RAND Journal of Economics
Vol. 37, No. 3, Autumn 2006
pp. 583–601

Multiproduct Cournot oligopoly

Justin P. Johnson*

and

David P. Myatt**

We study a Cournot industry in which each firm sells multiple quality-differentiated products. We use an upgrades approach, working not with the actual products but instead with upgrades from one quality to the next. The properties of single-product models carry over to the supply of upgrades, but not necessarily to the supply of complete products. Product line determinants and welfare results are presented. Strategic commitment to product lines is considered; firms may well choose to compete head-to-head.

1. Introduction

■ Competition between multiproduct firms abounds. While understanding such competition is important, only limited progress has been made. We enhance understanding by presenting a general analysis of oligopolistic competition in quantities between firms offering multiple quality-differentiated products. We address three broad questions. First, when do the insights of single-product Cournot models carry over to a multiproduct world? Second, what factors determine firms' product lines (that is, the qualities that they offer), and how do qualities differ from the socially efficient ones? Third, to what extent does the opportunity to precommit strategically to a product line influence our results?

The setting is a market populated by consumers with general preferences for quality, and an arbitrary number of firms. We follow earlier work (Johnson and Myatt, 2003) by taking an "upgrades approach." This approach recasts competition in terms of conceptual upgrades from one quality level to the next, instead of in terms of actual products.[1]

A benefit of the upgrades approach is that it helps answer our first question. The insights of single-product Cournot models carry over to a multiproduct world when we think in terms of upgrades, but may fail when we think of the actual products themselves. For instance, when firms are symmetric, an increase in the number of competitors will yield an expansion in the supply of each upgrade. The supply of a particular quality, however, might well decline. Similarly, in an asymmetric setting, firms with lower costs produce more of every upgrade, echoing results from single-product industries. Once again, however, this does not necessarily apply to the supply of a

* Cornell University; jpj25@cornell.edu.

** Oxford University; david.myatt@economics.ox.ac.uk.

We thank the Editor Joe Harrington and two anonymous referees for extremely helpful comments and suggestions.

[1] For example, selling separate low- and high-quality products is conceptually equivalent to selling a low-quality "baseline" product alongside a theoretical "upgrade" from low to high quality. A high-quality product is then obtained by combining the baseline product with an upgrade.

specific complete product; it is possible that lower-cost firms produce zero units of some qualities while higher-cost firms offer positive supplies.

To answer our second question, we show that the key determinants of product lines include the returns to quality (that is, the change in the ratio of cost to willingness to pay as quality increases) and the changes in demand elasticity as quality increases. In contrast, the qualities offered by a social planner are determined solely by returns to quality. The quality of products consumed is distorted downward from the first-best. Relative to the second-best, however, quality is too high.

In answering our third question, we first show that firms competing in quantities never precommit not to sell the highest-quality good, since with strategic substitutes this makes them soft and leads to an expansion of their rivals' output. Interestingly, a decision instead not to sell a lower-quality product has conflicting effects. It toughens the committing firm's stance in the higher-quality upgrade market but softens it in the lower-quality upgrade market. While the net effect on profits is indeterminate, there is a bias toward not committing so that multiproduct firms may compete head-to-head (that is, with the same qualities) even given the opportunity to avoid doing so. These results stand in stark contrast to what prevails in price-setting models of competition in quality-differentiated markets such as that of Champsaur and Rochet (1989), who showed that duopolists precommit to producing qualities ranges that do not intersect.[2]

The upgrades technique was used in earlier work (Johnson and Myatt, 2003) in which we considered a multiproduct incumbent's response to entry.[3] Multiproduct quantity competition was also considered by Gal-Or (1983) and De Fraja (1996). Others considered competition in prices when goods are horizontally differentiated,[4] and monopoly price discrimination (Mussa and Rosen, 1978; Maskin and Riley, 1984).

Section 2 lays out our model, Sections 3 through 5 contain our analysis, and in Section 6 we conclude. All proofs are found in the Appendix.

2. Supply and demand in a multiproduct world

■ Here we describe supply and demand in a market for quality-differentiated products. We look for pure-strategy Nash equilibria in which each multiproduct firm simultaneously chooses its outputs given those of other firms, and refer to such an equilibrium as a "multiproduct Cournot equilibrium." Our specification generalizes that in Johnson and Myatt (2003).

Formally, M distinct product qualities $q_M > \ldots > q_1 > 0$ are supplied by N firms, where $z_{ir} \geq 0$ is firm r's output of quality q_i, and $z_i \equiv \sum_{r=1}^{N} z_{ir}$ is the total industry supply of q_i.[5] We will make extensive use of the cumulative variables $Z_{ir} \equiv \sum_{j=i}^{M} z_{jr}$ (firm r's supply at quality q_i and above) and $Z_i \equiv \sum_{r=1}^{N} Z_{ir}$ (the corresponding industry supply). By construction, such cumulative variables satisfy the monotonicity constraints $Z_{1r} \geq Z_{2r} \geq \ldots \geq Z_{Mr} \geq 0$.

□ **Market demand.** A unit mass of consumers is indexed by a type parameter $\theta \in [0, \bar{\theta}]$. We write $H(z)$ for the type such that a mass $z \in [0, 1]$ of consumers value quality more highly.

[2] The price-setting models of Gabszewicz and Thisse (1979, 1980) and Shaked and Sutton (1982, 1983), while related, restrict each firm to a single product and hence are unable to offer insight into product lines.

[3] Itoh (1983) implicitly used an upgrade-type approach, and Fudenberg and Tirole (1998) explicitly did so. This technique was also used in one of several applications in our study of changing demand dispersion (Johnson and Myatt, 2006).

[4] Brander and Eaton (1984) and Klemperer (1992) studied duopolies in which firms commit to product lines before engaging in price competition. Caplin and Nalebuff (1991) presented an important equilibrium existence result. Eaton and Lipsey (1979) and Judd (1985) considered entry deterrence via brand proliferation. The issue of optimal variety was addressed by Dixit and Stiglitz (1977) for single-product firms. Anderson, de Palma, and Nesterov (1995) extended this analysis, while Anderson and de Palma (1992, 2006) considered multiproduct firms. Horizontal differentiation may be combined with vertical differentiation, as in Gilbert and Matutes (1993), Stole (1995), Verboven (1999), and Ellison (2005).

[5] We can allow M to be arbitrarily large and the increments between successive qualities to be arbitrarily small. Hence we can allow suppliers to offer a large menu of qualities drawn from an arbitrarily fine grid.

$H(z)$ is strictly decreasing and continuously differentiable, $H(0) = \bar{\theta}$ and $H(z) = 0$ for $z \geq 1$.[6] A consumer θ who pays a price p_i for product i enjoys utility $u(\theta, q_i) - p_i$, where $u(\theta, q)$ is strictly increasing in θ and q, twice continuously differentiable, and exhibits increasing differences, so that $u(\theta, q) - u(\theta, q')$ is strictly increasing in θ whenever $q > q'$. Furthermore, we assume that $u(\theta, 0) = u(0, q) = 0$ for all θ and q. This equates a product of zero quality with no consumption, and ensures that the lowest type gains no value from a purchase. Each consumer purchases a single unit of the product that maximizes $u(\theta, q_i) - p_i$, unless doing so yields strictly negative utility, in which case she purchases nothing.

To derive an inverse demand system, we begin with the case where $Z_1 = \sum_{i=1}^{M} z_i < 1$, so that there is partial market coverage. We require a set of positive prices such that exactly z_i consumers wish to purchase product i. Higher-quality products must carry a price premium and be purchased by consumers with higher types. Thus a consumer of type $H(Z_1)$ must be indifferent between purchasing quality q_1 and not purchasing at all, and hence $p_1 = u(H(Z_1), q_1)$. Similarly, a type $H(Z_i)$ with Z_i others above her must be just indifferent between products i and $i - 1$, and so $p_i = p_{i-1} + [u(H(Z_i), q_i) - u(H(Z_i), q_{i-1})]$. We define $q_0 = p_0 = 0$ for convenience, and note that $p_i - p_{i-1}$ is the price differential between products i and $i - 1$. This satisfies

$$p_i - p_{i-1} = P_i(Z_i), \quad \text{where } P_i(Z) \equiv u(H(Z), q_i) - u(H(Z), q_{i-1}). \tag{1}$$

It turns out that (1) also describes the system of inverse demand when there is complete market coverage, so that $Z_1 \geq 1$, and addresses the possibility that some products are in zero supply. In Johnson and Myatt (2003) we confirm that (1) is correct for all circumstances.

$P_i(Z_i)$ is the extra utility received by a type $\theta = H(Z_i)$ from increasing quality from q_{i-1} to q_i, and is independent of Z_j for $j \neq i$; this is the strength of the upgrades approach.[7] Equivalently, it is the price of an "upgrade" in quality. We offer this (conceptual) interpretation: Z_1 units of a "baseline" product of quality q_1 are supplied, together with successive upgrades in order to achieve higher qualities.

We might consider $Z_{ir} P_i(Z_i)$ to be the revenue earned by firm r from providing quality increases. If upgrades corresponded to actual physical products, a familiar regularity condition would be that marginal revenue is decreasing.

Assumption. Marginal revenue is decreasing in each upgrade market, so that

$$\frac{\partial \left[P_i(Z_i) + Z_{ir} P_i'(Z_i)\right]}{\partial Z_i} < 0. \tag{2}$$

We maintain this assumption throughout the entire article.[8] In a single-product world, (2) implies that a firm's best response is decreasing in the output of its competitors. In terms of complete physical products, a unit increase in Z_i corresponds to a unit increase in z_i and a unit decrease in z_{i-1}, so that (2) says that the marginal revenue of converting lower-quality units into higher-quality units is decreasing. It also implies that the marginal revenue of increasing the supply of a complete product i is decreasing.

□ **Market supply.** Firm r manufactures product i at a constant marginal cost of $c_{ir} > 0$, so that the constant marginal cost of a theoretical upgrade from quality q_{i-1} to quality q_i is

[6] It is equivalent to assume that consumers' types are drawn from the strictly increasing and continuous distribution $G(\theta) : [0, \bar{\theta}] \mapsto [0, 1]$. Given this specification, $H(z)$ satisfies $H(z) = G^{-1}(1 - z)$.

[7] In contrast, the complete product of quality q_i has a price p_i that depends upon the supplies of all products.

[8] If, following footnote 6, $G(\theta)$ is the distribution of types, with density $g(\theta)$, then (2) is equivalent to $\partial^2[u(\theta_i, q_i) - [(1 - G(\theta_i))/g(\theta_i)](\partial u(\theta_i, q_i)/\partial\theta)]/\partial\theta\partial q > 0$, where $\theta_i = H(Z_i)$, so that the virtual surplus of consumers is well behaved. This is the condition used by Champsaur and Rochet (1989). It ensures that pointwise maximization of virtual surplus generates a quality schedule that is nondecreasing in θ, so that solving a firm's maximization program pointwise is legitimate.

