented by a "one-step-at-a-time" dynamic. In fact, in this simplified state space, it is possible for the process to transit from a state into itself or to at most three others. Hence it is not possible to apply results from simple birth-death processes, as it was in Section 2.3.

Transition probabilities can still be found easily. The only additional consideration is the presence of the bad apple. With probability $1/n$, this agent is chosen. The probability that they choose to contribute differs from other agents, as their cost of contribution is larger. The simplest case is to assume that

$$\tilde{c}_n \sim N(C, \varepsilon\xi^2), \quad \text{and} \quad \tilde{v}_n - \tilde{c}_n \sim N(v - C, \varepsilon\kappa^2),$$

which is identical to Equation (1) except that the average cost parameter c is replaced by some $C > c$. An identical process to that presented in Section 2.3 then yields transition probabilities $\Pr[z_{t+1} \in Z_i^+ \mid z_t \in Z_j^-]$, $\Pr[z_{t+1} \in Z_i^- \mid z_t \in Z_j^-]$, $\Pr[z_{t+1} \in Z_i^+ \mid z_t \in Z_j^+]$, and $\Pr[z_{t+1} \in Z_i^- \mid z_t \in Z_j^+]$. For $\varepsilon > 0$, the ergodic distribution is not available in closed form. However, limiting results can be obtained — see Myatt and Wallace (2003b). For the purposes of the current paper, however, the ergodic distribution is of interest away from the limit, as well as when ε is small.

To this end, numerical simulations allow an examination of the ergodic distribution for various parameter values. Figures 5, 6, and 7 show the long run distribution of the process with $c = \xi = \kappa = 1$, $v = 4$, $m = 5$, and $C = 2$, for a variety of values of ε. In each case, the distribution on the left illustrates the ergodic distribution before the addition of a bad apple (and with $n = 6$), whilst the distribution on the right shows the effect of adding a bad apple to the population (and hence $n = 7$).

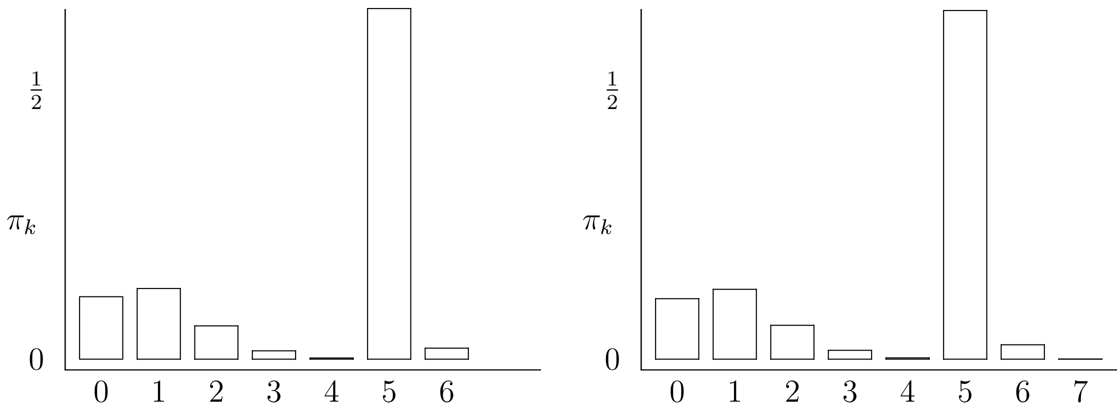

FIGURE 5. Bad Apple Ergodic Distributions for $\varepsilon = 1$.

Figure 5 shows the ergodic distributions for $\varepsilon = 1$. Section 2.4 argued that increasing the size of the population whilst keeping team size constant would have the effect of increasing the probability that a successful team forms. However, as can be seen from the figure, there is no appreciable increase in the probability of team formation — the two ergodic distributions are almost identical.

