A Theory of Stable Price Dispersion

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Abstract. We study the pricing of homogeneous products sold to customers who consider different sets of suppliers, and where there is an asymmetric price rigidity: initial price positions are followed by an opportunity for firms to cut (but not to raise) their final prices. We find (uniquely, in many cases) prices that are stable, in that no firm wishes to undercut any other, and different across firms. We apply several criteria for how such disperse prices emerge from collective choice and non-cooperative interaction. Our results apply across a broad range of consideration-set configurations, including those that arise endogenously from the actions of both suppliers and customers.

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1. Explaining Stable Price Dispersion

Seemingly identical products are often sold at different prices. To explain price dispersion, the established framework associated with the classic contributions including those by Varian (1980) and Burdett and Judd (1983), among others, and developed substantially by more recent work by Armstrong and Vickers (2022), specifies "consideration sets" for different customers, where such a set contains the firms that are accessible to a customer.[2] In the simplest setting, a customer is either "captive" to one firm or is a "shopper" who buys from the cheapest firm. A single-stage pricing game with this structure has no pure-strategy equilibrium, and so price dispersion is typically interpreted as the outcome from realizations of mixed-strategy equilibria.[3]

If pricing choices are made simultaneously and with commitment, then mixed-strategy predictions can be a useful way to think about the likely distribution of disperse prices. On the other

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[2]The "consideration set" terminology originates in the marketing literature, was used in economics by Eliaz and Spiegler (2011), Manzini and Mariotti (2014) and others, and is central to the economic environment we study.

[3]Varian (1980, 1981) constructed a symmetric equilibrium in which firms continuously randomize over an interval which extends downward from customers' reservation price. Baye, Kovenock, and de Vries (1992) characterized the full set of equilibria, including in settings where the firms vary according to the sizes of their captive customer populations. Similar properties hold in early papers by Shilony (1977) and Rosenthal (1980).

hand, such equilibria might not be satisfactory in all situations. A susceptibility to ex post deviations makes them inconsistent with any ability of retailers to slash prices to undercut others. Furthermore, the repeated play of a mixed-strategy equilibrium predicts dynamically uncorrelated variation in firms' prices. However, in many settings there appears to be an empirical lack of sufficiently frequent temporal variation in otherwise disperse prices.

Our response is to add to existing work by providing a theory (in Section 4) that builds upon a stylized feature of many scenarios: it is easy for a firm to cut, but hard for it to hike, an established price, at least in the short run. We assume, therefore, that once price positions have been established (by historical precedent, by collective choice, or by individual choices in a two-stage game) they then become upper bounds for any revised (and final) price offers.

For a price profile to be stable in this context requires only that no firm has an incentive to undercut a cheaper competitor. Undercut-proof profiles always exist, and satisfy certain properties. For example, if every pair of firms forms a consideration set for some group of customers then strictly positive undercut-proof prices are entirely distinct for every firm. Among the (large) set of undercut-proof prices, we identify those that are Pareto efficient for the industry. Such prices might result from the collective choice of firms, or a body representing them. An efficient profile exists and, in several natural settings, there is exactly one.

We then ask whether an undercut-proof and efficient profile of prices emerges from non-cooperative interaction. We first examine whether those prices are pure strategies on the path of a subgame-perfect equilibrium of a two-stage game in which firms simultaneously choose initial price positions prior to an opportunity to cut their price.[4] We also investigate whether a firm can gain from a small adjustment upward (i.e., a slight loosening of our no-upward-revisions assumption) in its initial price knowing that this might trigger others to undercut. In several settings these criteria pin down clear predictions of stable price dispersion.

Consider, for example, a simple duopoly (we describe this more fully in Section 3) in which two firms enjoy their own captive audiences but where shoppers compare their prices. Starting from high prices, firms undercut each other to capture the shoppers. Further best-replies walk the firms down a staircase of prices. At a sufficiently low price one firm abandons the shoppers and elevates its price back up to exploit its captive customers.[5] If raising price is impossible, then the "elevator" from a shopper-capturing low price to a captive-exploiting high price is disabled. A firm that occupies an initial low-price position has (one-sided) commitment power.[6] If such a "limit" price is set sufficiently low, then the competitor will be dissuaded from undercutting.

[4]One interpretation is that firms first choose list prices, followed by discounts. Others have also studied discounts to list prices. Anderson, Baik, and Larson (2023) studied mixed-strategy pricing for targeted price discrimination. Personalized offers are not observed by competing firms, and so a single-stage game seems to be an appropriate model. Gill and Thanassoulis (2016) studied a Hotelling duopoly in which some customers see only firms' list prices, while others also have access to second-stage discounts.

[5]This "Edgeworth cycle" logic (Maskin and Tirole, 1988a,b) rules out pure-strategy equilibria. If there were a unique lowest price then that firm would raise it; if multiple firms are cheapest then one would undercut.

[6]The mixed-strategy equilibria of single-stage pricing games implicitly rely on two-sided commitment power: some firms always have an ex post incentive to deviate, but their initial price choices are treated as binding. Our approach weakens the commitment power of firms by allowing for ex post price reductions.

In this setting we find that the unique efficient undercut-proof price profile is for the firm with fewer captive customers to offer a low limit price and serve shoppers; the other firm exploits its (larger) captive audience. This outcome also emerges from the play of a subgame-perfect equilibrium of a two-stage game in which initial prices are chosen (in a first stage) followed by discounting opportunities (in a second stage). The low-price firm earns the same expected profit as it does in a mixed-strategy equilibrium of a single-stage game. This means that it does not gain from offering a higher initial price; doing so would (in effect) place it back in such a single-stage pricing game. In summary, our approach predicts stable price dispersion with expected profits that match those from a single-stage model.

A duopoly necessarily restricts dispersion to two price points. However, our main analysis with n firms predicts fuller dispersion arising from more general consideration-set specifications.