$C_{ir} \equiv c_{ir} - c_{(i-1)r}$.[9] Fixing the supplies of its competitors, each quantity-setting firm chooses a vector of outputs to maximize the sum of profits earned across its entire product line. Equivalently, each firm maximizes the sum of profits associated with each upgrade:

$$\pi_r = \sum_{i=1}^{M} z_{ir}(p_i - c_{ir}) = \sum_{i=1}^{M}(Z_{ir} - Z_{(i+1)r})\left[\sum_{j\le i}(P_j(Z_j) - C_{jr})\right] = \sum_{i=1}^{M} Z_{ir}(P_i(Z_i) - C_{ir}).$$

Hence we may construct a multiproduct Cournot game in two ways. First, we may view each firm r as choosing a set of M product supplies, subject to the nonnegativity constraints $z_{ir} \geq 0$ for each i. Alternatively, we may view each firm as choosing a set of M upgrade supplies, subject to these forming a monotonic sequence $Z_{1r} \geq \ldots \geq Z_{Mr} \geq 0$, where these monotonicity constraints ensure that $z_{ir} = Z_{ir} - Z_{(i+1)r} \geq 0$. Furthermore, whenever a constraint binds, so that $Z_{ir} = Z_{(i+1)r}$, firm r is not supplying quality q_i. The advantage of the latter approach is that the direct effect of Z_{ir} is felt only in the market for the ith theoretical upgrade, and hence single-product Cournot logic may be exploited.

In summary, a "multiproduct Cournot equilibrium" (that is, a pure-strategy Nash equilibrium in quantities) is a collection of upgrade supplies Z^*_{ir} for each i and r, where firm r's supplies maximize the sum of the profits associated with each upgrade, subject to the appropriate monotonicity constraints. Before studying such equilibria, we first ensure that one exists. For the purposes of Proposition 1, we say that firms r and s are of the same type if $c_{ir} = c_{is}$ for each i, that is, if they face the same cost structure. An immediate corollary of this proposition is that there is a unique and symmetric equilibrium when firms are symmetric (that is, they all share the same technological capability).[10]

Proposition 1. A multiproduct Cournot equilibrium exists. It is unique if either (i) there are at most two different types of firms, or (ii) there are at most two quality levels.

In equilibrium, a firm's output satisfies first-order conditions that are straightforward extensions of those in single-product Cournot models. The only difference is that each firm r must satisfy multiple first-order conditions, with the exact number and their exact form depending on which products r supplies in positive quantities.[11]

□ **Discussion.** Here we discuss some of our modelling choices. First, we view firms' quantities as the key decision variables. This is most appropriate for an industry where production runs are chosen prior to final pricing decisions. Second, we do not explicitly model horizontal product differentiation. We acknowledge that this is a strong assumption in some circumstances, but note that multiproduct competition is inherently complex and that strong assumptions also would be required to make progress if we, for instance, instead specified a multiproduct price-setting model incorporating horizontal differentiation. Third, we do not incorporate fixed costs in our base model, yet certainly these may influence the qualities produced. We address fixed costs in a free-entry environment in Section 4. We note, however, that we allow for arbitrary firm-specific marginal

[9] The cost assumption we need for the upgrades approach is that the marginal costs of producing a given quality are constant with respect to quantity.

[10] Equilibrium is also unique if there is some subset of the M products such that in each potential equilibrium each firm is selling exactly these products. In such cases the upgrade markets can be treated independently, since the monotonicity constraints can be ignored. Hence, the assumption of decreasing marginal revenue (2) would be sufficient for uniqueness in each upgrade market, just as it is sufficient in a single-product setting. Incorporating the monotonicity constraints is what complicates the analysis.

[11] Suppose in equilibrium firm r produces good k in positive supply, so that $Z^*_{kr} > Z^*_{(k+1)r}$. If qualities in some range just lower than q_k are not supplied by r, then for some $i < k$ it is the case that $Z^*_{(i-1)r} > Z^*_{ir} = \cdots = Z^*_{kr} > Z^*_{(k+1)r}$. The relevant first-order condition for firm r regarding product q_k ensures that r is indifferent to raising slightly (and simultaneously) Z_{jr} for $j \in \{i, \ldots, k\}$: $\sum_{j=i}^{k}[P_j(Z^*_j) + Z^*_{jr}P'_j(Z^*_j)] = \sum_{j=i}^{k} P_j(Z^*_j) + Z^*_{kr}\sum_{j=i}^{k} P'_j(Z^*_j) = \sum_{j=i}^{k} C_j$. Equalization of marginal revenue and marginal costs still occurs, but the relevant marginal revenue expression is a sum of marginal revenue terms. There is one such condition for each good r produces in positive supply.

costs. Hence, we may take our base analysis as representing a "second stage" of competition following one in which firms invest in production facilities, since if a firm is unable to produce some quality q_i, that firm's marginal cost for that quality can be set arbitrarily high, ensuring that producing zero complete units of that quality is a dominant strategy.[12] Fourth, our assumption that quantities and qualities are chosen simultaneously may seem to neglect important strategic effects. In particular, if each firm could commit not to produce certain qualities prior to final quantity competition (for instance, by not building the relevant production facilities), the issue of whether firms would elect to specialize in different niches or instead compete head-to-head would be raised. We address this in Section 5, showing that firms indeed may choose to compete head-to-head. Furthermore, we again note that we may take our base analysis (with arbitrary upgrade costs) as representing the appropriate second stage of competition.

3. Multiproduct Cournot equilibrium

■ Here we answer our first main question by extending the insights of single-product Cournot models to a multiproduct world. Many such insights carry over to the supplies of upgrades (equivalently, to the total quantity of complete products at or above a given quality level), but not necessarily to the supply of a given complete product. First, we study an industry in which firms have symmetric technologies. Second, we study asymmetric industries.

□ **Symmetric industries.** When firms have symmetric technological capabilities, the unique multiproduct Cournot equilibrium (Proposition 1) is easy to characterize. Suppose for now that all products are offered in strictly positive supply, so that $Z_1^* > Z_2^* > \ldots > Z_M^* > 0$. Since the unique equilibrium must be symmetric, each firm in the industry supplies a $1/N$ share of each upgrade and monotonicity constraints do not bind. The equilibrium is determined by a familiar single-product Cournot first-order condition, equalizing marginal revenue and marginal cost, applied to each upgrade market:

$$P_i(Z_i^*) + \frac{Z_i^* P_i'(Z_i^*)}{N} = C_i \quad \text{or, equivalently, } P_i(Z_i^*)\left[1 - \frac{\eta_i(Z_i)}{N}\right] = C_i, \tag{3}$$

where $\eta_i(Z_i) \equiv -Z_i P_i'(Z_i)/P_i(Z_i)$ is the reciprocal of the (price) elasticity of demand for upgrade i and we have dropped the subscript r from C_{ir} because firms are symmetric. Inspecting (3), comparative statics emerge naturally. The upgrade supply Z_i^* is decreasing in its marginal cost C_i and increasing in the number of firms N. These results also hold when some qualities are not supplied.[13]

Our first formal result concerns the effects of entry. Recall that (2), the assumption of decreasing marginal revenue, is maintained throughout.

Proposition 2. In the unique multiproduct Cournot equilibrium of a symmetric industry, an increase in the number of firms results in an expansion of the supply of each upgrade, and hence a reduction in each of their prices. Industry profits fall.

Hence a basic lesson of single-product Cournot models holds in a multiproduct world so long as we think in terms of upgrades. Immediate corollaries flow from this proposition. First, since the price of a complete product is the sum of the component upgrade prices, its price falls with entry. Second, recall that the supply Z_i^* of upgrade i is the number of units supplied at or

[12] Moreover, note that since we do not constrain upgrade costs to be positive, this restriction does not imply that this firm does not produce higher-quality goods.

[13] If product i is not offered, then $z_i^* \equiv Z_i^* - Z_{i+1}^* = 0$. The constraint $Z_i^* \geq Z_{i+1}^*$ binds, and (3) may fail. However, since the neighboring upgrades i and $i+1$ are identically supplied, we combine them to obtain a new theoretical upgrade with a marginal cost of $\tilde{C}_i = C_i + C_{i+1} = c_{i+1} - c_{i-1}$, and inverse demand $\tilde{P}_i(Z_i) = P_i(Z_i) + P_{i+1}(Z_i)$ (recall that $Z_i = Z_{i+1}$ by assumption). Relabeling yields a new system with $M - 1$ upgrades. Given this procedure, (3) holds when we replace C_i with $\tilde{C}_i$ and $P_i(Z_i)$ with $\tilde{P}_i(Z_i)$.

above quality q_i, and so entry pushes upward the distribution of qualities sold. In particular, Z_1^* is increasing in N, so that more products (of all qualities combined) are supplied by the industry.

These familiar effects of entry may well fail for supplies of complete products. To see this, consider a two-quality world, and suppose that $Z_1^* > Z_2^* > 0$ so that (3) holds. Entry increases both Z_1^* and Z_2^*; this implies that both total production and the production of the highest-quality good rise. The supply $z_1^* \equiv Z_1^* - Z_2^*$ of quality q_1, however, can go either way. In particular, straightforward derivations lead to the following comparative static:

$$\frac{\partial z_1^*}{\partial N} = \frac{1}{N}\left[\frac{Z_1^*}{(N+1)-\rho_1(Z_1^*)} - \frac{Z_2^*}{(N+1)-\rho_2(Z_2^*)}\right], \quad \text{where } \rho_i(Z) = -\frac{ZP_i''(Z)}{P_i'(Z)}. \qquad (4)$$

Here the elasticity of its slope $\rho_i(Z)$ measures the curvature of the inverse demand curve.[14] If $\rho_1(Z) \geq \rho_2(Z)$ for each Z, so that the curvature is decreasing in quality, then the output of quality q_1 will rise following entry.[15] If, on the other hand, $\rho_2(Z_2^*) - \rho_1(Z_1^*)$ is sufficiently large, then the output of the second upgrade expands by more than the output of the baseline upgrade. A shift in supply from low to high quality overwhelms the overall expansion of the industry, meaning that the output of low-quality units falls even though total output rises.

Our next result concerns the effect of an industry-wide change in marginal costs.

Proposition 3. In the unique multiproduct Cournot equilibrium of a symmetric industry, an increase in the marginal cost of upgrade i results in a (weak) contraction in the supply of all upgrades, and a rise in all of their prices. If upgrade i is initially in strictly positive supply, then there is a strict reduction in its supply. Industry profits fall.

If all products are offered, then the effect of a local increase in C_i is confined to Z_i^*, since no monotonicity constraints bind.[16] If $i > 1$, then the total supply brought to market will remain constant. However, $z_i^* = Z_i^* - Z_{i+1}^*$ will fall and $z_{i-1}^* = Z_{i-1}^* - Z_i^*$ will rise. If the increase in C_i is sufficiently large, then the monotonicity constraint $Z_i \geq Z_{i+1}$ will bind, and product i will be eliminated from the product line of firms.[17]

Similar logic reveals that an increase in the complete product cost c_i has intuitive effects on output.[18] To see this, note that such an increase corresponds to an increase in C_i and a corresponding decrease in C_{i+1}. Presuming that all products are in positive supply, a small increase in c_i reduces Z_i and increases Z_{i+1}, which is equivalent to an increase in z_{i-1} and z_{i+1}, and a decrease in z_i that offsets these increases.

