Decreasing Returns to Sampling without Replacement

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March 2026.[1]

Abstract. We study sampling from a finite population without replacement when seeking an extreme (lowest or highest) value. An example is a buyer searching for the lowest price. It is well known that there are decreasing returns to sampling from continuous populations: the expected minimum is a decreasing and discretely convex function of the sample size. We show that this holds when sampling without replacement from a finite population. We also state a sufficient condition on population values for the properties to hold for other order statistics.

Order statistics are relevant to many calculations. A classic case from economics is a buyer searching for the best deal by randomly sampling competing price offers: such a buyer pays the lowest price (that is, the first-order or lowest-order statistic) from a sample of quotations.[2] The expected price paid (the expectation of that lowest-order statistic) clearly falls as the sample size increases. A natural hypothesis is that there are decreasing returns to sampling, so that the expected lowest-order statistic is (discretely) convex in the sample size. This is a known result whenever the sampled population distribution is continuous (so that replacement is immaterial).[3] A recent and elegant paper by Watt (2025) shows that this is also true for other lower-order statistics when the reverse hazard rate of the (continuous) population distribution satisfies an appropriate monotonicity condition.

Naturally, sampling in some settings is made without replacement from a finite population. For example, in our recent work (Myatt and Ronayne, 2025) we study oligopolistic pricing and search when firms set non-randomized prices for buyers who seek distinct quotations from them, which we treat as

[1]Ronayne is grateful for financial support from Deutsche Forschungsgemeinschaft via CRC TRR 190 (project 280092119). We thank Joe Harrington for efficiently handling our paper. Most importantly, we owe an immense debt of gratitude to Mitch Watt for generously providing a clean and general proof of a result (Lemma 1, which verifies convexity properties of a class of upper-tail sums) that we had previously obtained via numerical verification for a finite range of population sizes. We used his suggested approach to extend that enabling result to all population sizes.

[2]Foundational work includes that by Burdett and Judd (1983) and Stigler (1961). Empirical work, such as that by Baye, Morgan, and Scholten (2004), has also relied on the properties of expected order statistics from distributions of prices.

[3]In economic applications the distribution from which draws are made is usually assumed to be continuous, as in auction theory (Krishna, 2013, Appendix C). Similarly, in the canonical model of pricing with search (Burdett and Judd, 1983) a continuum of firms' equilibrium price choices form a continuous distribution from which buyers draw. Their analysis, as well as that of many subsequent studies in that tradition, utilize the known properties of the expected value of extreme order statistics when draws are from a continuous population. See, for example, Janssen and Moraga-González (2004, fn. 12, p. 1097) and Moraga-González, Sándor, and Wildenbeest (2013, fn. 4, p. 1208).

sampling without replacement.[4] Our results here show that there are decreasing returns to searching for the best deal: the expected lowest-order statistic is a discretely convex function of the sample size.

A second relevant setting (considered by Watt (2025) when bidders are drawn from a continuous population) is an auctioneer's decision of how many bidders to invite. Under private valuations, an Anglo-Japanese auction efficiently generates surplus equal to the highest valuation. Our main results imply that the expectation of this surplus is discretely concave (as well as increasing) in the number of bidders invited. Further, our supplementary results (which apply to other order statistics) show that the expected revenue (equal to the second-highest valuation) is increasing and (subject to a condition on the pattern of valuations) discretely concave in the number of bidders.[5] When that is so, the auctioneer's problem (with any cost that is weakly convex in the number of bidders) is nicely behaved.

There is comprehensive work on order statistics when sampling is from finite populations in the statistics literature.[6] However, the full properties of the expected values of those statistics appear not to be reported. A partial exception is a recent paper by O'Neill (2025). He noted (p. 664) that the topic "has not received much attention in the literature" and that statistical texts (David and Nagaraja, 2003, for example) focus instead on order statistics from IID sampling from a continuous distribution. He also offered a result corresponding to our own when population values are uniformly distributed.

The core contribution of this note is to confirm that the result known for continuous populations also holds when random draws are made without replacement from a finite population: the expected lowest (highest) order statistic is decreasing (increasing) and discretely convex (concave) in the sample size. This holds without restrictions on the population values.