One case of interest (studied in Section 5) is when firms are symmetric in the sense that the mass of customers for each consideration set depends only on the number of firms considered, and so firms are "exchangeable" within such consideration sets. We identify a unique profile of efficient undercut-proof prices. Those (entirely) distinct prices are chosen as pure strategies along the equilibrium path of our two-stage pricing game. A broadening to "quasi-exchangeability" allows for firms with differently sized captive audiences. The efficient price profile orders the firms so that those with larger captive bases set higher prices. Moreover, we find that (in a two-stage local-adjustment pricing model) no firm can gain from locally raising its initial price.

Our "exchangeable" framework readily encompasses the classic captive-and-shopper or model-of-sales world (covered explicitly in Section 6). The unique efficient undercut-proof prices emerge again via pure-strategy play of our two-stage pricing game. The firm with the smallest captive audience sets a distinctly lower "on sale" price and serves all shoppers, while other firms exploit their captives with prices that collapse together at the monopoly "regular" price. This means that price dispersion in the simplest setting reduces back to two price positions.[7]

In a second extended setting (in Section 7) we build upon models of informative advertising (Butters, 1977; Grossman and Shapiro, 1984) in which customers are randomly, independently, and asymmetrically aware of each firm. We find that an efficient price profile places the firm with the greatest awareness at the highest price. All these profiles generate the same profit for each firm, and for each, no firm can benefit by locally raising its initial price position. Moreover, if firms are "awareness-ordered" such that those with greater awareness adopt higher prices, then these prices are supported by the on-path play of pure strategies in our two-stage game.

We cover a rich set of consideration-set specifications, but that coverage is necessarily incomplete. For traditional single-stage models, we do not know of a complete analysis of the single-stage pricing game with no restriction placed on the structure of consideration sets.[8]

[7] Similarly, the relevant single-stage game studied in the literature in this setting predicts (for generic parameters) three distinct prices (Baye, Kovenock, and de Vries, 1992, Section V). As we discuss more in Section 6, these stark predictions are a result of the stark consideration sets in the captive-shopper setting.

[8] The early literature offered a comprehensive analysis of duopoly (Narasimhan, 1988) and of situations in which customers are either captive to one firm or shop among all of them (Baye, Kovenock, and de Vries,

Such single-stage models are nested within our framework, and so this suggests that we cannot achieve full results for all possible specifications.[9] Nevertheless, some results (in Section 4) place minimal restrictions on consideration sets, while further results that exploit more structure (in Sections 5 to 7) do allow for both the relative importance of differently sized consideration sets (under exchangeability) and asymmetry across firms (under independent awareness).

Despite this breadth, it is important to expose the complexities of the environment and the limitations of our approach. We do that (in Section 8) via three triopoly examples. The first example (using our quasi-exchangeable specification) finds efficient undercut-proof prices that are not supported by pure-strategy play of our (unrestricted) two-stage pricing game. The second example (using our independent awareness specification) has multiple efficient profiles that are supported by such pure-strategy play. The third example specifies a prominent firm with captive customers together with less-prominent firms that have audiences that always overlap with that prominent firm. We find that the prominent firm is the most expensive, but that the non-prominent firm with a larger audience (the one that is most tempting to undercut) is the cheapest.[10] This reveals that stable prices are not always ordered by notions of firm size.

The key primitive for our work is the constellation of customers' consideration sets. Of course, consideration sets can be influenced by the endogenous actions of firms and customers. As our final step (in Section 9) we illustrate how our framework can be embedded into deeper models of such endogenous decisions, via two duopoly examples. Firstly, we allow firms to influence the awareness of customers via costly advertising. We find that symmetric firms make different advertising decisions, resulting in a clear dispersion of their prices. Secondly, we allow customers to pay for price quotations à la Burdett and Judd (1983). Their search decisions are strategic complements (increased search by others results in greater price dispersion and so raises the incentive for a customer to seek a second quotation), which readily generates multiple equilibria. A (stable) equilibrium with high search exhibits substantial stable price dispersion.

Before describing our theory, we discuss (in Section 2 that follows) empirical regularities of price dispersion and constraints to price movements. We then (informally) outline our theory for the duopoly case (in Section 3) before developing the full theory (in Section 4) and applying it to exchangeability (Section 5), captive-shopper (Section 6), independent awareness (Section 7), and selected triopoly (Section 8) settings, before closing with our two consideration-endogenizing applications (Section 9). We include proofs of most key results in an appendix within the main paper (Appendix A). Two further appendices in our online supplemental material describe some tangential extensions (Appendix B) and collect together proofs that repeat established techniques or deal with our later examples and results in other appendices (Appendix C).

1992; Rosenthal, 1980; Varian, 1980). Other papers considered "independent awareness" specifications (Ireland, 1993; McAfee, 1994). In a recent paper, Armstrong and Vickers (2022) offered a comprehensive analysis of mixed-strategy equilibria for the triopoly case, along with various particular (and natural) n-firm specifications.

[9]If all initial prices are chosen to be equal to the monopoly price (or indeed any common price) then the second-stage subgame played by firms is the same as the usual single-stage pricing game.

[10]The first claim holds for the ordered-search model of Arbatskaya (2007) and in the search-and-prominence duopoly model of Moraga-González, Sándor, and Wildenbeest (2021). In contrast, Armstrong, Vickers, and Zhou (2009) used a sequential-search model to predict that a prominent firm offers the lowest price.

2. Empirical Regularities

Before explaining our theoretical framework and our formal results, we describe the price dispersion that we seek to explain and the empirical justification for our assumptions.

Price Dispersion. Empirical studies have identified extensive price dispersion. For example, Kaplan and Menzio (2015) used the large Kilts-Nielsen panel of 50,000 households to show that the standard deviation (relative to the mean) of prices at brick-and-mortar stores ranges from 19% (when products are defined narrowly) to 36% (when defined broadly).