□ **Asymmetric industries.** Turning to asymmetric industries, recall that a robust property of a single-product Cournot equilibrium is that a firm with lower marginal costs produces more than a less efficient firm, and earns higher profits. To carry this result over, we must impose a more stringent notion of efficiency. Firm r has lower costs than firm s if $C_{ir} \leq C_{is}$ for all i, so that firm r faces a lower marginal cost of increasing the quality of a product. This is a sorting condition on firms' cost functions, similar to that on consumer preferences. It implies that the complete products of firm r cost less, so that $c_{ir} \leq c_{is}$ for each i.

Proposition 4. Consider any multiproduct Cournot equilibrium. If firm r has lower marginal costs for raising quality than firm s, so that $C_{ir} \leq C_{is}$ for all i, then $Z_{ir}^* \geq Z_{is}^*$ for all i and firm r earns higher profits than firm s.

---

14 Robinson (1933) referred to this measure as the "adjusted concavity" of inverse demand.

15 A special case of this is when demand for each upgrade is linear, so that $\rho_i = 0$ for each i. In terms of primitives, setting $u(\theta, q) = \theta q$ and $\theta \sim U[0, 1]$ yields $P_i(Z_i) = (q_i - q_{i-1})(1 - Z_i)$.

16 Recall that an increase in C_i is equivalent to an identical increase in c_j for all $j \geq i$.

17 Once a product is eliminated, we may then, following the procedure described in footnote 13, combine identically supplied neighboring upgrades to obtain a new system in which no constraints bind.

18 The effect on profits of c_i is ambiguous. See the discussion following Proposition 5, and also see Section 5.

Hence a firm with lower costs sells more of each upgrade or, equivalently, more complete units at and above any particular quality level. An implication is that a more efficient firm sells more products in total than a less efficient firm, and also more of the highest quality. However, Proposition 4 does not say that a firm with lower costs supplies more of a complete product $i < M$. Simply, the reason is that $z_{ir} = Z_{ir} - Z_{(i+1)r}$, and as both of these upgrades are higher for lower-cost firms, in general their difference is indeterminate.[19]

Proposition 4 confirms that lower-cost firms earn higher profits. However, if the sorting condition on the firms' cost functions does not hold, so that (for example) some firm is more efficient than others at producing high-quality goods but less efficient at producing low-quality goods, then it is not possible to rank profits or upgrade outputs.[20]

We now turn to comparative statics results. Such results require a unique equilibrium, and we focus on the case in which there are two possible qualities.[21] Suppose that the marginal cost of upgrade 2 of firm r increases. Intuitively, the partial effect is a reduction in Z_{2r}, to which others respond by raising their outputs of upgrade 2. If this causes the constraint $Z_{1s} \geq Z_{2s}$ to bind for some s, then that firm's output in upgrade market 1 will also increase. In the resulting equilibrium, we expect that the firm whose costs have increased should face (weakly) higher rivals' output in each upgrade market, and hence be strictly worse off.

Proposition 5. Consider the unique multiproduct Cournot equilibrium of an industry in which there are two possible qualities. If upgrade i is in positive supply for firm r (so that $Z^*_{ir} > 0$), then an increase in r's marginal cost C_{ir} for upgrade i results in a strict contraction of the industry's supply of upgrade i (and hence a reduction in the industry supply of qualities i or higher), a strict contraction of r's output of upgrade i, and a strict reduction in the profits of r. Finally, rivals' equilibrium output $\sum_{s \neq r} Z^*_{js}$ increases for $j \in \{1, 2\}$.

A change in upgrade costs has intuitive effects, but changes to a complete product's cost need not. The reason is that an increase in the cost of the low quality product is equivalent to a simultaneous increase in upgrade 1's cost and a decrease of the same magnitude in upgrade 2's cost.[22] Consequently, a change in the cost of a complete product will have conflicting effects. For example, a firm's profits could either increase or decrease, depending on how the outputs of other firms in the different upgrade markets equilibrate.[23]

4. Product lines and welfare

■ We turn to our second main question. Referring to the set of qualities that a firm offers in positive supply as its product line, we ask, what factors determine firms' product lines, and how do equilibrium qualities differ from those that would be offered by a social planner?

□ **Product lines: returns to quality and price sensitivity.** Consider a symmetric industry. Suppose that product i is offered, so that $Z^*_i > Z^*_{i+1}$. Marginal revenue in upgrade market i must

[19] In fact, a lower-cost firm might not sell product i, even though a higher-cost firm does so. This may arise because even though a more efficient firm is pushed toward producing more of upgrade i due to its absolute cost advantage in its production, it also has an absolute advantage in the production of upgrade $i + 1$, which pushes it toward producing more of the complete product $i + 1$ and less of i.

[20] For instance, a firm that is very efficient in producing high-quality units but less efficient at producing low-quality units will fare well compared to other firms if consumers place relatively high values on the upgrade to high quality, whereas the reverse will hold if the upgrade is not valued highly.

[21] For a given set of products, if there is an equilibrium in which each firm sells exactly these products, then there is no other equilibrium in which each firm sells exactly these products (see footnote 10). Restricting to such equilibria, comparative statics for local changes in costs mirror those from single-product markets.

[22] To see this algebraically, suppose that firm r's cost of producing the low-quality product rises by κ, but the cost of the high-quality product doesn't change. This firm's cost for upgrade 1 rises by κ, whereas its cost for upgrade 2 is $c_{2r} - (c_{1r} + \kappa) = (c_{2r} - c_{1r}) - \kappa$ (where c_{ir} denotes the initial product costs for $i \in \{1, 2\}$); the cost of upgrade 2 decreases by κ as the cost of product 1 increases by κ.

[23] In Section 5 we show how a similar logic prevails when we consider the incentives of firms to precommit strategically to a range of products before choosing exact quantities.

exceed marginal cost for industry output $Z < Z_i^*$, while the marginal cost in upgrade market $i+1$ must exceed marginal revenue for $Z > Z_{i+1}^*$. So long as $C_i > 0$ and $C_{i+1} > 0$, these observations can be combined as follows:

$$Z \in (Z_{i+1}^*, Z_i^*) \quad \Rightarrow \quad \frac{P_i(Z)}{C_i}\left[1 - \frac{\eta_i(Z)}{N}\right] > 1 > \frac{P_{i+1}(Z)}{C_{i+1}}\left[1 - \frac{\eta_{i+1}(Z)}{N}\right],$$

where $\eta_i(Z)$ is the reciprocal of the (price) elasticity of demand for upgrade i. We are now able to identify two factors that result in the supply of quality q_i. First, if $P_i(Z)/C_i > P_{i+1}(Z)/C_{i+1}$, then increasing quality reduces the ratio of the marginal consumer's valuation for an upgrade and its cost, so that there are "decreasing returns to quality." Second, if $\eta_i(Z) < \eta_{i+1}(Z)$, then the demand for higher upgrades is less elastic than for lower upgrades, so that "price sensitivity is decreasing in quality."[24] Intuitively, product i is sold whenever firms restrict the supply of upgrade $i+1$, which happens when that upgrade is relatively expensive to manufacture (decreasing returns to quality) and its demand is relatively inelastic (decreasing price sensitivity).

Proposition 6. Consider the unique multiproduct Cournot equilibrium of a symmetric industry. Suppose $C_i > 0$ and $C_{i+1} > 0$. If product i is offered, then evaluated for any $Z \in (Z_i^*, Z_{i+1}^*)$, either (i) there are decreasing returns to quality, or (ii) price sensitivity is decreasing in quality. Conversely, if (i) and (ii) hold for every Z (with one holding strictly), then product i will be offered.

Proposition 6 is of theoretical interest in identifying forces that limit product lines even in the absence of fixed costs.[25] However, there is also empirical content when economies of scope are strong, meaning that incremental per-product fixed costs are relatively small over a range of quality. For example, consider an automobile manufacturer, who surely faces substantial overall fixed costs. Nonetheless, given a basic platform, the car's quality can easily be varied, for instance by putting in a more powerful engine, a better sound system, or leather upholstery. Moreover, while such quality increases are costly, they are reasonably taken to be marginal cost increases. (Of course, longer ranges of quality require further fixed investments, such as when BMW chooses to offer cars in a 3 series and also a 5 and a 7 series.) Thus, Proposition 6 is empirically relevant when analyzing such a "submarket," meaning a (potentially limited) range of qualities that can be produced by a firm that bears a single fixed cost (that is, a fixed cost that does not depend on the number of different qualities actually produced in that range).

Proposition 6 also allows for a welfare comparison. Consider the choice of a social planner, supposing that there are increasing returns to quality: $P_i(Z)/C_i < P_{i+1}(Z)/C_{i+1}$ for each Z. Take any consumer type θ, where $\theta = H(Z)$ for some Z. Suppose it were socially efficient for the planner to supply quality q_i to this consumer, necessitating $P_i(Z) \geq C_i$. However, from increasing returns it must also be true that $P_{i+1}(Z) > C_{i+1}$, so that the planner strictly prefers to supply upgrade $i+1$. This contradiction implies that it is never socially optimal to supply the complete product i. Nevertheless, a symmetric multiproduct Cournot industry may supply this product, if price sensitivity is sufficiently decreasing in quality.

□ **First-best welfare analysis: equilibrium quality distortion.** A classic result from the analysis of monopoly price discrimination is that the quality consumed by individuals is distorted downwards from the socially efficient level (Mussa and Rosen, 1978). Here we extend this result.

We define socially efficient consumption as follows. A social planner with access to the production technologies of the N (potentially asymmetric) firms produces for each consumer θ

[24] In terms of the underlying utility function $u(\theta, q)$, $\eta_i(Z) < \eta_{i+1}(Z)$ if and only if $\partial[(u(\theta, q_{i+1}) - u(\theta, q_i))/(u(\theta, q_i) - u(\theta, q_{i-1}))]/\partial\theta > 0$ for $\theta = H(Z)$. This says that higher types value higher upgrades proportionally more than lower types do.

[25] Asymmetries in technological capabilities are also important. For instance, when a firm enjoys a comparative advantage in upgrade market $i+1$, it will tend to produce more of upgrade $i+1$, which may lead it to eliminate quality q_i from its product line.

a single unit of the product that maximizes that consumer's utility less the costs of producing the unit, so long as this is a non-negative value.

Proposition 7. Suppose that some firm s has the lowest costs, so that $C_{is} \leq C_{ir}$ for each i and r, and consider any multiproduct Cournot equilibrium. The quality consumed by each type θ is weakly lower than a social planner would choose. Moreover, for a positive measure of types, the quality consumed is strictly lower than a social planner would choose.

A complete proof is in the Appendix. Here we argue the claim when firms are symmetric and produce all M products. Recall from Section 3 that in this case the (unique) equilibrium first-order condition for each firm for a given upgrade i is

$$P_i(Z_i^*) + \frac{Z_i^* P_i'(Z_i^*)}{N} = C_i.$$

Since $P_i(Z_i)$ is downward sloping, the price in each upgrade market exceeds the marginal cost of that upgrade. Hence, a planner prefers to increase the supply of each upgrade, meaning that she prefers to increase weakly quality for each consumer and to increase strictly it for some. This does not say that the supply of complete physical products of a given quality q_i should be higher, although it does say that the total number of products (of all qualities together) should be. Once again, the insights of single-product models (firms produce too little relative to the social optimum) extend to upgrades, but not to complete products.