Pushing further to study other order statistics (the kth lowest or kth highest realization) we identify a simple condition on population values, inspired by Watt (2025) and which is satisfied under the uniform-distribution setting studied by O'Neill (2025), such that the discrete-convexity result holds in our sampling-without-replacement environment.

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[4] Our solution concept returns distinct prices that are undercut-proof and such that no firm wishes to raise price given that competitors can cut prices in response. The pricing games used in the earlier literature instead involve mixed-strategy solutions which (from a buyer-search perspective) are equivalent to sampling from a continuous population.

[5] Such results also apply to a uniform-price auction for k objects when the price equals the $(k+1)$st highest valuation.

[6] See, for example, Wilks (1962, Section 8.8), Nagaraja (1992, Section 7), Arnold, Balakrishnan, and Nagaraja (1992, Section 3.7), and Evans, Leemis, and Drew (2006).

1. Extreme Order Statistics

Core Result. Consider a finite population of $n \geq 3$ values indexed such that $x_1 \leq x_2 \leq \cdots \leq x_n$. The population is sampled uniformly and without replacement. Let a sample of $q \in \{1 \ldots, n\}$ values be drawn, the random variable equal to the lowest population rank in the sample be $R_q \in \{1, \ldots, n\}$, its corresponding value be X_q, and the expected value of the lowest-order statistic be $\mu_q \equiv \mathrm{E}[X_q]$.[7]

Proposition 1 (Monotonicity and Convexity of the Sample Minimum). *The expected sample minimum is at least weakly decreasing in the sample size. That is, for $q \in \{1, \ldots, n-1\}$,*

$$\mu_q \geq \mu_{q+1}. \tag{1}$$

If the lowest population value is unique ($x_1 < x_2$) then μ_q is strictly decreasing, and so eq. (1) *is strict.*

The expected sample minimum is discretely convex in the sample size. That is, for $q \in \{1, \ldots, n-2\}$,

$$\mu_q - \mu_{q+1} \geq \mu_{q+1} - \mu_{q+2}. \tag{2}$$

If the second and third lowest population values are distinct ($x_2 < x_3$), then μ_q is strictly discretely convex so that eq. (2) *holds with a strict inequality.*

Illustration. Consider $n = 3$ and $x_1 < x_2 < x_3$. With $q = 1$, each value is equally likely, so that $\mu_1 = (x_1 + x_2 + x_3)/3$. If $q = 2$, then the three possible (and equally likely) samples are $\{x_1, x_2\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$. In the first two cases the minimum is x_1 whereas in the third case it is x_2 so $\mu_2 = (2x_1 + x_2)/3$. Finally, if $q = 3$, then the lowest value is always found and so $\mu_3 = x_1$. We have

$$\mu_1 - \mu_2 = \frac{x_3 - x_1}{3} \quad \text{and} \quad \mu_2 - \mu_3 = \frac{x_2 - x_1}{3}. \tag{3}$$

By inspection, the expected minimum strictly falls as the sample size increases and there are (strictly) decreasing returns (that is, $\mu_1 - \mu_2 > \mu_2 - \mu_3$) to such an increase.

Proof of Proposition 1. Defining $x_0 = 0$, the expectation of the sample minimum satisfies

$$\mu_q = \sum_{j=1}^{n} x_j \Pr[R_q = j] = \sum_{j=1}^{n} (x_j - x_{j-1}) \Pr[R_q \geq j]. \tag{4}$$

$\Pr[R_q \geq j]$ is the probability that the entire sample belongs to the $n - j + 1$ highest ranks. It satisfies

$$\Pr[R_q \geq j] = \binom{n-j+1}{q} \Big/ \binom{n}{q} = \Pr[R_{q-1} \geq j] \times \frac{n-j-q+2}{n-q+1}, \tag{5}$$

where the second equality applies for $q \geq 2$ and follows from considering the factorial terms in the component binomial terms; this generates a convenient recursive form for the probabilities. We note

[7] If there are ties in the population values then (without loss of generality) we maintain distinct population ranks for them. We note here that the analysis of O'Neill (2025, Theorem 1) applies to $\mathrm{E}[R_q]$ which corresponds to $\mathrm{E}[X_q]$ when the population values satisfy $x_i = i$ or are, more generally, uniformly distributed.