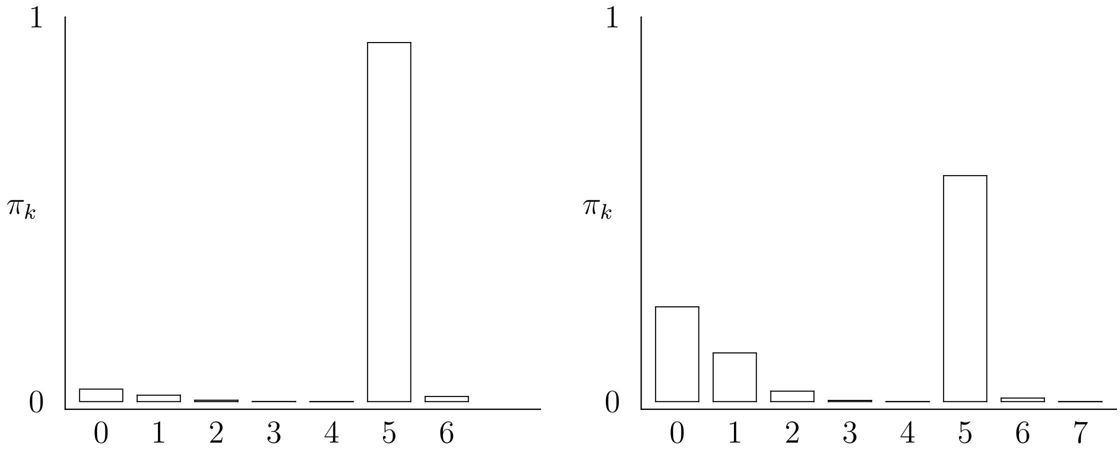

FIGURE 6. Bad Apple Ergodic Distributions for $\varepsilon = 0.5$.

Figure 6 goes one step further. As ε decreases (to 0.5), the probability of a successful team forming is substantially reduced by the introduction of a bad apple. Even though the population increases from $n = 6$ to $n = 7$, $\sum_{k \geq m} \pi_k$ is reduced.

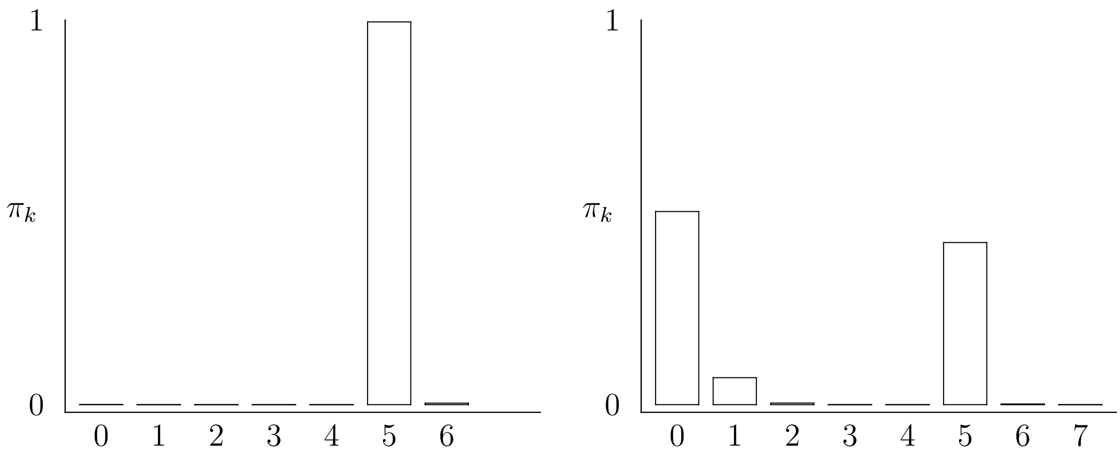

FIGURE 7. Bad Apple Ergodic Distributions for $\varepsilon = 0.25$.

The smaller ε becomes, the more pronounced this effect. Figure 7 illustrates this for $\varepsilon = 0.25$. Now the introduction of an extra (bad apple) agent results in a complete reversal of the team's fortunes. More weight now lies at Z_0 than at Z_m. Without a bad apple, on the other hand, the process spends almost all its time at Z_m.