We note that this proposition does not hold with arbitrary upgrade costs. For example, suppose that there are two firms and two products. If firm 1 is more efficient at producing product 1 but only firm 2 is capable of producing product 2, then it is possible that in equilibrium some consumers purchase product 2 even though society would be better off if these consumers instead used a lower quality product built by firm 1. However, even for this counter-example the spirit of the proposition remains valid: a social planner prefers a consumer to receive a weakly higher quality when constrained to use the production technology of the consumer's original supplier.

□ **Second-best welfare analysis: free entry and social inefficiency.** Here we extend the insight of Mankiw and Whinston (1986) regarding excessive entry.[26] This leads to a second-best welfare comparison regarding not only entry levels but also the qualities supplied to consumers, complementing the first-best comparison of Proposition 7.

We study the following two-stage model. In stage one, N firms enter, where N is taken to be continuous. In stage two, firms compete in quantities as in our previous analysis. Entering requires the fixed cost $F > 0$ and allows the firm to produce any of the M products with upgrade costs $\{C_i\}_{i=1}^M$. Thus entrants have symmetric production capabilities, and moreover there are strong economies of scope so that firms that have invested F need bear no further significant per-product fixed costs. Many, although by no means all, industries exhibit strong economies of scope across some range.[27] Drawing upon the discussion following Proposition 6, our results here at least apply to entry into a submarket of the overall quality ladder within which quality changes do not require further significant fixed costs.

Let $\pi(N)$ denote the variable profits of a firm. This function is well specified, continuous, and strictly decreasing, which ensures a unique equilibrium value N^e.[28] At the equilibrium entry level, the variable profits of entrants exactly offset the fixed cost F, so $\pi(N^e) = F$.

Our social efficiency benchmark is as follows. In stage one a social planner interested in maximizing consumer utility less total costs chooses a number of firms N to enter, but in stage

[26] The question of whether excessive entry occurs in multiproduct environments has been explored in price-setting models of product variety by Anderson and de Palma (1992, 2006).

[27] The automobile industry provides one of many examples from manufacturing. Other examples include "damaged goods" (Deneckere and McAfee, 1996).

[28] The function is well specified because the equilibrium is unique given N by Proposition 1. Other properties follow from Proposition 2 and inspection of the relevant first-order condition (3).

two firms compete as in our previous analysis; we will thus be making a second-best welfare comparison. Let $Z_i^*(N)$ denote the equilibrium industry output of upgrade i given entry level N, and let N^* denote an optimal value. The planner's objective function is

$$W(N) = \sum_{i=1}^{M} \int_0^{Z_i^*(N)} [P_i(Z) - C_i]\, dZ - FN.$$

Comparing the following result to Proposition 7, we see that although quality is undersupplied relative to the first-best, it is oversupplied relative to the second-best.

Proposition 8. In the free-entry equilibrium in which the entry level N is a continuous variable and in which bearing the entry cost F allows a firm to produce all available products at upgrade costs $\{C_i\}_{i=1}^M$, the level of entry is higher than in the second-best: $N^e > N^*$. Furthermore, quality is oversupplied relative to the second-best: the quality consumed by each type θ is weakly higher than in the second-best, and for positive measure of types the quality consumed is strictly higher than in the second-best.

The intuition behind Proposition 8 is quite similar to that in Mankiw and Whinston (1986). They showed that when there is a business-stealing effect in the market for a homogeneous good, marginal entrants value their entry more than the social planner does, and consequently equilibrium entry and industry output are excessive. The same forces operate in a multiproduct setting, with the equilibrium industry supply of each upgrade being excessively high. Recalling once again that the supply of each upgrade i equals the total number of complete units of quality q_i or higher, it follows that quality is too high relative to the second-best.

5. Strategic product line choice

■ Here we take up our remaining question by exploring the possibility of strategic product line commitment, and ask how this influences our earlier results.[29] We discuss at the end of this section how our results differ substantially from those in the price-setting literature.

The number of firms is fixed at N. Each firm, prior to setting quantities, simultaneously decides whether to impose restrictions on its production capabilities; equivalently, firms decide which products they are capable of producing. Mixed strategies are allowed. Each firm's realized choice is observed by all other firms, and then (pure-strategy) quantity competition occurs. This two-stage formulation of multiproduct quality competition captures the possibility that there may be a long-run component to product line choice, causing strategic effects. Since we are interested in understanding these effects, we ignore direct costs associated with the first stage. We look at subgame-perfect Nash equilibria.

We restrict attention to the two-quality case. In the first stage, therefore, each firm (potentially at random) chooses whether to restrict itself to selling just the high-quality product or just the low-quality product, or instead not to restrict itself. Existence of an equilibrium is ensured.[30] To rule out uninteresting cases, we suppose that if a firm s has not restricted itself to selling only the low-quality product, then $Z_{2s}^* > 0$.

□ **Committing to low quality.** Suppose that some firm r commits to producing only the low-quality product; this amounts to the restriction that $Z_{2r} = 0$ in the second stage. Intuitively, in partial equilibrium (that is, fixing the outputs of other firms), firm r is forced to lower Z_{2r} and may also choose to lower Z_{1r} if $Z_{1r}^* = Z_{2r}^*$ originally. Firms $s \neq r$, meanwhile, will respond to the

[29] Our analysis in this section also could be applied more generally to firms making strategic investments that influence their marginal production costs. Thus, it may be extended to understand the strategic tradeoffs a firm faces when deciding whether to become a "niche" player specializing in a certain quality range.

[30] Our earlier uniqueness result for the case of two qualities is easily extended to this setting, ensuring a unique equilibrium in each subgame. Since firms have a finite number of choices in the first stage, standard techniques ensure that an equilibrium of the overall two-stage game exists (potentially in mixed strategies).

lowering of Z_{2r} by expanding their own outputs in upgrade market 2. This may lead them to also raise Z_{1s}, and Z_{1s} may also increase if Z_{1r} decreases. The net equilibrium effect is an increase in $\sum_{s \neq r} Z^*_{js}$ for $j \in \{1, 2\}$.

We conclude that there may well be strategic effects associated with committing to selling only the low-quality product. However, these effects work strictly against the firm that so commits: they lead to an increase in the sum of other firms' outputs in each upgrade market and also constrain the firm's ability to respond optimally.

Proposition 9. In the two-stage game with two possible quality levels, in equilibrium no firm restricts itself to selling only the low-quality product.

Another way of thinking about Proposition 9 is that committing to selling only the low-quality product is equivalent to an increase in C_{2r} by some large amount κ. Proposition 5 indicates that this leads to a decrease in profits for firm r, and indeed it can be shown that the current proposition is a corollary of the earlier one.

□ **Committing to high quality.** We turn now to the possibility that some firm r might commit to selling only the high-quality product, which is equivalent to imposing the constraint $Z_{1r} = Z_{2r}$. This induces conflicting effects, and it is optimal in some circumstances but not in others. To understand this, first note that the partial equilibrium effect is for r to lower Z_{1r} and raise Z_{2r} from their initial levels. Firms $s \neq r$ will tend to respond by raising Z_{1s} and lowering Z_{2s}.

Proposition 10. Consider the two-stage game with two possible quality levels. Suppose that firms $s \neq r$ are not restricting their product lines in stage 1, and that when firm r does not do so, $Z^*_{1s} > Z^*_{2s}$ for all s. Then when firm r moves from not restricting itself to only selling product 2, the second-stage equilibrium changes as follows: (i) $\sum_{s \neq r} Z^*_{1s}$ strictly increases and $\sum_{s \neq r} Z^*_{2s}$ strictly decreases, (ii) Z^*_{1r} strictly decreases and Z^*_{2r} strictly increases, and (iii) Z^*_1 strictly decreases and Z^*_2 strictly increases.

This proposition can be stated in terms of complete products. For instance, the first claim says that total rivals' output of the low-quality good increases while their output of the high-quality good decreases, and the second claim says that the opposite happens for firm r.

Committing to selling only the high-quality good has conflicting effects on r. On the one hand, r is "tougher" in upgrade market 2, leading others to curb production there to the benefit of r. On the other hand, r is "softer" in upgrade market 1, causing others to increase production to the detriment of r.

Heuristically, this also can be understood by supposing that firm r's upgrade cost in market 1 increases by some amount $\kappa > 0$ while its cost in upgrade market 2 decreases by a like amount. A suitably chosen value of κ leads to the same equilibrium outputs and profits for all firms as does directly imposing the constraint $Z_{1r} = Z_{2r}$. To understand the total effect on profits from increasing κ, denote firm r's profits by $\pi_r(\kappa)$ and consider the marginal change in profits. Using the envelope theorem, we have precisely

$$\frac{d\pi_r(\kappa)}{d\kappa} = (Z^*_{2r} - Z^*_{1r}) + Z^*_{1r} P'_1(Z^*_1) \sum_{s \neq r} \frac{\partial Z^*_{1s}}{\partial \kappa} + Z^*_{2r} P'_2(Z^*_2) \sum_{s \neq r} \frac{\partial Z^*_{2s}}{\partial \kappa}.$$

The first term $(Z^*_{2r} - Z^*_{1r})$ is negative, since the increase in upgrade market 1's costs is felt over more units than the decrease in market 2's costs. The second term is the strategic effect on profits in upgrade market 1. For this upgrade, firm r's reaction curve is pulled down, causing its rivals to expand: the strategic price effect is negative. The third term is the strategic effect on profits in upgrade market 2, and is positive because r's reaction curve pushes outward, lowering the output of other firms so that the strategic price effect is positive.

The overall impact on profits from a marginal increase in κ is ambiguous and depends on the magnitudes of the slopes of firms' reaction functions in the two upgrade markets, upon which we have put little structure. It follows that the total change in profits brought about by imposing the constraint $Z_{1r} = Z_{2r}$ (equivalently, by raising κ) is ambiguous. However, we do see that for a firm to gain by restricting itself to only selling the high-quality good, the positive strategic effect in market 2 must not only overwhelm the negative effect in market 1, but also overcome the effect of being constrained. Note further that because r produces more in upgrade market 1, for the strategic profit effect in upgrade market 2 to overwhelm that in upgrade market 1, a necessary condition is that the strategic price effect in market 2 strictly exceeds that in upgrade market 1:

$$\left| P_2'(Z_2^*) \sum_{s \neq r} \frac{\partial Z_{2s}^*}{\partial \kappa} \right| > \left| P_1'(Z_1^*) \sum_{s \neq r} \frac{\partial Z_{1s}^*}{\partial \kappa} \right|.$$

We now describe the complete equilibrium outcome of the two-stage game in the special case of multiplicative preferences and uniformly distributed consumer valuations. This yields linear upgrade demand curves $P_i(Z_i) = (q_i - q_{i-1})(1 - Z_i)$ for $i \in \{1, 2\}$. In this case, equilibrium exhibits firms competing head-to-head, that is, each firm selling both products.[31]

Proposition 11. Let demand be linear in both upgrade markets. Suppose, if no firm restricts itself in the first stage, that $Z_{1s}^* > Z_{2s}^*$ for each s. Then there exists a unique equilibrium to the two-stage game. In the equilibrium, no firm restricts its product line in the first stage and each firm produces positive quantities of both complete products in the second stage.