(of course) that $\Pr[R_q \geq j] = 0$ for $j > n - q + 1$ because a sample (without replacement) of size q must include at least one observation from outside the highest $q - 1$ population values.

Differencing with respect to q, and for sample sizes satisfying $q \in \{1, \ldots, n-1\}$,

$$\Delta\mu_q \equiv \mu_{q+1} - \mu_q = \sum_{j=1}^{n} (x_j - x_{j-1}) \left(\Pr[R_{q+1} \geq j] - \Pr[R_q \geq j] \right) \tag{6}$$

$$= -\sum_{j=1}^{n} (x_j - x_{j-1}) \Pr[R_q \geq j] \left(\frac{j-1}{n-q} \right), \tag{7}$$

following from eq. (5). Each summand is non-negative, which gives the first result of the proposition. If $x_1 < x_2$, then the summand for $j = 2$ is strictly positive and the proposition's second claim follows.

We now turn to matters of discrete convexity. For $q \in \{1, \ldots, n-2\}$,

$$\Delta\mu_{q+1} \equiv \mu_{q+2} - \mu_{q+1} = -\sum_{j=1}^{n} (x_j - x_{j-1}) \Pr[R_{q+1} \geq j] \left(\frac{j-1}{n-q-1} \right) \tag{8}$$

$$= -\sum_{j=1}^{n} (x_j - x_{j-1}) \Pr[R_q \geq j] \left(\frac{n-j-q+1}{n-q} \right) \left(\frac{j-1}{n-q-1} \right). \tag{9}$$

Differencing again with respect to q, where once again $q \in \{1, \ldots, n-2\}$,

$$\Delta^2\mu_q \equiv \Delta(\Delta\mu_q) = \Delta\mu_{q+1} - \Delta\mu_q = \sum_{j=2}^{n} (x_j - x_{j-1}) \Pr[R_q \geq j] \left(\frac{j-1}{n-q} \right) \left(\frac{j-2}{n-q-1} \right). \tag{10}$$

Each summand is non-negative, which gives the third claim of the proposition. If $x_2 < x_3$ then the summand for $j = 3$ is strictly positive and therefore $\Delta^2\mu_q > 0$, which is our final claim. $\square$

Ties for the Population Minimum. We now consider ties for the lowest value in the population. Let the lowest $t \in \{2, \ldots, n-1\}$ population values be tied, so that $x_1 = \cdots = x_t < x_{t+1} \leq \cdots \leq x_n$. If the sample size satisfies $q > n - t$ then that sample is guaranteed to find the lowest population value, and so any further increase in the sample size has no effect. For smaller sample sizes, inspection of the relevant terms in the proof of Proposition 1 provides the implications, which we state below.

Proposition 2 (The Effect of Tied Lowest Values). *Suppose that there are $t > 1$ tied minimum population values so that $x_1 = \cdots = x_t < x_{t+1} \leq \cdots \leq x_n$.*

For smaller sample sizes $q \in \{1, \ldots, n-t\}$ equations (1) *and* (2) *hold strictly so that μ_q is strictly decreasing and strictly discretely convex over such q.*

For larger sample sizes $q \in \{n-t+1, \ldots, n\}$ the lowest value is drawn with certainty so that $\mu_q = x_1$.

The Highest Order Statistic. Proposition 1 also tells us that the expected value of the sample maximum is (weakly) increasing in the sample size, and strictly so if the highest population value is unique. Similarly, the expected sample maximum is discretely concave in the sample size, and strictly so if the second and third highest population values are not equal. A result analogous to Proposition 2 also follows when there are tied maximum population values so that the statistic is strictly increasing and strictly discretely concave for smaller sample sizes $q \in \{1, \dots, n-t\}$ and flat thereafter.

2. Other Order Statistics

We now turn to the expectation of the kth lowest ranked member of a sample with $k \in \{2, \dots, n\}$. Consider samples with a size $q \in \{k, \dots, n\}$ and (re)define R_q to be the population rank of the kth lowest observation drawn, where X_q and μ_q are its value and expected value, respectively.