In all three cases, $Z_m^+ \approx 0$. That is, the bad apple spends little time in successful teams when they form. There are, after all, six other agents with lower cost parameters to choose from. Successful teams do not include the bad apple. Nevertheless, it is the bad apple's presence that destroys the successful team. The reason is this: it is very much more likely that a contributing bad apple in a successful team finds it optimal to cease contributing if called upon to update their behaviour. In fact, following the intuition of Section 2.3, the difficulty of taking such a step is only $[(v - C)/\kappa]^2$ rather than $[(v - c)/\kappa]^2$. Roughly speaking, team formation still requires $m - 1$ steps, each with difficulty $[c/\xi]^2$ (there is no need for bad apples to help form teams). So whilst it has become easier to break a successful team, it has become no easier to form one.

Of course, this intuition (as discussed in Section 2.4) applies particularly to the limiting case as $\varepsilon \to 0$. Further away from the limit, the bad apple effect becomes less pronounced, and the population size relative to the team size begins to play the critical role. This is apparent from Figures 5, 6, and 7. As ε increases in size, the bad apple effect continues to hamper successful team formation, but to a lesser degree.

4. Conclusions

This paper presents a model of the evolution of team formation. A population of players repeatedly update their decision to either contribute to a potential team or not. If sufficiently many agents choose to contribute in any given round, a successful team is formed, and *every* agent in the population receives the benefits that accrue. Each contributor, regardless of whether the team is successful, must pay a cost.

The game has multiple equilibria. In particular, if no-one contributes, then revising players have no incentive to contribute, as they alone cannot form a team and hence derive any benefit. Likewise if a successful team is formed, contributing players have no incentive to stop contributing, as this would result in team collapse, and lower payoffs; and non-contributing players have no incentive to join the team, as they already receive the full benefits of successful coordination without paying any cost.

Initially, each player's benefit and contribution cost are drawn from distributions with identical means and variances. Thus, on occasion, players find it optimal to play against the flow of play described in the previous paragraph. As the distributions collapse to their means, and hence noise is driven from the model, a precise condition arises to select between the multiple strict equilibria. Equation (5) gives the condition for the updating process described above to spend nearly all its time in the equilibria associated with successful team formation.

High benefits, low costs and low team size all make team formation easier. More surprisingly, perhaps, when there is very little noise, population size does not matter.

However, as noise is reintroduced into the model, an increase in the population size does improve the fortunes of teams. In each updating period their are more non-contributors to choose from who may potentially join a team.

Increasing the pool of agents might not always be such a good idea, however. Once asymmetry is introduced into the model, a single "bad apple" (a player with a higher average cost than the rest of the population) can have a detrimental effect upon team formation. As Section 3 shows, even though the bad apple would not play a part in any successful team, their mere presence in the population can overturn the selection result of the symmetric case. This is most apparent when there is little noise in the process, but can still have an impact away from the limit. At a first glance, increasing the population might seem to improve the chances of successful team formation, but the nature of the agents being added to the pool plays a critical role.

An interesting extension of the ideas proposed in this paper would allow for general asymmetries across players in the population. This is the direction taken in a companion paper, Myatt and Wallace (2003b). The updating procedure described in Section 2 can longer be represented as a simple birth-death process. Limiting results are still available, via the use of Freidlin and Wentzell (1984)-style rooted tree methods, but these go well beyond the analysis required for the ideas presented here.