Our results differ from those in the literature on commitment with price competition.[32] Champsaur and Rochet (1989) found that price-setting multiproduct duopolists avoid competing head-to-head in order to reduce the intensity of price competition; equilibrium involves each firm committing to an interval of qualities that does not intersect that offered by the other. For similar reasons, firms hold back on the breadth of their product portfolios in the analyses of Anderson and de Palma (1992, 2006). The underlying theoretical reason for why we reach different conclusions is simply that prices tend to be strategic complements, while quantities tend to be strategic substitutes.

One might then wonder whether the possibility we identify of firms choosing to compete head-to-head is merely a theoretical curiosity. It is not; there are many examples of firms so competing with vertically differentiated product lines. For example, in the market for plasma televisions, manufacturers such as Mitsubishi, Samsung, and Sony each offer product lines spanning the set of high-resolution possibilities.[33] Similarly, the microprocessors of AMD and Intel compete at many points in quality space,[34] as do the cars of Audi, BMW, Mercedes-Benz, and Jaguar.[35] Note further that in these industries, capacity choices are important, and developing new products takes time. Thus, the spirit of our formulation in this section seems appropriate.

If the softening of price competition is a main objective in designing a line of quality-differentiated products, the outcomes in these industries are less intuitive. However, if we entertain the possibility that competition in these industries is reasonably approximated by quantity setting, then these observations are not surprising.

[31] The additional restriction in the hypothesis of Proposition 11 only serves to rule out multiple equilibria that are equivalent as far as second-period outcomes. Suppose the assumption that $Z_{1s}^* > Z_{2s}^*$ whenever no firm restricts itself in the first stage fails for some firm. Then, whether this firm restricts itself in the first stage or not has no effect on any firm's outputs or payoffs in the second stage. However, strictly speaking there would exist multiple equilibria categorized by this firm's first-stage decision.

[32] The idea that the intensity of price competition may be influenced by product characteristics of single-product firms was pursued by Gabszewicz and Thisse (1979) and Shaked and Sutton (1982).

[33] Each offers "enhanced-definition" televisions that display 480 horizontal lines of resolution, as well as "high-definition" televisions that display approximately either 720 or 1,080 horizontal lines. These resolutions effectively span what would be useful to consumers given current source material and transmission standards.

[34] Their products range from low-end desktop PC chips (the Sempron for AMD and the Celeron for Intel) to 64-bit dual-core workstation processors (such as elements in AMD's Opteron and Intel's Xeon series).

[35] Audi's A4, A6, and A8 are positioned, respectively, against BMW's 3, 5, and 7 series; Mercedes-Benz's C, E, and S classes; and Jaguar's X, S, and XJ types. Each of the twelve models has its own platform, distinguishing this example from that of within-platform scope economies discussed after Proposition 6.

We conclude that product line commitments introduce interesting strategic effects. Nonetheless, in some cases firms will not gain from strategically omitting products, providing further justification for the simultaneous-move game specified in our base analysis of earlier sections.

6. Concluding remarks

■ A general model of multiproduct Cournot competition was presented and analyzed. Since our model specified in terms of upgrades is an isometry of the model specified in terms of complete products, any results we have derived using our approach can be derived using the complete-products approach. However, a benefit of the upgrades approach is that intuition from single-product industries can be applied easily to multiproduct ones.

A number of empirical implications emerge from our study. First, following our analysis in Section 3, many comparative statics of single-product industries carry over to multiproduct ones when we think in terms of upgrades. Given the definition of upgrades, these predictions involve the distribution of qualities consumed. Second, following our analysis in Section 4, the quality consumed is lower than in the first-best, but (at least when economies of scope are strong) quality is too high relative to the second-best. Finally, Section 5's analysis suggests that strategic product line choice involves tradeoffs that differ sharply from other models of competition with vertically differentiated product lines, and is consistent with observed head-to-head competition.

Appendix

■ Proofs of Propositions 1–11 follow.

We prove Proposition 1 in steps. We begin by confirming that an equilibrium exists.

Proof of Proposition 1 (existence). Without loss of generality, we restrict to $Z_{1r} \leq 1$ for each r. Each firm's choice set is a compact and convex subset of a Euclidean space. A firm's profits are continuous in the choices of its opponents, and (given decreasing marginal revenue) concave in its own choice and hence quasi-concave in its own choice. It is well known that these results are sufficient for the existence of an equilibrium; see, for example, Fudenberg and Tirole (1991, p. 34). *Q.E.D.*

To prove uniqueness for cases (i) and (ii), we will suppose that there are two distinct equilibria with equilibrium supplies Z^*_{ir} and Z^{**}_{ir}, and then argue by contradiction. Before proceeding, we present two lemmas. From the maintained assumption that marginal revenue is decreasing in every upgrade market, the following is immediate.

Lemma A1. Suppose that $Z^{**}_i \geq Z^*_i$ and $Z^{**}_{ir} \geq Z^*_{ir}$. If one of these inequalities holds strictly, neglecting monotonicity constraints, if firm r has a weak incentive to expand its supply of upgrade i in the $(**)$-equilibrium, then it will have a strict incentive to do so in the $(*)$-equilibrium. Similarly, if it has a weak incentive to contract its supply of upgrade i in the $(*)$-equilibrium, then it will have a strict incentive to do so in the $(**)$-equilibrium.

The next lemma says that two distinct equilibria cannot have ranked industry output profiles.

Lemma A2. There cannot be two distinct equilibria satisfying $Z^{**}_i \geq Z^*_i$ for every upgrade i.

Proof. We argue by contradiction. If the equilibria differ and $Z^{**}_i \geq Z^*_i$ for every i, then there is some firm r that, for at least one upgrade, produces strictly more in the $(**)$-equilibrium than in the $(*)$-equilibrium. For such a firm, we can find a sequence of neighboring upgrade markets $\{j, \ldots, k\}$ for which $Z^{**}_{ir} > Z^*_{ir} \geq 0$ for all $i \in \{j, \ldots, k\}$. We can extend this sequence downward until either $j = 1$, or $Z^*_{(j-1)r} \geq Z^{**}_{(j-1)r}$. This ensures that there is no upward constraint on Z^*_{jr}; this is trivially true when $j = 1$, and for $j > 1$ we have $Z^*_{(j-1)r} \geq Z^{**}_{(j-1)r} \geq Z^{**}_{jr} > Z^*_{jr}$. Next, we extend the sequence upward until either $k = n$ or $k < n$ and $Z^*_{(k+1)r} \geq Z^{**}_{(k+1)r}$. This ensures that there is no downward constraint on Z^{**}_{kr}.

Under the $(*)$-equilibrium, firm r faces no binding upward constraint in market j. It could, therefore, simultaneously expand (that is, increase each upgrade output by an identical small amount) in all markets $i \in \{j, \ldots, k\}$. It does not, and so must have at least a weak incentive to contract simultaneously in all such markets. Since $Z^{**}_i \geq Z^*_i$ and $Z^{**}_{ir} > Z^*_{ir}$ for these i, under the $(**)$-equilibrium it has a strict incentive to contract simultaneously in all of these markets. It does not, and hence there must be a binding downward constraint in market k, so that $Z^{**}_{kr} = Z^{**}_{(k+1)r}$. This is a contradiction. *Q.E.D.*

Following Lemma A2, if there are two equilibria when there are only two qualities, then the industry outputs in these equilibria must satisfy either $Z^{**}_1 > Z^*_1$ and $Z^*_2 > Z^{**}_2$, or $Z^{**}_1 < Z^*_1$ and $Z^*_2 < Z^{**}_2$. We can label the firms to ensure that we deal with the former configuration. These observations lead to a proof of Proposition 1 for case (ii).

Proof of Proposition 1 (uniqueness for case (ii)). Lemma A2 implies that two distinct equilibria must satisfy $Z_1^{**} - Z_1^* > 0 > Z_2^{**} - Z_2^*$. Consider the subset of firms $R \subseteq \{1, \dots, N\}$ that satisfy $Z_{1r}^{**} > Z_{1r}^*$. Under the $(*)$-equilibrium, firm $r \in R$ faces no upward constraint in market $i = 1$, and hence must face a weak incentive to lower Z_{1r}. Since $Z_1^{**} > Z_1^*$ and $Z_{1r}^{**} > Z_{1r}^* \geq 0$, firm r faces a strict incentive to contract under the $(**)$-equilibrium, and so its monotonicity constraint must bind: $Z_{1r}^{**} = Z_{2r}^{**}$. Recalling that $Z_{2r}^* \leq Z_{1r}^*$, we have

$$r \in R \quad \Rightarrow \quad Z_{2r}^{**} - Z_{2r}^* \geq Z_{1r}^{**} - Z_{1r}^* > 0. \tag{A1}$$

Next, consider the subset of firms $S \subseteq \{1, \dots, N\}$ that satisfy $Z_{2s}^{**} < Z_{2s}^*$. Using an argument equivalent to that given above, we can conclude that $Z_{1s}^{**} = Z_{2s}^{**}$, and

$$s \in S \quad \Rightarrow \quad Z_{1s}^* - Z_{1s}^{**} \geq Z_{2s}^* - Z_{2s}^{**} > 0. \tag{A2}$$

R contains all firms satisfying $Z_{1r}^{**} > Z_{1r}^*$, and no members of S. Since $Z_1^{**} > Z_1^*$, this implies

$$\sum_{r \in R}(Z_{1r}^{**} - Z_{1r}^*) > \sum_{s \in S}(Z_{1s}^* - Z_{1s}^{**}) \geq \sum_{s \in S}(Z_{2s}^* - Z_{2s}^{**}), \tag{A3}$$

where the weak inequality follows from (A2). Similarly, since $Z_2^* > Z_2^{**}$,

$$\sum_{s \in S}(Z_{2s}^* - Z_{2s}^{**}) > \sum_{r \in R}(Z_{2r}^{**} - Z_{2r}^*) \geq \sum_{r \in R}(Z_{1r}^{**} - Z_{1r}^*), \tag{A4}$$

where this time the weak inequality follows from (A1). Chaining the inequalities of (A3) and (A4) together, we reach the contradiction $\sum_{r \in R}(Z_{1r}^{**} - Z_{1r}^*) > \sum_{r \in R}(Z_{1r}^{**} - Z_{1r}^*)$. *Q.E.D.*

Proof of Proposition 1 (uniqueness for case (i)). By assumption, there are at most two types of firms. We partition the firms into subsets R and S according to these types. Firms of the same type share the same cost structure, and hence offer the same outputs in equilibrium (this is implied by Proposition 4). For instance, if $\{r, r'\} \subseteq R$, then $Z_{ir}^* = Z_{ir'}^*$ and $Z_{ir}^{**} = Z_{ir'}^{**}$ for all i. Suppose that j is the first upgrade for which two equilibria differ. Without loss of generality, suppose that $Z_j^{**} \geq Z_j^*$. Given that the equilibria differ at j, one subset of firms must be producing strictly more in the $(**)$-equilibrium than in the $(*)$-equilibrium. Suppose it is subset R, so that $Z_{jr}^{**} > Z_{jr}^*$ for $r \in R$. Since j is the first upgrade where the equilibria differ, then either $j = 1$ or $j > 1$ and $Z_{(j-1)r}^* = Z_{(j-1)r}^{**} \geq Z_{jr}^{**} > Z_{jr}^*$, and hence firm $r \in R$ faces no upward constraint on Z_{jr}^*. Thus, it has a weak incentive to lower Z_{jr}^*. Since $Z_{jr}^{**} > Z_{jr}^* \geq 0$ and $Z_j^{**} \geq Z_j^*$, it has a strict incentive to lower Z_{jr}^{**}, yet does not. If $j = n$, this is impossible and we have reached a contradiction. If $j < n$, however, then it must face a binding downward constraint, $Z_{jr}^{**} = Z_{(j+1)r}^{**}$, which in turn implies that $Z_{(j+1)r}^{**} - Z_{(j+1)r}^* \geq Z_{jr}^{**} - Z_{jr}^* > 0$. There are now two possibilities. The first is that $Z_{j+1}^{**} \geq Z_{j+1}^*$, and the second is that $Z_{j+1}^{**} < Z_{j+1}^*$.