An initial result is the monotonic effect of an increase in the sample size on the order statistics.

Proposition 3 (Monotonicity of Lower-Order Statistics). *The distribution of the kth lowest order statistic X_q undergoes at least a weak first-order stochastic shift downwards following an increase in the sample size q, and so its expectation μ_q is a weakly decreasing function of q. If the population values are each unique so that $x_1 < x_2 < \cdots < x_n$, then the claims hold strictly.*

Proof. We couple the random variables R_q and R_{q+1} on the same probability space: take a random permutation of the n population ranks, and then obtain R_q and R_{q+1} from the first q and $q+1$ members of that permutation. The addition of one more rank cannot raise the kth lowest rank, and so $R_{q+1} \le R_q$. We conclude that R_q first-order stochastically dominates R_{q+1} and that μ_q is non-increasing in q. If the population values are all unique (no ties) then there is positive probability that the added rank results in $X_{q+1} < X_q$. This implies a strict first-order domination and that μ_q strictly decreases in q. □

Proposition 3 also tells us that the distribution of the kth highest order statistic is stochastically ordered so that its expectation is increasing in q, and strictly so when population values are distinct.

We now consider the (discrete) convexity of μ_q. $\Pr[R_q = j]$ takes a hypergeometric form:[8]

$$\Pr[R_q = j] = \binom{j-1}{k-1}\binom{n-j}{q-k}\Big/\binom{n}{q} = \frac{k}{j}\left[\binom{j}{k}\binom{n-j}{q-k}\Big/\binom{n}{q}\right], \tag{11}$$

which applies to $j \in \{k, \dots, n-q+k\}$. This probability is zero if $j < k$ (k draws within the sample rank weakly below j) or if $j > n-q+k$ ($q-k$ draws rank strictly above j).

[8]Of course $\Pr[R_q = j] = 0$ if either $j < k$ or if $n - j < q - k$, given that either inequality implies that the lowest k members of the sample must include population values ranked strictly above j.

Watt (2025) provided a condition for (discrete) convexity when samples are made from a continuous distribution $F(\cdot)$ with density $f(\cdot)$. His sufficient condition is a monotonic reverse hazard rate (MRHR) so that $f(x)/F(x)$ is decreasing (Watt, 2025, Theorem 1).

A sufficient condition for MRHR is that the density $f(x)$ is decreasing, so that $F(x)$ is concave. This, in turn, is equivalent to convexity of the inverse $F^{-1}(z)$. For a finite population, the analog of this is that the sequence of population values is discretely convex, so that $x_i - x_{i-1}$ is increasing in i.[9]

Definition. *The population values have* increasing spacings *if they form a discretely convex function of the index i. This holds if $y_i \equiv x_{i+1} - x_i$ is increasing in i, so that $y_{i+1} \geq y_i$ for $i \in \{1, \ldots, n-2\}$.*

This condition allows us to extend our earlier result to non-extreme order statistics. We begin by defining $S_{nq} \equiv 0$ and $S_{jq} \equiv \sum_{\ell=j+1}^{n} \Pr[R_q \geq \ell]$ for $j \in \{1, \ldots, n-1\}$. This also satisfies

$$S_{jq} \equiv \sum_{\ell=j+1}^{n} \Pr[R_q \geq \ell] = \sum_{\ell=j+1}^{n} (\ell - j) \Pr[R_q = \ell] = \mathrm{E}\left[\max\left\{(R_q - j), 0\right\}\right]. \tag{12}$$

This expression, the expectation of a convex hinge function $\max\{(R_q - j), 0\}$ which is zero until j and then kinks to be linearly increasing, is useful because (as we show below) we can, when population values have increasing spacings, write the expected kth order statistic as a weighted sum of such terms. These terms are themselves discretely convex in q because (heuristically) an increase in the sample size pushes the distribution of R_q downward, from where further increases in q are less effective.