However, intertemporal changes seem not to explain most of the variation we see in prices.[11] This differs from a literal interpretation of an equilibrium in mixed strategies.[12] Using data from the Bureau of Labor Statistics, Nakamura and Steinsson (2008) estimated the median duration of a (regular) price in the US to be between 8–11 months. The European Central Bank (ECB, 2005) gathered survey data from 11,000 European firms and found the "median firm changes its price once a year." Using Norwegian retail data, Wulfsberg (2016) and Moen, Wulfsberg, and Aas (2020) found a high persistence of price dispersion: prices on average last 6-16 months depending on the product category and macro environment, and stores charging prices in a particular quartile of the distribution stay there with high probability (0.83–0.93) month-to-month. Related, Gorodnichenko, Sheremirov, and Talavera (2018) examined daily online pricing data. They reported (pp. 1764–1766) that "although online prices change more frequently than offline prices, they nevertheless exhibit relatively long spells of fixed prices," and so "online price setting is characterized by considerable frictions." Specifically, prices are fixed for long spells of 7–20 weeks and "clearly do not adjust every instant." They concluded that prices tend to vary in the cross section rather than over time.

A stylized summary of empirical findings is that in many (but of course not all) settings, firms persistently occupy different high and low pricing positions (with occasional dynamic changes) rather than rapidly flipping among them.[13] In principle, long price spells could be because of a paucity of opportunity for firms to change their prices. But on that point, it is notable firms do not change their prices at every opportunity, even though they often set the schedule of opportunities. For example, 43% of Euro-area firms reviewed prices at least four times a year, but only 14% changed price that often (ECB, 2005, see also ECB, 2019). This summary calls for a theory that can explain stable price dispersion.

[11] Kaplan and Menzio (2015) decomposed the variation and found the intertemporal component (their "transaction" component) accounted for a substantial fraction, but less than half. Also with Kilts-Niesen data, Kaplan, Menzio, Rudanko, and Trachter (2019) reported that "a sizeable fraction of the variance of prices for the same good is caused by persistent differences in the price that different stores set for that good [...]."

[12] If, for example, the symmetric mixed equilibrium of Varian (1980) were played each period, the variance of prices through time for a particular firm (the transaction component) would explain all of the observed variation.

[13] Many industry-specific studies are also consistent with this summary, e.g., those on prescription drugs (Sorensen, 2000), illicit drugs (Galenianos, Pacula, and Persico, 2012), memory chips (Moraga-González and Wildenbeest, 2008), and textbooks (Hong and Shum, 2010). A contrasting conclusion was provided by Lach (2002), who emphasized that (p. 433) "stores move up and down the cross-sectional price distribution." And in some industries, gasoline for example, substantial dynamic price movements have been documented (see, e.g., Chandra and Tappata, 2011; Pennerstorfer, Schmidt-Dengler, Schutz, Weiss, and Yontcheva, 2020).

Constraints to Price Rises. The key assumption in our theory is that firms are able to price below some initial level, but find it difficult to price above it. An initial price could be interpreted as a list or recommended retail price, to which a discount (but not a price hike or an "over") can be applied. It could also be interpreted as an established expectation of a price. There are at least two applied motivations for our approach.

Firstly, legal constraints may force a firm to meet any published offer. In a study of dispersed prices for prescription drugs, Sorensen (2000, p. 837) reported that "price-posting legislation dictates that any posted price must be honored at the request of the consumer." Charging an "over" at the point of sale can fall under many authorities' definitions of deceptive pricing. For example, the UK's Advertising Standards Authority advises that there should be availability of a product at a listed price.[14] There can also be limitations on price rises. Obradovits (2014) documented an Austrian gasoline market regulation that prohibited more than one price rise per day (which must be implemented at noon), while price cuts were freely permitted.[15]

Secondly, customers may see any attempt to charge above some initial, preceding, or established price as unfair, socially unacceptable, or may otherwise reduce their demand. For example, Anderson and Simester (2010) found (from a field experiment with 50,000 customers) that customers shunned firms from whom they had bought a product when they observed the product being sold subsequently (by the same seller) for less.[16] The associated "consumer antagonism" discussed in their paper might have many roots. It may be that an initial price sets a reference point for loss-aversion arguments (Kahneman and Tversky, 1979), which suggests a higher elasticity of demand above the initial price than below (Ahrens, Pirschel, and Snower, 2017). The role of fairness concerns as a constraint on profit-seeking was central to the work of Kahneman, Knetsch, and Thaler (1986). Relating their ideas to Okun (1981), they reported "the hostile reaction of customers to price increases that are not justified by increased costs and are therefore viewed as unfair." The importance of fairness considerations in pricing is central to many marketing studies (e.g., Campbell, 1999, 2007; Bolton, Warlop, and Alba, 2003; Xia, Monroe, and Cox, 2004). Naturally, firms are not ignorant of these concerns: the ECB reports cited above found that a firm's "implicit contract" with their customers (that their prices will not rise) was a primary reason behind the observed price stickiness.[17]

Taken together, the evidence leads us to the view that prices can be thought of (at least in the short run) as difficult to adjust upwards and so form *asymmetric commitments* to customers.

[14]Similar advice is provided by Ireland's Competition and Consumer Protection Division.
[15]Obradovits (2014) reported that similar regulations were proposed in New York State and in Germany.
[16]Related, marketing research documents how "advertised reference prices" set value perceptions and purchase intentions, e.g., Urbany, Bearden, and Weilbaker (1988); Lichtenstein, Burton, and Karson (1991); Grewal, Monroe, and Krishnan (1998); Alford and Engelland (2000); Kan, Lichtenstein, Grant, and Janiszewski (2013).
[17]Of course prices sometimes rise, but typically with inflation, unlike cuts (e.g., Nakamura and Steinsson, 2008).

3. An Illustrative Duopoly Example

We now illustrate the economic environment and the tools of our framework through the example of a duopoly, before turning to a full specification of the theory in Section 4.