We deal with the first possibility here. If $Z_{j+1}^{**} \geq Z_{j+1}^*$, then, since $Z_{(j+1)r}^{**} > Z_{(j+1)r}^*$ and firm r is free to expand in markets j and $j+1$ under the $(*)$-equilibrium, we can repeat our argument and conclude that $Z_{(j+2)r}^{**} = Z_{(j+2)r}^{**}$ and $Z_{(j+2)r}^{**} - Z_{(j+2)r}^* \geq Z_{(j+1)r}^{**} - Z_{(j+1)r}^* > 0$. In fact, we iterate the same argument until either we reach an upgrade $k = n$, and therefore a contradiction, or we find an upgrade k where $Z_k^{**} - Z_k^* \geq 0 > Z_{k+1}^{**} - Z_{k+1}^*$ and $Z_{(k+1)r}^{**} - Z_{(k+1)r}^* \geq Z_{kr}^{**} - Z_{kr}^* > 0$ for all firms $r \in R$.

Finding such an upgrade k corresponds precisely to the second possibility. Since $Z_{k+1}^{**} < Z_{k+1}^*$ while $Z_{(k+1)r}^{**} > Z_{(k+1)r}^*$ for all $r \in R$, it must be the case that $Z_{(k+1)s}^{**} < Z_{(k+1)s}^*$ for all $s \in S$. Furthermore, given the other inequalities derived above,

$$\sum_{s \in S}(Z_{(k+1)s}^* - Z_{(k+1)s}^{**}) > \sum_{r \in R}(Z_{(k+1)r}^{**} - Z_{(k+1)r}^*) \geq \sum_{r \in R}(Z_{kr}^{**} - Z_{kr}^*) \geq \sum_{s \in S}(Z_{ks}^* - Z_{ks}^*),$$

which implies that $Z_{(k+1)s}^{**} < Z_{ks}^{**}$. We now see that under the $(*)$-equilibrium, any firm $s \in S$ is unconstrained upward. We can use identical logic to show that this means it has a strict incentive to contract $Z_{(k+1)s}^* > 0$. Once again, if $k+1 = n$, then we have reached a contradiction. If not, then we can continue as before, only this time working with the subset of firms S. This procedure only stops once we reach upgrade n and find that one subset of firms faces binding downward monotonicity constraints on its strictly positive supply of that upgrade; this is where we always reach a contradiction. *Q.E.D.*

Proof of Proposition 2. We write Z_i^* and Z_i^{**} respectively for equilibrium industry output of upgrade i before and after an increase in the number of firms. Following Proposition 1, each equilibrium will be symmetric. Hence we are able to work only with industry outputs and will not need to refer to individual firm outputs. For any upgrade i satisfying $Z_i^* = 0$, it is automatically true that $Z_i^{**} \geq Z_i^*$. We will show that if $Z_i^* > 0$, then $Z_i^{**} > Z_i^*$.

Suppose not. We may find a sequence of neighboring upgrades $\{j, \dots, k\}$ for which $Z_i^{**} \leq Z_i^*$ and $Z_i^* > 0$ for all $i \in \{j, \dots, k\}$. We extend this sequence downward (redefining j as required) until either $j = 1$ or $Z_{j-1}^{**} > Z_{j-1}^*$. If $j = 1$, then there is no upward constraint on Z_j^{**}. If $j > 1$, then we combine inequalities to obtain $Z_{j-1}^{**} > Z_{j-1}^{*} \geq Z_j^{*} \geq Z_j^{**}$. This ensures that $Z_{j-1}^{**} > Z_j^{**}$, and once again there is no upward constraint on Z_j^{**}. Next, we extend the sequence upward (here redefining k as required) until either $k = n$ or $Z_{k+1}^{**} > Z_{k+1}^{*}$. If $k = n$, there is no downward constraint on Z_k^{*}. If $k < n$, then we may combine inequalities to obtain $Z_k^{*} \geq Z_k^{**} \geq Z_{k+1}^{**} > Z_{k+1}^{*}$. Again, since $Z_k^{*} > Z_{k+1}^{*}$, there is no downward constraint on Z_k^{*}.

Under the $(**)$-equilibrium, each firm faces no upward constraint in market j, and so must face a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Under the $(*)$-equilibrium, industry output for each $i \in \{j, \ldots, k\}$ is weakly higher. Since there are strictly fewer firms, each individual firm's output for each one of these upgrades must be strictly higher. Hence, each firm faces a strict incentive to contract in all of these markets. No firm does, and so there must be a binding downward constraint in market k. This is a contradiction.

We have shown that the industry's output of each positively supplied upgrade strictly increases following entry, and hence the upgrade's price must strictly fall. The baseline upgrade (that is, $i = 1$) is always in strictly positive supply, and hence its price strictly falls.

Given that output in each upgrade market increases following entry, it is easy to show that the total output of rivals increases from any firm's perspective, and hence it is worse off. Furthermore, the industry's profits must also fall. The industry is producing more than a monopolist would, and hence further output expansions must reduce industry profits. *Q.E.D.*

Proof of Proposition 3. We write Z_i^{*} and Z_i^{**} respectively for equilibrium industry output of upgrade i before and after a weak increase in upgrade costs. If the proposition is false, then we may find a sequence $\{j, \ldots, k\}$ for which $Z_i^{**} > Z_i^{*}$ for all $i \in \{j, \ldots, k\}$. We extend the sequence downward (redefining j as required) until either $j = 1$ or $Z_{j-1}^{**} \leq Z_{j-1}^{*}$. An argument similar to that used in the proof of Proposition 2 ensures that there is no upward constraint on Z_j^{*}. We extend the sequence upward (here redefining k as required) until either $k = n$ or $Z_{k+1}^{**} \leq Z_{k+1}^{*}$. Again, an argument similar to that used in the proof of Proposition 2 ensures that there is no downward constraint on Z_k^{**}.

Under the $(*)$-equilibrium each firm faces no upward constraint in market j, and so must face a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Under the $(**)$-equilibrium and for all such $i \in \{j, \ldots, k\}$, industry output is strictly higher, firm outputs are strictly higher, and costs are weakly higher (since we have not assumed that an upgrade cost for some i within this sequence $\{j, \ldots, k\}$ has increased). Each firm must face a strict incentive to contract for all $i \in \{j, \ldots, k\}$. No firm does so, and hence there must be a binding downward constraint in market k. This is a contradiction. Hence, we conclude that $Z_i^{**} \leq Z_i^{*}$ for each $i \in \{j, \ldots, k\}$. The rise in the price of each upgrade follows directly.

We now show that if an upgrade is in positive supply and its cost strictly increases, then its supply must strictly fall. Suppose not. We know that the supply of upgrades cannot strictly rise. Thus, if the claim is false, we must be able to find a sequence $\{j, \ldots, k\}$ for which $Z_i^{**} = Z_i^{*} > 0$ for all $i \in \{j, \ldots, k\}$ and for which costs are strictly higher for some $i \in \{j, \ldots, k\}$. Once again, we expand the sequence to ensure that there is no upward constraint on Z_j^{*} and downward constraint on Z_k^{**}. Under the $(*)$-equilibrium, each firm faces a weak incentive to simultaneously contract for all $i \in \{j, \ldots, k\}$. Due to the strictly higher cost for some upgrade in this sequence, there must be a strict incentive to contract under the $(**)$-equilibrium. Once again, this yields a contradiction.

The final claim is that industry profits fall following a change in costs. We omit a complete proof for space considerations, but describe the process. Amalgamate products as necessary, and apply the implicit function theorem to compute the change in output following an increase in costs. Then the change in profits is easily computed, for example using the envelope theorem. It is easily shown that our definition of decreasing marginal revenue precludes a profit increase. The effect of discrete changes in costs may be calculated by integrating. If the set of products being offered changes as costs do, the integral must be computed over different regions, but the result is the same. *Q.E.D.*

Proof of Proposition 4. Suppose that the first part of the result is false. Take the lowest j for which $Z_{jr}^{*} < Z_{js}^{*}$. Since this is the lowest such j, then either $j = 1$ or $j > 1$ and $Z_{(j-1)r}^{*} \geq Z_{(j-1)s}^{*}$, which implies that $Z_{(j-1)r}^{*} > Z_{jr}^{*}$. For both of these cases, firm r faces no upward monotonicity constraint in market j. Next, take the highest k for which $Z_{ks}^{*} = Z_{js}^{*}$. Since this is the highest k with this property, and since $Z_{js}^{*} > Z_{jr}^{*} \geq 0$, firm s faces no downward monotonicity constraint in market k. Hence firm s has a weak incentive to raise simultaneously the supply of all upgrades $i \in \{j, \ldots, k\}$. Since $Z_{is}^{*} > Z_{ir}^{*}$ for all $i \in \{j, \ldots, k\}$, and firm r has weakly lower costs, firm r must have a strict incentive to raise simultaneously all of these outputs. Since this is feasible by construction, we have a contradiction. It now follows that lower-cost firms face lower output of rivals in each upgrade market. Since they also have lower costs, they must be earning higher profits. *Q.E.D.*

Proof of Proposition 5. We consider an increase in C_{1r}. (The proof for an increase in C_{2r} follows from similar logic.) We write Z_{is}^{*} and Z_{is}^{**} for equilibrium upgrade outputs before and after this increase, and consider an exhaustive list of possibile cases.