Lemma 1 (Convex Upper-Tail Sum). *Suppose population values are each unique so that $x_1 < x_2 < \cdots < x_n$. The term $S_{jq} = \mathrm{E}\left[\max\left\{(R_q - j), 0\right\}\right]$ is weakly discretely convex in q.*

The proof of this lemma is contained within our supplemental online appendix.[10]

Proposition 4 (Convex Lower-Order Statistics). *If the population values have increasing spacings and the population values are each unique so that $x_1 < x_2 < \cdots < x_n$, then the expectation of the kth lowest-ranked observation is strictly discretely convex in the sample size q.*

Proof. Developing further the expression from eq. (4) in the proof of Proposition 1,

$$\mu_q = \sum_{j=k}^{n} x_j \Pr[R_q = j] \quad \text{(where we begin at } k \text{ because } \Pr[R_q = j] = 0 \text{ for } j < k) \tag{13}$$

[9]This is the analog of convexity of $F^{-1}(x)$. An analog of the MRHR condition is that $i(x_i - x_{i-1})$ is increasing in i. Ideally we would obtain Proposition 4 under this weaker condition, but we have not been able to construct a proof.

[10]As noted within our supplemental appendix, the proof of Lemma 1 is based on detailed and very generous suggestions by Mitch Watt. An earlier version of the paper provided code to check this property (that S_{jq} is discretely convex in q) for any finite n, and we confirmed that we had checked for a large (but finite) range of population sizes.

$$= x_k + \sum_{j=k+1}^{n} y_{j-1} \Pr[R_q \geq j] \quad \text{(recalling that we defined } y_j \equiv x_{j+1} - x_j) \tag{14}$$

$$= x_k + \sum_{j=k+1}^{n} y_{j-1}(S_{(j-1)q} - S_{jq}) \quad \left(\text{recalling that we defined } S_{jq} \equiv \sum_{\ell=j+1}^{n} \Pr[R_q \geq \ell]\right) \tag{15}$$

$$= x_k + y_k S_{kq} + \sum_{j=k+1}^{n-1} (y_j - y_{j-1}) S_{jq} \quad \text{(where we eliminated } y_{n-1}S_{nq} \text{ because } S_{nq} = 0) \tag{16}$$

If population values have increasing spacings then $y_j - y_{j-1} \geq 0$ for all j and so μ_q is (given the discrete convexity of each S_{jq}) the sum of discretely convex functions and is discretely convex.

Last, we have the following (where the final equality in eq. (17) is from eq. (39)):

$$S_{kq} = S_{(k-1)q} - Pr[R_q \geq k] = S_{(k-1)q} - 1 = k\left(\frac{n+1}{q+1} - 1\right), \tag{17}$$

and so we see that S_{kq} is strictly discretely convex in q so that μ_q is also strictly discretely convex. □

AI USE DECLARATION

The AI tool `refine.ink` was used to check the manuscript for consistency and clarity. We subsequently edited as needed and take full responsibility for the content of the published article.

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Supplemental Online Appendix for "Decreasing Returns to Sampling without Replacement."

David P. Myatt · David Ronayne · Mitchell Watt · February 2026

This proof approach is based on detailed suggestions from Mitch Watt (who acted as a superb referee) to whom we owe our thanks, and whom deserves joint credit for this supplement. ***DPM*** *and* ***DR****.*

Proof of Lemma 1. Using eq. (12), $S_{jq} \equiv \sum_{\ell=j+1}^{n} \Pr[R_q \geq \ell] = \sum_{\ell=j+1}^{n} \sum_{m=\ell}^{n} \Pr[R_q = m]$. The discrete convexity of S_{jq} corresponds to $(S_{j(q+2)} - S_{j(q+1)}) - (S_{j(q+1)} - S_{j(q)}) \geq 0$, or equivalently $S_{j(q+2)} - 2S_{j(q+1)} + S_{j(q)} \geq 0$. This is equivalent to

$$\sum_{\ell=j+1}^{n} \sum_{m=\ell}^{n} (\Pr[R_{q+2} = m] - 2\Pr[R_{q+1} = m] + \Pr[R_q = m]) \geq 0. \tag{18}$$