Consider a homogeneous-good, zero-cost duopoly in which $p_i \in [0, v]$ is the final retail price of firm $i \in \{1, 2\}$. Each customer wants one unit and is willing to pay $v > 0$ for it. A mass $\lambda_i > 0$ of customers always buy from i, while $\lambda_S > 0$ "shoppers" buy from a cheapest firm.

Our approach is to envisage a situation in which initial price positions are adopted, after which firms are able to cut but not raise those prices. Specifically, suppose that

($t = 1$) initial prices $\bar{p}_i \in [0, v]$ are chosen (either collectively or non-cooperatively), and then
($t = 2$) firms (after observing those initial prices) choose their final retail prices $p_i \in [0, \bar{p}_i]$.

For some pairs of initial prices, $p_i = \bar{p}_i$ is the best each firm i can do in the second stage. This is true for positive prices only if the lower price is sufficiently below the higher price. In essence, the lower price has to be pushed far enough below the higher price in order to dissuade an undercut. If $\bar{p}_1 \geq \bar{p}_2$, then the "no undercutting" constraint is $\bar{p}_1\lambda_1 \geq \bar{p}_2(\lambda_1 + \lambda_S)$. These different initial prices are stable in the sense that they are robust to undercutting opportunities.

This leaves open the question of how initial price positions are formed. One approach is to note that the firms would collectively wish these prices to be as high as possible (for a given ranking of prices). In the example above (which labels firm 2 as cheaper), the highest possible undercut-proof prices satisfy $\bar{p}_1 = v$ and $v\lambda_1 = \bar{p}_2(\lambda_1 + \lambda_S)$. There are therefore two pairs of "maximal" undercut-proof prices, one for each of the two possible orders of prices:

$$\bar{p}_1 = v, \quad \bar{p}_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S} \quad \text{and} \quad \bar{p}_1 = \frac{v\lambda_2}{\lambda_2 + \lambda_S}, \quad \bar{p}_2 = v. \tag{1}$$

If $\lambda_1 > \lambda_2$ then the first undercut-proof pair of prices in (1) generates more profit for each firm than the second, and so is (Pareto) efficient from the perspective of firms.

We next ask: what happens if initial prices are chosen non-cooperatively? Our answer is that there is a subgame-perfect equilibrium (of the game in which firms simultaneously choose $\bar{p}_i \in [0, v]$ at stage $t = 1$ and $p_i \in [0, \bar{p}_i]$ at $t = 2$) in which they choose the pair of prices $\bar{p}_1 = v$ and $\bar{p}_2 = v\lambda_1/(\lambda_1 + \lambda_S)$ at $t = 1$; and then $p_i = \bar{p}_i$ at $t = 2$. That is, efficient undercut-proof prices are predicted as pure strategies on a (subgame-perfect) equilibrium's path.

The key to constructing this equilibrium is to ask what happens when the cheaper firm deviates upward to a higher value of $\bar{p}_2$ in the first stage. Such a deviation generates a second-stage incentive for the first firm to undercut the second firm. However, we show that in the (unique, in this case) Nash equilibrium of the subgame, each firm plays a (continuously) mixed strategy over prices in $[v\lambda_1/(\lambda_1 + \lambda_S), v)$, which gives the same expected profit as on the equilibrium path. (Because these upwards unilateral deviations (of any size) by firm 2 are unprofitable, the initial prices are also (automatically) robust to small upwards unilateral adjustments.)

4. A Theory of Stable Price Dispersion

The Economic Environment. On the supply side, $n > 1$ firms indexed by $i \in \{1, \ldots, n\}$ produce a homogeneous product with the same constant marginal cost which, without (further) loss of generality, we set to zero. A firm's profit is (as usual) its price multiplied by its sales.

On the demand side, each customer is willing to pay at most $v > 0$ for a single unit. A customer's *consideration set* lists those firms from whom they are able to buy. Customers buy from the cheapest firm in their consideration sets; ties can be broken in any interior way.[18]

We write $\lambda(B) : 2^N \mapsto \mathcal{R}^+$ for the mass of customers who consider firms within $B \subseteq \{1, \ldots, n\}$, and we write $B_i = \mathbb{I}[i \in B] \in \{0, 1\}$ for the indicator of whether firm i is a member of B. We also use the shorthand $\lambda_i = \lambda(\{i\})$ for the mass of those who are "captive" to a single firm i. To set aside uninteresting cases, we assume that each firm i has some captive customers ($\lambda_i > 0$) and that a positive mass consider i together with at least one other firm $j \neq i$.[19]

A special case is the classic "captive and shopper" model in which a mass of $\lambda_S \equiv \lambda(\{1, \ldots, n\})$ customers are "shoppers" who consider every firm; all other non-singleton consideration sets have zero mass. The latter feature guarantees that a single-stage pricing game with symmetric firms (with equally sized captive audiences) has (infinitely) many equilibria, and a game with asymmetric firms involves mixing by only the two smallest firms (in terms of captives), while others charge v (Baye, Kovenock, and de Vries, 1992).

In their analysis of a single-stage pricing game in which consideration sets are formed randomly and symmetrically, Johnen and Ronayne (2021) noted the key property of "twoness" which means that consideration pairs have positive mass. Such a single-stage game has a unique mixed-strategy equilibrium if and only if twoness holds. We state a formal definition here.

Definition (Twoness). *The property of "twoness" holds if* $\lambda(\{i, j\}) > 0$ *for* $i \neq j$.

Our exposition is simplified appreciably if we maintain this property as an assumption, and so we do so throughout this and the next section. Fundamentally, our claims hold without this assumption, but our statements can become tortuous. Of course, the "twoness" property does not hold for a strict captive-and-shopper world (in which any price comparisons by shoppers are made between all firms) and so we consider that case separately (with no meaningful change in our results, and in the interest of completeness) in Section 6.