Case (i): $Z_1^{**} \geq Z_1^{*}$ and $Z_2^{**} \geq Z_2^{*}$. Suppose that $Z_{1s}^{**} > Z_{1s}^{*} \geq 0$ for some firm s. It must have a weak incentive to lower Z_{1s}^{*} and hence a strict incentive to lower Z_{1s}^{**}, implying $Z_{1s}^{**} = Z_{2s}^{**} > Z_{2s}^{*} \geq 0$. Now, firm s must have a weak incentive to simultaneously raise Z_{1s}^{**} and Z_{2s}^{**} and thus a strict incentive to raise Z_{1s}^{*} and Z_{2s}^{*}. It does not, and we have a contradiction. We conclude that $Z_{1s}^{**} \leq Z_{1s}^{*}$ for all s. If $Z_{1s}^{**} < Z_{1s}^{*}$, then this would imply that $Z_1^{**} < Z_1^{*}$, which contradicts the original assumption that $Z_1^{**} \geq Z_1^{*}$. We conclude that $Z_{1s}^{**} = Z_{1s}^{*}$ for all s. Similar arguments ensure that $Z_{2s}^{**} \leq Z_{2s}^{*}$ for all s. Combining this fact with the assumption that $Z_2^{**} \geq Z_2^{*}$, we conclude that $Z_{2s}^{**} = Z_{2s}^{*}$ for all s. This is specifically true for firm r, and additionally we know that $Z^*_{1r} > 0$ by assumption. Firm r has a weak incentive to lower Z^*_{1r}. Since C_{1r} is strictly higher in the (∗∗)-equilibrium, it must have a strict incentive to lower Z^{**}_{1r}. This is a contradiction if $Z^{**}_{1r} > Z^{**}_{2r}$. Suppose instead that $Z^{**}_{1r} = Z^{**}_{2r} > 0$, so that $Z^*_{1r} = Z^*_{2r} > 0$. Firm r must have a weak incentive to simultaneously lower Z^*_{1r} and Z^*_{2r}. Once again, in the (∗∗)-equilibrium, where it produces the same output but has strictly higher costs, it must have a strict incentive to engage in simultaneous contraction, and this final contradiction rules out case (i) as a possibility.

Case (ii): $Z^{**}_1 \geq Z^*_1$ and $Z^{**}_2 < Z^*_2$. We claim that $Z^{**}_{2s} - Z^*_{2s} \geq Z^{**}_{1s} - Z^*_{1s}$ for all s. Once this claim is established, we will show that it leads to a contradiction. Recalling that $Z^*_{1s} \geq Z^*_{2s}$ is always satisfied, we note that the claim is automatically true when $Z^{**}_{1s} = Z^{**}_{2s}$. If $Z^{**}_{1s} > Z^*_{1s}$, then the argument at the beginning of case (i) ensures that $Z^{**}_{1s} = Z^{**}_{2s}$. This means that the only situation in which the claim might fail is when $Z^*_{1s} \geq Z^{**}_{1s} > Z^{**}_{2s}$. Firm s has a weak incentive to lower Z^{**}_{2s}. Since $Z^*_2 > Z^{**}_2$, if $Z^*_{2s} > Z^{**}_{2s} \geq 0$, then firm s would have a strict incentive to lower Z^*_{2s}. This would be a contradiction, and hence $Z^{**}_{2s} \geq Z^*_{2s}$ while $Z^{**}_{1s} \leq Z^{**}_{1s}$ and hence $Z^{**}_{2s} - Z^*_{2s} \geq Z^{**}_{1s} - Z^*_{1s}$. We have proven that $Z^{**}_{2s} - Z^*_{2s} \geq Z^{**}_{1s} - Z^*_{1s}$ for all s. However,

$$0 > Z^{**}_2 - Z^*_2 = \sum\nolimits_s (Z^{**}_{2s} - Z^*_{2s}) \geq \sum\nolimits_s (Z^{**}_{1s} - Z^*_{1s}) = Z^{**}_1 - Z^*_1 \geq 0.$$

The final inequality is the case (ii) assumption that $Z^{**}_1 \geq Z^*_1$. The contradiction $0 > 0$ rules out case (ii) as a possibility.

We ruled out cases (i) and (ii), and must conclude that $Z^{**}_1 < Z^*_1$. This proves the claim that the industry's output Z_1 is strictly decreasing in C_{1r} when $Z^*_{1r} > 0$. For cases (iii) and (iv) we now show that firm r's competitors weakly expand in both upgrade markets.

Case (iii): $Z^{**}_1 < Z^*_1$ and $Z^{**}_2 < Z^*_2$. We can employ the logic used for case (i) to show that $Z^{**}_{1s} \geq Z^*_{1s}$ and $Z^{**}_{2s} \geq Z^*_{2s}$ for any $s \neq r$, and hence $\sum_{s \neq r}(Z^{**}_{is} - Z^*_{is}) \geq 0$ for $i \in \{1, 2\}$.

Case (iv): $Z^{**}_1 < Z^*_1$ and $Z^{**}_2 \geq Z^*_2$ with $Z^{**}_{2r} \leq Z^*_{2r}$. We will show that $Z^{**}_{1s} - Z^*_{1s} \geq Z^{**}_{2s} - Z^*_{2s}$ for any firm $s \neq r$. If $Z^*_{1s} = Z^*_{2s}$, then, since $Z^{**}_{1s} \geq Z^{**}_{2s}$, this is automatically true. Suppose instead that $Z^*_{1s} > Z^*_{2s}$. Firm s has a weak incentive to raise Z^*_{1s}. If $Z^{**}_{1s} \leq Z^*_{1s}$, then, since $Z^{**}_1 < Z^*_1$, it would have a strict incentive to raise Z^{**}_{1s}, a contradiction. Thus $Z^{**}_{1s} > Z^*_{1s}$. A similar argument ensures that $Z^{**}_{2s} \leq Z^*_{2s}$. This implies that $Z^{**}_{1s} - Z^*_{1s} \geq Z^{**}_{2s} - Z^*_{2s}$. Hence,

$$\sum_{s \neq r}(Z^{**}_{1s} - Z^*_{1s}) \geq \sum_{s \neq r}(Z^{**}_{2s} - Z^*_{2s}) \geq 0,$$

where the second inequality follows directly from $Z^{**}_2 \geq Z^*_2$ with $Z^{**}_{2r} \leq Z^*_{2r}$.

For cases (iii) and (iv) we have shown that firm r's competitors weakly expand in both upgrade markets following an increase in C_{1r}. In the (∗∗)-equilibrium firm r faces weakly more competition and strictly higher costs, and hence must be strictly worse off given that $Z^*_{1r} > 0$. Furthermore, using similar logic to that employed before, it is straightforward to confirm that $Z^{**}_{1r} < Z^*_{1r}$ for both cases, and $Z^{**}_{2r} \leq Z^*_{2r}$ for case (iii). This completes the proof for (iii) and (iv); we need eliminate only case (v).

Case (v): $Z^{**}_1 < Z^*_1$ and $Z^{**}_2 \geq Z^*_2$ with $Z^{**}_{2r} > Z^*_{2r}$. As for case (iv), $Z^{**}_{1s} - Z^*_{1s} \geq Z^{**}_{2s} - Z^*_{2s}$ for any firm $s \neq r$. By assumption, $Z^{**}_2 - Z^*_2 \geq 0 > Z^{**}_1 - Z^*_1$. Combining these observations, it must be the case that $Z^{**}_{1r} - Z^*_{1r} < Z^{**}_{2r} - Z^*_{2r}$. Since $Z^{**}_{1r} \geq Z^{**}_{2r}$, it must be the case that $Z^*_{1r} > Z^*_{2r}$. Firm r has a weak incentive to lower Z^*_{2r}, and since $Z^{**}_2 \geq Z^*_2$ with $Z^{**}_{2r} > Z^*_{2r} \geq 0$, it must have a strict incentive to lower Z^{**}_{2r}. This is a contradiction. *Q.E.D.*

Proof of Proposition 6. This follows from the argument in the text. *Q.E.D.*

Proof of Proposition 7. Fix a consumer θ who purchases product i from firm r. This implies that firm r supplies product i, and so $Z^*_{ir} > Z^*_{(i+1)r}$. For any $k < i$, this inequality implies that firm r is free to simultaneously lower its supply of all upgrades $j \in \{k + 1, \ldots, i\}$. It chooses not to do so, and hence

$$\sum_{j=k+1}^{i} C_{jr} \leq \sum_{j=k+1}^{i} [P_j(Z^*_j) + Z^*_{jr} P'_j(Z^*_j)] < \sum_{j=k+1}^{i} P_j(Z^*_j) = p_i - p_k \leq u(\theta, q_i) - u(\theta, q_k), \tag{A5}$$

where the last inequality follows since θ purchases product i rather than k.

Consider now a social planner who is free to use the production technology of any firm in the industry. By assumption, firm s has the lowest costs. Hence the planner would supply all consumers from firm s. Now, using (A5), for any $k < i$,

$$u(\theta, q_i) - u(\theta, q_k) > \sum_{j=k+1}^{i} C_{jr} \geq \sum_{j=k+1}^{i} C_{js},$$

which implies that the social planner would always have firm s supply a quality weakly greater than q_i to type θ.

For the final claim of the proof, set $\theta = H(Z^*_i)$. Since $u(\theta, q)$ is continuous in θ, for sufficiently small ε the argument given above proves that a consumer of type $\theta - \varepsilon$ would be assigned a quality of at least q_i by the social planner, but this type in fact purchases a quality of at most $q_{i-1} < q_i$. *Q.E.D.*

Proof of Proposition 8. Let N^* denote the socially optimal level of entry, satisfying

$$W'(N^*) = \sum_{i=1}^{M} \frac{dZ_i^*(N^*)}{dN} \left[P_i(Z_i^*(N^*)) - C_i\right] - F = 0.$$

By Proposition 2, $Z_i^*(N)$ increases with N, and moreover per-firm output falls with N. There is therefore a business-stealing effect: $Z_i^*(N)/N \geq dZ_i^*(N)/dN \geq 0$, with these being strict for $Z_i^*(N) > 0$. Hence the total profits of an entrant evaluated at N^* are

$$\sum_{i=1}^{M} \frac{Z_i^*(N^*)}{N^*} \left[P_i(Z_i^*(N^*)) - C_i\right] - F > \sum_{i=1}^{M} \frac{dZ_i^*(N^*)}{dN} \left[P_i(Z_i^*(N^*)) - C_i\right] - F = W'(N^*) = 0,$$

where the inequality will follow if we can argue that either $P_i(Z_i^*(N^*)) - C_i \geq 0$ for each i, or $Z_i^*(N^*) = dZ_i^*(N^*)/dN = 0$ whenever $P_i(Z_i^*(N^*)) - C_i < 0$. To see that this is the case, take $P_i(Z_i^*(N^*)) - C_i < 0$ for some upgrade i, and extend this sequence to all adjacent upgrades j such that $Z_j^*(N^*) = Z_i^*(N^*)$, yielding a sequence $\{j, \ldots, k\}$. Amalgamate this sequence as suggested in footnote 13. Either this amalgamated upgrade has a nonnegative price-cost margin, or the margin is negative but supply is zero. In the latter case, supply is also zero for a local increase in N, meaning $dZ_i^*(N^*)/dN = 0$ (for this amalgamated upgrade). *Q.E.D.*