Using the hypergeometric expression from eq. (11), we note that

$$\Pr[R_{q+1} = m] = \frac{(q+1)(n-m-q+k)}{(n-q)(q+1-k)} \Pr[R_q = m] \quad \text{and} \tag{19}$$

$$\Pr[R_{q+2} = m] = \frac{(q+2)(n-m-q-1+k)}{(n-q-1)(q+2-k)} \Pr[R_{q+1} = m] \tag{20}$$

$$= \frac{(q+2)(n-m-q-1+k)}{(n-q-1)(q+2-k)} \times \frac{(q+1)(n-m-q+k)}{(n-q)(q+1-k)} \Pr[R_q = m]. \tag{21}$$

Substituting these expressions into each summand of eq. (18), we obtain

$$\Pr[R_{q+2} = m] - 2\Pr[R_{q+1} = m] + \Pr[R_q = m] \tag{22}$$

$$= \frac{N_{qm} \times \Pr[R_q = m]}{(q+2-k)(n-q-1)(n-q)(q+1-k)} \tag{23}$$

$$\text{where} \quad N_{qm} = (q+2)(n-m-q-1+k)(q+1)(n-m-q+k) \tag{24}$$

$$- 2(q+1)(n-m-q+k)(n-q-1)(q+2-k) \tag{25}$$

$$+ (n-q-1)(q+2-k)(n-q)(q+1-k). \tag{26}$$

The denominator of eq. (23) is strictly positive (for q in a permissible range) and is independent of the summation index m. This means that our desired criterion eq. (18) is equivalently

$$Y_j \equiv \sum_{\ell=j+1}^{n} X_\ell \geq 0 \quad \text{where} \quad X_\ell \equiv \sum_{m=\ell}^{n} N_{qm} \Pr[R_q = m] \geq 0, \tag{27}$$

where the notation X_ℓ and Y_j suppresses the dependence on q. We note that $R_q \leq n - q + k$ (given that $q - k$ ranks must be strictly above R_q) so that $\Pr[R_q = m] = 0$ for $m > n - q + k$. Similarly, $R_q \geq k$ and so $\Pr[R_q = m] = 0$ for $m < k$. Thus we in fact require

$$Y_j \equiv \sum_{\ell=j+1}^{n-q+k} X_\ell \geq 0 \quad \text{where} \quad X_\ell \equiv \sum_{m=\max\{\ell,k\}}^{n-q+k} N_{qm} \Pr[R_q = m] \geq 0, \tag{28}$$

which we need to check for $j \in \{k-1, \ldots, n-q+k-1\}$.

To show that $Y_j \geq 0$ for all such j we will show that Y_j is single-peaked, or discretely quasi-concave, in the index j, and so must achieve its minimum at an endpoint. We will then confirm that it is weakly positive at its endpoints, which in turn implies that it must be weakly positive for all other j.

To proceed, we begin by establishing the properties of X_ℓ for $\ell \in \{k, \ldots, n-q+k\}$. We note that $X_{\ell+1} - X_\ell = -N_{q\ell} \Pr[R_q = \ell]$, and that (given that $\Pr[R_q = \ell] > 0$ for ℓ in this range) the sign of this difference is determined by the sign of $-N_{q\ell}$. This is explicitly

$$-N_{q\ell} = -(q+2)(n-\ell-q-1+k)(q+1)(n-\ell-q+k) \tag{29}$$
$$2(q+1)(n-\ell-q+k)(n-q-1)(q+2-k) \tag{30}$$
$$-(n-q-1)(q+2-k)(n-q)(q+1-k). \tag{31}$$

The coefficient for the ℓ^2 term is $-(q+1)(q+2) < 0$, and so $-N_{q\ell}$ (and so $X_{\ell+1} - X_\ell$) is a concave quadratic function of ℓ. This can potentially go from negative to positive and back to negative as ℓ increases across its range. Evaluating at the lower end of the relevant range for $\ell = k$, we note that

$$-N_{qk} = -k(k-1)(n-q)(n-q-1) < 0 \quad \Rightarrow \quad X_{k+1} < X_k. \tag{32}$$

This implies that X_ℓ is decreasing and then (at least potentially) increasing and then decreasing again.