Appendix A. Omitted Proofs

Proof of Proposition 1. The "switching ratios" of Equation (4) can be employed in the first place to eliminate states that will not feature in the limiting distribution. For instance, consider the relative probability of 0 versus 1 contributor. Formally:

$$\frac{\pi_0}{\pi_1} = \frac{\Pr[z_{t+1} \in Z_0 \mid z_t \in Z_1]}{\Pr[z_{t+1} \in Z_1 \mid z_t \in Z_0]} = \frac{(1/n)\Phi[c/\xi\sqrt{\varepsilon}]}{(n/n)(1-\Phi[c/\xi\sqrt{\varepsilon}])} \to \infty, \text{ as } \varepsilon \to 0.$$

Similarly, for $k \neq 0$ and $k \neq m$, it is the case that $\lim_{\varepsilon\to 0} \pi_k = 0$. Thus, for vanishingly small ε, the only states that have positive weight in the ergodic distribution are the ones that correspond to pure strategy Nash equilibria, $z \in Z_0 \cup Z_m$.

Equation (4) is again used, this time to compare the pure Nash equilibrium states. π_0/π_m involves the product of m fractions. Substituting with the transition probabilities in Equations (2) and (3) yields the following expression:

$$\frac{m!(1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}])\Phi[c/\xi\sqrt{\varepsilon}]^{m-1}}{(n!/m!)(1-\Phi[c/\xi\sqrt{\varepsilon}])^{m-1}\Phi[(v-c)/\kappa\sqrt{\varepsilon}]} \propto \frac{1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}]}{(1-\Phi[c/\xi\sqrt{\varepsilon}])^{m-1}}.$$

The latter term can be decomposed into ratios of densities and hazard rates for the Normal distribution. The expression becomes

$$\frac{1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}]}{(1-\Phi[c/\xi\sqrt{\varepsilon}])^{m-1}} = \frac{\phi[(v-c)/\kappa\sqrt{\varepsilon}]}{(\phi[c/\xi\sqrt{\varepsilon}])^{m-1}} \times \frac{(\phi[c/\xi\sqrt{\varepsilon}]/(1-\Phi[c/\xi\sqrt{\varepsilon}]))^{m-1}}{\phi[(v-c)/\kappa\sqrt{\varepsilon}]/(1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}])}. \quad (6)$$

Rewriting the first term in the right hand side of Equation (6) explicitly gives

$$\frac{\phi[(v-c)/\kappa\sqrt{\varepsilon}]}{(\phi[c/\xi\sqrt{\varepsilon}])^{m-1}} = (2\pi)^{(m-2)/2} \exp\left(-\frac{1}{2\varepsilon}\left[\frac{(v-c)^2}{\kappa^2} - \frac{(m-1)c^2}{\xi^2}\right]\right). \quad (7)$$

Now consider the denominator of the second term in the right hand side of Equation (6). Notice that $(v-c)/\kappa > 0$, and hence $(v-c)/\kappa\sqrt{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Since the hazard rate of the Normal distribution is asymptotically linear, it follows that

$$\lim_{\varepsilon\to 0}\left[\frac{\phi[(v-c)/\kappa\sqrt{\varepsilon}]}{1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}]}\right] = \lim_{\varepsilon\to 0}\left[\frac{v-c}{\kappa\sqrt{\varepsilon}}\right].$$

A similar argument holds for the numerator, so that

$$\lim_{\varepsilon\to 0}\left[\frac{\phi[c/\xi\sqrt{\varepsilon}]}{1-\Phi[c/\xi\sqrt{\varepsilon}]}\right]^{m-1} = \lim_{\varepsilon\to 0}\left[\frac{c}{\xi\sqrt{\varepsilon}}\right]^{m-1}.$$

Both of these terms are polynomial in ε, whereas the first term of Equation (6) is exponential in ε. The exponential term dominates the polynomial in the limit, and hence determines behaviour as $\varepsilon \to 0$. Examining Equation (7) it is clear that

$$\frac{(v-c)^2}{\kappa^2} > (m-1)\frac{c^2}{\xi^2} \quad \Rightarrow \quad \lim_{\varepsilon\to 0}\left[\frac{1-\Phi[(v-c)/\kappa\sqrt{\varepsilon}]}{(1-\Phi[c/\xi\sqrt{\varepsilon}])^{m-1}}\right] = 0,$$

which yields the desired result. □

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