Proof of Proposition 9. Suppose that firm r commits to omitting the high-quality product, which is only relevant if $Z_{2r}^* > 0$ initially. This commitment is equivalent to imposing a sufficiently high cost c_{2r} of producing quality q_2, or imposing a sufficiently high cost C_{2r}. By Proposition 5, such an increase will strictly harm firm r. The only subtlety is that in Proposition 5 no other firm s has restricted itself to sell only the high-quality or only the low-quality product. However, such restrictions would correspond to setting either c_{1s} or c_{2s} to sufficiently high levels; these possibilities are accommodated by Proposition 5. *Q.E.D.*

Proof of Proposition 10. As discussed in the text, a commitment by firm r to sell only quality q_2 is equivalent to adding $\kappa > 0$ to c_{1r} where κ is sufficiently large. Equivalently, firm r operates with upgrade costs of $C_{1r} + \kappa$ and $C_{2r} - \kappa$. Beginning from $\kappa = 0$, all firms sell both products, and hence first-order conditions are satisfied in both upgrade markets. A marginal increase in κ results in an expansion in Z_{1s}^* and a contraction in Z_{2s}^* for each $s \neq r$. This ensures that we may safely ignore the monotonicity constraints of firms $s \neq r$ as we increase κ. It is straightforward to confirm that Z_1^* and Z_{1r}^* shrink while Z_2^* and Z_{2r}^* expand as κ increases. We keep increasing κ until the constraint $Z_{1r}^* \geq Z_{2r}^*$ binds. *Q.E.D.*

Proof of Proposition 11. We will show that, no matter what the decisions of others, firm r will never wish to restrict itself in the first stage. As noted in the text, one way of restricting itself to producing only the high quality would be for firm r to increase C_{1r} by κ while reducing C_{2r} by κ, where $\kappa > 0$ is chosen to be sufficiently large. We will show the stronger result that any increase in κ harms firm r so long as $Z_{1r}^* > Z_{2r}^*$ initially. Consider any second-stage equilibrium. For each firm s, summing over $i \in \{1, 2\}$ gives

$$\sum\nolimits_{i=1}^{2} \left[(q_i - q_{i-1})(1 - Z_i^* - Z_{is}^*)\right] = C_{1s} + C_{2s} = c_{2s}. \tag{A6}$$

This is just the first-order condition for firm s if $Z_{1s}^* = Z_{2s}^*$, or the sum of its two first-order conditions if $Z_{1s}^* > Z_{2s}^*$. Summing (A6) over s and rearranging, we obtain

$$\sum\nolimits_{i=1}^{2} (q_i - q_{i-1}) Z_i^* = \frac{Nq_2}{N+1} - \frac{\sum_s c_{2s}}{N+1}. \tag{A7}$$

Note that (A7) does not depend on c_{1s} for any s, and in particular c_{1r} for firm r. It follows that whatever firm r does, $\sum_{i=1}^{2}(q_i - q_{i-1})Z_i^*$ is identical across each second-stage equilibrium.

Compare the second-stage equilibrium in which no firm has restricted itself in the first stage to that in which a single firm, say firm 1, has restricted itself. Using arguments similar to those in other proofs, the effect is a strict decrease in Z_1^* and a strict increase in Z_2^*. For each firm $s \neq 1$, Z_{1s}^* has increased and Z_{2s}^* has decreased, so that it is still the case that $Z_{1s}^* > Z_{2s}^*$. Now consider also restricting firm $s = 2$ to offer only the high-quality product. An inspection of firm 1's first-order condition reveals that firm 1 does not change its equilibrium output, since $\sum_{i=1}^{2}(q_i - q_{i-1})Z_i^*$ is the same. From this it can be argued again that Z_1^* has decreased, Z_2^* has increased, and firms $s > 2$ are still choosing $Z_{1s}^* > Z_{2s}^*$. Continuing iteratively, we conclude that whenever firm r has not restricted itself, it must be choosing $Z_{1r}^* > Z_{2r}^*$.

Now consider a marginal increase in κ. Following the arguments used in earlier proofs, this leads to a decrease in Z_1^* and an increase in Z_2^*, where these are strict changes unless κ is large enough that firm r is setting $Z_{1r}^* = Z_{2r}^*$, in which case any further increase in κ has no effect. Now, as argued just above, any firm $s \neq r$ that has restricted itself does not change its equilibrium outputs. For any unrestricted firm $s \neq r$, we know that $Z^*_{1s} > Z^*_{2s}$, so this firm's two first-order conditions hold. Manipulating them, we see

$$\frac{\partial Z^*_{is}}{\partial \kappa} = -\frac{\partial Z^*_i}{\partial \kappa} \qquad \text{for } i \in \{1, 2\}.$$

We now compute the equilibrium change on firm r's profits from a small increase in κ (in a region where firm r chooses $Z^*_{1r} > Z^*_{2r}$). Denote by S the set of firms $s \neq r$ that are not restricted in this subgame, and by $|S|$ the number of such firms in S. Then

$$\begin{aligned}
\frac{d\pi_r(\kappa)}{d\kappa} &= (Z^*_{2r} - Z^*_{1r}) + Z^*_{1r} P'_1(Z^*_1) \sum_{s \neq r} \frac{\partial Z^*_{1s}}{\partial \kappa} + Z^*_{2r} P'_2(Z^*_2) \sum_{s \neq r} \frac{\partial Z^*_{2s}}{\partial \kappa} \\
&= (Z^*_{2r} - Z^*_{1r}) + |S| \left[Z^*_{1r}(q_1 - q_0) \frac{\partial Z^*_1}{\partial \kappa} + Z^*_{2r}(q_2 - q_1) \frac{\partial Z^*_2}{\partial \kappa} \right] \\
&= (Z^*_{2r} - Z^*_{1r}) + |S|(q_1 - q_0) \frac{\partial Z^*_1}{\partial \kappa} \left(Z^*_{1r} - Z^*_{2r} \right) < Z^*_{2r} - Z^*_{1r} < 0.
\end{aligned}$$

The third equality is implied by the fact that $\sum_{i=1}^{2}(q_i - q_{i-1})Z^*_i$ is constant as κ changes. This implies, in turn, that $(q_1 - q_0)(\partial Z^*_1/\partial \kappa) = -(q_2 - q_1)(\partial Z^*_2/\partial \kappa)$. The penultimate inequality follows from $Z^*_{1r} > Z^*_{2r}$ and $\partial Z^*_1/\partial \kappa < 0$. Hence an increase in κ strictly harms firm r. It follows that firm r is worse off in each possible subgame, when it restricts itself. *Q.E.D.*

References

Anderson, S. and de Palma, A. "Multiproduct Firms: A Nested Logit Approach." *Journal of Industrial Economics*, Vol. 40 (1992), pp. 261–276.

——— and ———. "Market Performance with Multiproduct Firms." *Journal of Industrial Economics*, Vol. 54 (2006), pp. 95–124.

———, ———, and Nesterov, Y. "Oligopolistic Competition and the Optimal Provision of Products." *Econometrica*, Vol. 63 (1995), pp. 1281–1301.

Brander, J.A. and Eaton, J. "Product Line Rivalry." *American Economic Review*, Vol. 74 (1984), pp. 323–334.

Caplin, A. and Nalebuff, B. "Aggregation and Imperfect Competition: On the Existence of Equilibrium." *Econometrica*, Vol. 59 (1991), pp. 25–59.

Champsaur, P. and Rochet, J.-C. "Multiproduct Duopolists." *Econometrica*, Vol. 57 (1989), pp. 533–557.

De Fraja, G. "Product Line Competition in Vertically Differentiated Markets." *International Journal of Industrial Organization*, Vol. 14 (1996), pp. 389–414.

Deneckere, R.J. and McAfee, R.P. "Damaged Goods." *Journal of Economics and Management Strategy*, Vol. 5 (1996), pp. 149–174.

Dixit, A.K. and Stiglitz, J.E. "Monopolistic Competition and Optimum Product Diversity." *American Economic Review*, Vol. 67 (1977), pp. 297–308.

Eaton, B.C. and Lipsey, R.G. "The Theory of Market Pre-emption: The Persistence of Excess Capacity and Monopoly in Growing Spatial Markets." *Economica*, Vol. 46 (1979), pp. 149–158.

Ellison, G. "A Model of Add-on Pricing." *Quarterly Journal of Economics*, Vol. 120 (2005), pp. 585–637.

Fudenberg, D. and Tirole, J. *Game Theory*. Cambridge, Mass.: MIT Press, 1991.

——— and ———. "Upgrades, Tradeins and Buybacks." *RAND Journal of Economics*, Vol. 29 (1998), pp. 235–258.

Gabszewicz, J.J. and Thisse, J.-F. "Price Competition, Quality and Income Disparities." *Journal of Economic Theory*, Vol. 20 (1979), pp. 340–359.

——— and ———. "Entry (and Exit) in a Differentiated Industry." *Journal of Economic Theory*, Vol. 22 (1980), pp. 327–338.

Gal-Or, E. "Quality and Quantity Competition." *Bell Journal of Economics*, Vol. 14 (1983), pp. 590–600.

Gilbert, R.J. and Matutes, C. "Product Line Rivalry with Brand Differentiation." *Journal of Industrial Economics*, Vol. 41 (1993), pp. 223–240.

Itoh, M. "Monopoly, Product Differentiation and Economic Welfare." *Journal of Economic Theory*, Vol. 31 (1983), pp. 88–104.

Johnson, J.P. and Myatt, D.P. "Multiproduct Quality Competition: Fighting Brands and Product Line Pruning." *American Economic Review*, Vol. 93 (2003), pp. 748–774.

——— and ———. "On the Simple Economics of Advertising, Marketing, and Product Design." *American Economic Review*, Vol. 96 (2006), pp. 756–784.

Judd, K.L. "Credible Spatial Preemption." *RAND Journal of Economics*, Vol. 16 (1985), pp. 153–166.

Klemperer, P.D. "Equilibrium Product Lines: Competing Head-to-Head May Be Less Competitive." *American Economic Review*, Vol. 82 (1992), pp. 740–755.

Mankiw, N.G. and Whinston, M.D. "Free Entry and Social Inefficiency." *RAND Journal of Economics*, Vol. 17 (1986), pp. 48–58.

Maskin, E. and Riley, J.G. "Monopoly with Incomplete Information." *RAND Journal of Economics*, Vol. 15 (1984), pp. 171–196.

Mussa, M. and Rosen, S. "Monopoly and Product Quality." *Journal of Economic Theory*, Vol. 18 (1978), pp. 301–317.

Robinson, J. *The Economics of Imperfect Competition*, London: Macmillan, 1933.

Shaked, A. and Sutton, J. "Relaxing Price Competition Through Product Differentiation." *Review of Economic Studies*, Vol. 49 (1982), pp. 3–13.

——— and ———. "Natural Oligopolies." *Econometrica*, Vol. 51 (1983), pp. 1469–1483.

Stole, L.A. "Nonlinear Pricing and Oligopoly." *Journal of Economics and Management Strategy*, Vol. 4 (1995), pp. 529–562.

Verboven, F. "Product Line Rivalry and Market Segmentation." *Journal of Industrial Economics*, Vol. 47 (1999), pp. 399–425.