[18] There are, of course, other environments that generate dispersed pure-strategy prices. Reinganum (1979) offered a version of the classic Diamond (1971) model in which firms have different costs and so set different monopoly prices. Anderson and De Palma (2005) studied customers who (exogenously) consider firms in a random order until they find a price that they are willing to pay, removing any meaningful price comparison. Each firm prices based on demand that has been appropriately modified to account for buyers' behavior. Arnold (2000) studied two firms with capacity constraints and single-search customers who see prices but not whether a firm is stocked out. Firms trade off price for the (endogenous) number of buyers that buy from them. For certain values of willingness to pay, there is an equilibrium in which the firms choose different prices.

[19] If i is not considered with another firm, then i's audience is entirely captive and it sets the monopoly price. If there are no captive customers, then Bertrand competition forces all profits down to zero.

Undercut-Proof Prices. We seek prices that are *stable* in the sense that no firm is left with a profitable unilateral deviation. In the context of a single-stage simultaneous-move pricing game, a price profile is stable in this sense if and only if it is a pure strategy Nash equilibrium.

We think of prices being easy to cut but hard to raise. We assume that any price change must be downward to below a reference level, which acts as a price ceiling for that firm.

Specifically, we write $\bar{p}_i \in [0, v]$ for an initial, established, or reference price of firm i. We think of this as a list price, a recommended retail price, or some settled expectation of what firm i will charge. A deviation for firm i is to any final retail price $p_i \in [0, \bar{p}_i)$. If no such deviation is profitable, then the initial price profile is *undercut-proof.*

Definition (Undercut-Proof Prices). *A profile of initial prices, where $\bar{p}_i$ is the price for firm i, is* undercut-proof *if no firm can strictly gain from a price cut.*

The trivial profile of zero prices is undercut-proof. On the other hand, some profiles are never undercut-proof: if there are ties of positive prices then they are pairwise compared (given the "twoness" assumption) which generates an incentive to undercut.[20] We conclude that any strictly positive undercut-proof prices must be entirely distinct, stated in claim (i) of Lemma 1.

Let us now (without loss of generality) label firms in decreasing order of initial price, so that $\bar{p}_1 \geq \cdots \geq \bar{p}_n$, and (from the argument above) the inequality $\bar{p}_i \geq \bar{p}_{i+1}$ is strict if $\bar{p}_{i+1} > 0$.

If firm j charges $\bar{p}_j > 0$ then it wins all price comparisons that involve only firms from the j most expensive. These are the consideration sets $B \subseteq \{1, \ldots, j\}$. We need to include only those in which firm j is considered, which is achieved via the indicator variable $B_j \in \{0, 1\}$. Hence firm j earns $\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)$. If firm j undercuts a cheaper firm $i > j$ then it wins all price comparisons which involve the i most expensive firms. To avoid a profitable undercut we need $\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B) \geq \bar{p}_i \sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)$, or equivalently

$$\bar{p}_i \leq \frac{\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)}. \tag{2}$$

This must hold for every $j < i$, which gives the characterization (3) in Lemma 1.

Lemma 1 (Basic Properties of Undercut-Proof Price Profiles). *Without loss of generality, label firms in order of their prices from highest to lowest so that $\bar{p}_1 \geq \cdots \geq \bar{p}_n$.*

(i) Any profile of strictly positive undercut-proof prices is strictly ordered: $\bar{p}_1 > \cdots > \bar{p}_n > 0$.

(ii) A profile of prices is undercut-proof if and only if

$$\bar{p}_i \leq \min_{j \in \{1,\ldots,i-1\}} \left\{ \frac{\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)} \right\} \quad \textit{for all } i \in \{2, \ldots, n\}, \tag{3}$$

[20] More generally, if a "k-ness" property holds, so that any consideration set containing $k > 1$ firms has positive mass (formally: $|B| = k \Rightarrow \lambda(B) > 0$) then at most $k - 1$ strictly positive undercut-proof prices can be tied.

The arguments in the text preceding the statement of Lemma 1 form the proof of its two claims. For other formal results that we report in the paper, any proof not following directly from the argument in main text is contained within Appendix A.

Once a profile of undercut-proof prices is established, then the absence of undercutting opportunities means it is the only prediction of a game in which firms can only cut their prices.

Lemma 2 (Equilibrium with Undercut-Proof Price Limits). *For any undercut-proof profile of prices, consider a single-stage game in which firm i chooses $p_i \in [0, \bar{p}_i]$. This game is dominance solvable, and the unique Nash equilibrium has $p_i = \bar{p}_i$ for all i.*

Lemma 1 describes the set of undercut-proof prices, but the set of such price profiles is large.

We now specify three criteria which we can use to refine further the set of price profiles.

Efficiency. Our first criterion poses this question: to which undercut-proof prices would firms collectively agree? Presumably, they would wish to raise prices as high as possible, and looking across all such profiles, would prefer those with Pareto superior profits. We use these definitions:

Definition (Maximal and Efficient Price Profiles). *The* maximal undercut-proof prices *for an ordering of firms are those that are higher than all other undercut-proof profiles that place firms' prices in the same order. An undercut-proof price profile is* Pareto efficient *if there is no other Pareto superior (in terms of firms' profits) undercut-proof profile.*

Maximal undercut-proof prices are readily identified. Strictly positive undercut-proof prices remain so if we raise the highest price to $\bar{p}_1 = v$. We then iteratively raise each successively lower price so that (3) binds. Each price rise is a Pareto improvement for the firms: a price goes up, and the allocation of demand across firms is preserved. Lemma 3 summarizes.

Lemma 3 (Maximal and Efficient Prices). *Maximal undercut-proof prices satisfy*

$$\bar{p}_1 = v, \textit{ and iteratively, } \bar{p}_i = \min_{j \in \{1,\dots,i-1\}} \left\{ \frac{\bar{p}_j \sum_{B \subseteq \{1,\dots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\dots,i\}} B_j \, \lambda(B)} \right\} \textit{ for all } i \in \{2, \dots, n\}. \quad (4)$$

Efficient price profiles are a (non-empty) subset of the $n!$ maximal price profiles defined via (4).