Next, we evaluate X_ℓ at the lower bound of the relevant range, at $\ell = k$, to obtain

$$X_k = \sum_{m=k}^{n-q+k} N_{qm} \Pr[R_q = m] \tag{33}$$
$$\propto \sum_{m=k}^{n-q+k} \left[\Pr[R_{q+2} = m] - 2\Pr[R_{q+1} = m] + \Pr[R_q = m]\right] \tag{34}$$
$$= \left[\left(\sum_{m=k}^{n-q+k} \Pr[R_{q+2} = m]\right) - 2\left(\sum_{m=k}^{n-q+k} \Pr[R_{q+1} = m]\right) + \left(\sum_{m=k}^{n-q+k} \Pr[R_q = m]\right)\right] \tag{35}$$
$$= 1 - 2 + 1 = 0. \tag{36}$$

For the second line the constant of proportionality is the denominator of eq. (23).

We conclude that X_ℓ begins at $X_k = 0$ and then strictly decreases to $X_{k+1} < 0$. It then (for larger ℓ) can potentially increase and then decrease again, which means that (again potentially) X_ℓ can become positive and then negative again. However, evaluating at the top of the range, $\ell = (n-q+k)$

$$N_{q(n-q+k)} = (n-q-1)(q+2-k)(n-q)(q+1-k) \quad \Rightarrow \quad X_{n-q+k} > 0. \tag{37}$$

This implies that X_ℓ does not become negative again on the relative range, and so crosses once from negative to positive across the relevant range $\ell \in \{k+1, \ldots, n-q+k\}$.

We evaluate the characteristics of Y_j for $j \in \{k-1, \ldots, n-q+k-1\}$, where $Y_j = 0$ for $j \geq n-q+k$.

We note that $Y_{j+1} - Y_j = -X_{j+1}$, which is evaluated for $j+1 \in \{k, \ldots, n-q+k\}$. We have $Y_k - Y_{k-1} = -X_k =$, and $Y_{k+1} - Y_k = -X_{k+1} > 0$. It follows that Y_j is initially increasing. We have established that X_{j+1} goes from weakly negative to strictly positive as j increases, and so $-X_{j+1}$ goes from weakly positive to strictly negative. This implies that Y_j is single-peaked across this range.

To establish that $Y_j \geq 0$, it is sufficient to check the endpoints. At the upper bound $j = n-q+k-1$ we have $Y_j = X_{n-q+k}$, and

$$m = n-q+k \quad \Rightarrow \quad N_{q(n-q+k)} = (q-k+1)(q-k+2)(n-q)(n-q-1) \geq 0. \tag{38}$$

The lower bound at $j = k-1$ corresponds to checking the discrete convexity of $S_{(k-1)q}$. Note that

$$S_{(k-1)q} = \mathrm{E}[R_q] - (k-1) = 1 + k\left(\frac{n+1}{q+1} - 1\right). \tag{39}$$

The last equality above follows from Theorem 1 of O'Neill (2025), but is readily verified directly:

$$\mathrm{E}[R_q] = \sum_{j=k}^{n} j \Pr[R_q = j] = \sum_{j=k}^{n-q+k} \frac{j\binom{j-1}{k-1}\binom{n-j}{q-k}}{\binom{n}{q}} \tag{40}$$

$$= \frac{(n+1)k}{q+1} \sum_{j=k}^{n-q+k} \frac{\binom{j}{k}\binom{n-j}{q-k}}{\binom{n+1}{q+1}} = \frac{(n+1)k}{q+1} \sum_{j=k+1}^{n-q+k+1} \frac{\binom{j-1}{k+1}\binom{(n+1)-j}{(q+1)-(k+1)}}{\binom{n+1}{q+1}} = \frac{(n+1)k}{q+1}. \tag{41}$$

The final equality holds because each summand (in the summation before the final equality) is the probability that the $(k+1)$th lowest observation from a sample of size $q+1$ from a population of size $n+1$ has rank j, and the summation is over the full permissible range for such j.

By inspection $S_{(k-1)q}$ is discretely convex in q and so $Y_{k-1} \geq 0$. Summarizing, we have shown (by inspection of the properties of X_ℓ) that Y_j is (at least weakly) increasing and then decreasing in j and so is minimized at one of its endpoints, and we have shown that both endpoints are at least weakly positive. This implies that $Y_j \geq 0$ for all j and so S_{jq} is (weakly) discretely convex, as required. □