This lemma allows us to construct $n!$ profiles (one for each ordering of firms' prices) that are candidates for Pareto efficiency. For some specifications (Sections 5 and 6) we find a unique Pareto efficient profile; but in others (Sections 7 and 8) we find multiple efficient profiles.

Two-Stage Pricing. Our central assumption is that upward price deviations, relative to an initial, established, or reference price, are difficult (by assumption, impossible) for a firm. This leaves open the mechanism for how such initial prices are set. The notion of (Pareto) efficiency, discussed above, is concerned with how prices might be collectively chosen. Here, however, we ask: can initial prices be established non-cooperatively?

To investigate this formally, we consider a two-stage perfect-information pricing game in which

$(t = 1)$ firms simultaneously choose their initial price positions $\bar{p}_i \in [0, v]$; and then
$(t = 2)$ firms simultaneously choose their final retail prices, which must satisfy $p_i \in [0, \bar{p}_i]$.

A natural solution concept is subgame-perfect equilibrium. Our two-stage game includes the single-stage pricing game studied in the literature, as a subgame. Specifically, if $\bar{p}_i = v$ for all firms, then they play the classic game in the second stage. Such a subgame has (of course) no pure-strategy Nash equilibrium, and so necessarily a subgame-perfect solution must include the play of mixed strategies for some (but not all) subgames. However, we seek (and tend to find) equilibria in which a profile of prices is played via pure strategies on the equilibrium path.

Definition (Pure-Strategy Play). *A profile of prices is* supported by the equilibrium play of pure strategies *if there is a subgame-perfect equilibrium of the two-stage pricing game in which that profile is played on the equilibrium path in both the first and second stage.*

An undercut-proof price profile is robust to (necessarily downward) deviations at the second stage. The definition asks also that the price profile is robust to deviations (in either direction) in a first stage of play in which initial pricing positions are established.

A price profile that satisfies this definition must be undercut-proof (which means, from Lemma 2, that those prices are played as the unique and pure-strategy equilibrium of the on-path second-stage subgame). Additionally, if the prices are not maximal then there is a profitable deviation for some firm in the first stage: a firm with a price that is "slack" with respect to the undercut-proofness constraints documented in (3) could raise its price (at least slightly) at the first stage. This would lead to a higher and more profitable set of initial prices at the second stage, and we know that those prices (given that they are undercut-proof) form the (unique and pure-strategy) Nash equilibrium of that second-stage subgame. This argument supports a lemma.

Lemma 4 (Pure-Strategy Play of Maximal Prices). *If a profile of prices is supported by the equilibrium play of pure strategies, then it comprises maximal undercut-proof prices.*

We have already seen that to identify efficient prices we can restrict attention to the maximal prices for each ordering of firms. Lemma 4 helpfully allows us to do the same as we search for (stable and disperse) prices that are supported by the equilibrium play of pure strategies.

Local Price Adjustments. Our two-stage game shows how prices might become established non-cooperatively. More generally, however, firms may face established reference prices from past behavior, precedent, or some other channel that fixes customers' expectations. The "no hike of an established price" constraint is underpinned (in Section 2) by either legal or social acceptability constraints. We can imagine, however, that small upward movements in an established price might be feasible. Allowing such movements constitutes a slight relaxation of our primary assumption. This prompts us to look for price profiles that are robust to local upward adjustments: is a small rise in an initial undercut-proof price beneficial for a firm?

Fix a profile of undercut-proof prices, $\bar{p}_i$ for each i. For $\bar{\Delta} > 0$, $\Delta_i \in [0, \bar{\Delta}]$ is a small upward price adjustment by firm i. In a two-stage local price-adjustment game (of perfect information)

($t = 1$) firms simultaneously choose their upward adjustments $\Delta_i \in [0, \bar{\Delta}]$; then
($t = 2$) firms simultaneously choose final retail prices which must satisfy $p_i \in [0, \bar{p}_i + \Delta_i]$.

Definition (Robustness to Local Deviations). *An undercut-proof price profile is* robust to local deviations *if, following a sufficiently small increase in one firm's initial price, there is a Nash equilibrium of the subsequent pricing game which does not increase the deviator's profit.*

Equivalently, for $\bar{\Delta}$ sufficiently small, there is a subgame-perfect equilibrium of the two-stage local price-adjustment game in which $\Delta_i = 0$ and $p_i = \bar{p}_i$ for all i on the equilibrium path.

This is a weaker requirement than support by the equilibrium on-path play of pure strategies in the general two-stage game, simply because we are restricting the extent of first-stage deviations.

We have two motivations for studying robustness to local deviations. The applied motivation is that, once price expectations are established, large deviations (specifically, large price rises) may be very difficult for a firm based on some of the reasons documented in Section 2. The pragmatic motivation is simply that we are able to prove results across a wider group of settings for the weaker requirement than we are for the stronger requirement.

Just above, we argued that any price profile that is supported by the equilibrium play of pure strategies must consist of maximal prices. The argument that we used there also applies when we consider only local deviations in the first stage of our game.

Lemma 5 (Maximal Prices and Local Deviations). *If an undercut-proof profile of prices is robust to local deviations, then it comprises maximal undercut-proof prices.*

Once again, in practical terms this means that we may restrict our attention to the maximal undercut-proof prices across the $n!$ orderings of firms when seeking a prediction for stable prices.

Local Deviations from Maximal Prices. Most results that follow place further restrictions on the structure of consideration sets. However, one result holds under more general conditions.

Our various criteria all lead to maximal undercut-proof prices. These satisfy the binding no-undercutting constraints reported in equation (4). Each constraint checks to see which firm $j < i$ is most tempted to undercut firm i. If one of the binding "temptation" constraints is generated by the firm immediately above, so that firm $i - 1$ is (one of the) most tempted to undercut firm i, then we are able to show that such prices are robust to local deviations.

Proposition 1 (A Sufficient Condition for Robustness to Local Deviations). *Fix an ordering for firms' prices, and consider the unique profile of maximal undercut-proof prices. If*

$$i-1 \in \arg\min_{j\in\{1,\ldots,i-1\}} \left\{ \frac{\bar{p}_j \sum_{B\subseteq\{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B\subseteq\{1,\ldots,i\}} B_j \, \lambda(B)} \right\} \quad \textit{for all } i \in \{2,\ldots,n\}, \tag{5}$$

so that firm $i-1$ is one of the most tempted amongst $\{1,\ldots,i-1\}$ to undercut the price of firm i, then that profile of maximal prices is robust to local deviations.

To prove this result (in Appendix A) we construct an equilibrium of the second-stage pricing subgame following a local deviation ($\Delta_i > 0$) upward by some firm $i > 1$, where the expected profit of firm i matches what it earns on the equilibrium path. If i does deviate upwards, then a lower-indexed (and so higher priced) firm is tempted to undercut i's new (higher) price. Equation (5) says that the non-undercutting constraint binds for firm $i-1$. Our approach is to construct a mixed-strategy equilibrium in which the two firms i and $i-1$ mix continuously, or "tango" (Baye, Kovenock, and de Vries, 1992), over an interval including $\bar{p}_i$.

Plan of the Paper. In the following sections, we derive maximal and Pareto efficient undercut-proof price profiles in several settings. We show that condition (5) holds, and so the efficient prices are robust to local deviations. We also demonstrate situations in which those prices are supported by the equilibrium play of pure strategies in our two-stage pricing game.

5. Exchangeability

Here we allow differently sized consideration sets to have arbitrary masses, but all consideration sets of the same size have the same mass. We call this property exchangeability.

Exchangeable Consideration Sets. In addition to captive customers, $\lambda_i > 0$ for each i, $I_m \geq 0$ customers know $m \in \{2,\ldots,n\}$ prices. Such consideration is random and symmetric across firms so that the mass I_m comprises equal shares of every combination of m firms: consideration sets of the same size have equal mass. For $B \subseteq \{1,\ldots,n\}$ with $|B| \geq 2$ members,

$$\lambda(B) = I_{|B|} \Big/ \binom{n}{|B|} \tag{6}$$

Firms differ by the size of their captive audiences, but are otherwise "exchangeable". We reserve the term "fully exchangeable" for situations in which all captive audiences are the same, so that $\lambda_i = \lambda_j$ for all $i \neq j$, which includes (for example) the classic symmetric Varian (1980) model of sales.[21] The more general "quasi-exchangeable" specification allows firms' captive shares to differ. An interpretation is that non-singleton consideration sets arise from shoppers who obtain a set of price quotations with a search technology that does not bias toward any particular firm. On the other hand, the singleton consideration sets include some (possibly different in mass) local, loyal, or non-shopper customers who are exogenously locked in to a specific supplier.

[21]The full exchangeability setting was analysed with a single-stage game by many, e.g., Burdett and Judd (1983); Johnen and Ronayne (2021); Lach and Moraga-González (2017); Nermuth, Pasini, Pin, and Weidenholzer (2013).

We retrieve the captive-shopper setting with $\lambda_S \equiv I_n$ and $I_m = 0$ for $1 < m < n$, and so this specification encompasses the classic model of sales. However (and as noted in the previous section) exposition is much smoother if we abstract from zero masses of comparison shoppers and assume that $I_m > 0$ for all m.[22] (We study the classic setting comprehensively in Section 6.)

Maximal and Efficient Undercut-Proof Prices. Setting $I_m > 0$ for all m implies that $I_2 > 0$, the "twoness" property. We know, therefore, that any profile of maximal undercut-proof prices contains n distinct prices. We label the firms so that: $\bar{p}_1 > \cdots > \bar{p}_n > 0$.

It is convenient to denote by X_i, the mass of customers (excluding captives) buying from $i \geq 2$:

$$X_i \equiv \sum_{m=2}^{i} I_m \left[\binom{i-1}{m-1} \Big/ \binom{n}{m} \right] \quad \text{and so using this notation} \quad \sum_{B \subseteq \{1,\ldots,i\}} B_i \lambda(B) = \lambda_i + X_i. \tag{7}$$

The term X_i sums over the relevant consideration-set sizes (no sale is made if $m > i$ because then m is cheaper than i). For each m, there are $\binom{n}{m}$ equally-sized consideration sets. Firm m makes a sale only if compared to $m-1$ others from the $i-1$ competitors with higher prices. There are $\binom{i-1}{m-1}$ such sets. We define $X_1 = 0$ for completeness.

To find the maximal undercut-proof prices for this ordering of firms, we can apply Lemma 3:

$$\bar{p}_1 = v \quad \text{and} \quad \bar{p}_i = \min_{j<i} \left\{ \bar{p}_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \quad \text{for } i > 1. \tag{8}$$

Because cheaper firms have more sales ($X_j < X_i$), the term $(\lambda + X_j)/(\lambda + X_i)$ increases in λ. This means that a firm with fewer captives has a greater incentive to undercut. To keep prices high, therefore, it is helpful to place larger firms (with more captives) higher in the ladder of prices. Moreover, this also means that more captive customers are pushed to higher prices. This informal discussion suggests that the Pareto efficient undercut-proof pricing profile will order firms so that $\lambda_1 \geq \cdots \geq \lambda_n$: firms with more captives are more expensive.

Our proof of this claim works by contradiction. We proceed down the list of firms until we find the lowest k where $\lambda_k < \lambda_{k+1}$. We can then show that switching the positions of those two firms results in a Pareto superior profile for firms. Once this is established, we can also show that the binding "no undercutting" constraint for each price $\bar{p}_i$ is the one corresponding to firm $i-1$ undercutting firm i. We can then use (8) to solve recursively for prices.

Proposition 2 (Stable Prices under Quasi-Exchangeability). *In the quasi-exchangeable setting, there is a unique undercut-proof Pareto efficient profile of prices in which all prices are distinct, higher prices are charged by firms with more captive customers, and:*

$$\bar{p}_i = \begin{cases} v & \text{if } i = 1 \\ v \displaystyle\prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j} & \text{if } i \geq 2. \end{cases} \tag{9}$$

We next assess whether this efficient price profile can be established non-cooperatively.

[22] Those zero-mass cases are cumbersome to carry, but our results extend naturally (continuously) to them.

Two-Stage Play under Full Exchangeability. Let $\lambda_i = \lambda > 0$ for all i. Proposition 2 applies, but, because firms are symmetric, we cannot predict which firm charges which price. Making this choice arbitrarily, consider our two-stage pricing game. We construct a subgame-perfect equilibrium in which the firms charge the prices, $\bar{p}_i$, reported in Proposition 2 on the equilibrium path. In the second stage, we have $p_i = \bar{p}_i$ from Lemma 2.

Consider a first-stage deviation by firm $k > 1$ (there is no opportunity for an upward deviation by the highest price firm) and write $\hat{p}_k > \bar{p}_k$ for that deviant initial price. This deviation means that $\hat{p}_k \in (\bar{p}_{i+1}, \bar{p}_i]$ for some higher-priced competitor $i < k$. In the subgame that follows there is no pure-strategy Nash equilibrium. We illustrate the strategies of a Nash equilibrium of that subgame below and place details of the proof of Lemma 6 in Appendix A.

The most straightforward case is when $\hat{p}_k \in (\bar{p}_k, \bar{p}_{k-1}]$. In this case, the deviation by firm k does not (strictly) upset the ordering of firms' initial prices. We can construct an equilibrium in which firms k and $k-1$ continuously (and symmetrically) mix over the interval $[\bar{p}_k, \hat{p}_k)$ via

$$F_k(p) = F_{k-1}(p) = \frac{(\lambda + X_k)(p - \bar{p}_k)}{p(X_k - X_{k-1})}, \tag{10}$$

which (by construction) gives the firms the same expected profit as on the equilibrium path. If $\hat{p}_k = \bar{p}_{k-1}$, the solution is continuous up to the common upper bound, and satisfies $F_k(\bar{p}_{k-1}) = F_{k-1}(\bar{p}_{k-1}) = 1$. If $\hat{p}_k < \bar{p}_{k-1}$ then the firms place residual mass at their initial prices.

A more complex case is when k deviates further upward. We discuss an example of this here: suppose $\hat{p}_k = \bar{p}_{k-2}$. To cope with this, we construct an equilibrium in which the three firms $k-2$, $k-1$, and k all mix (symmetrically) over the interval $[\bar{p}_k, \bar{p}_{k-1})$. Firm $k-1$ then places an atom with remaining mass at its constraining initial price $\bar{p}_{k-1}$. Firms k and $k-2$ then begin mixing again at some price $\tilde{p} \in (\bar{p}_{k-1}, \bar{p}_{k-2})$, and the construction continues. We can repeat this process similarly for higher deviations. Doing so, the proof of the next lemma shows that profits from the constructed Nash equilibrium match those from $p_i = \bar{p}_i$ for all i. (The complexity of the relevant strategies varies with the deviation, $\hat{p}_k$; the proof contains a complete treatment.)

Lemma 6 (Second-Stage Subgames: Full Exchangeability). *In the full exchangeability setting, consider the subgame following the first-stage prices reported in Proposition 2, with the exception of firm $k > 1$, which deviates to an initial price $\hat{p}_k > \bar{p}_k$. There is a Nash equilibrium of that subgame in which each firm i earns a profit of $\bar{p}_i(\lambda + X_i) = v\lambda$.*

To complete a subgame-perfect equilibrium's specification, we allow any equilibrium to be played in second-stage subgames that are further off-path. We then arrive at Proposition 3.

Proposition 3 (Stable Prices in a Two-Stage Game: Full Exchangeability). *In the full exchangeability setting, the unique undercut-proof Pareto efficient profile of Proposition 2 is supported by the equilibrium play of pure strategies in a two-stage pricing game.*

This shows that in the full exchangeability setting, the unique Pareto efficient undercut-proof profile of prices is robust to any (thereby including local) unilateral deviations.

Quasi-Exchangeability. We now assume that captive masses are distinct: $\lambda_i \neq \lambda_j$ if $i \neq j$.

Proposition 2 maps firms to prices, uniquely: firms with more captives are more expensive. We label firms so that $\lambda_1 > \cdots > \lambda_n$. With the strict ordering of captive audiences, the binding undercut-proofness constraint is always local (from the next most expensive firm) whereas non-local constraints have slack. These are the conditions that we need to apply Proposition 1.

Specifically, suppose some firm $k > 1$ deviates to $\hat{p}_k \in (\bar{p}_k, \bar{p}_{k-1}]$. Just as we did when developing Proposition 1, we construct the following (asymmetric) mixed-strategy profile:

$$F_{k-1}(p) = \frac{(p - \bar{p}_k)(\lambda_k + X_k)}{p(X_k - X_{k-1})} \quad \text{and} \quad F_k(p) = \frac{(p - \bar{p}_k)(\lambda_{k-1} + X_k)}{p(X_k - X_{k-1})}, \tag{11}$$

which generates on-path expected profits. The firms place any remaining mass at their initial prices. Each firm $i < k - 1$ has more captive customers than $k - 1$ and k, and does not have a profitable deviation into the interval in which $k - 1$ and k mix. This leads us to Proposition 4.

Proposition 4 (Stable Prices in a Two-Stage Game: Quasi-Exchangeability). *In the quasi-exchangeability setting when firms have distinct captive audiences, the unique undercut-proof Pareto efficient profile of prices, from Proposition 2, is robust to local deviations and so is supported by the equilibrium play of pure strategies in a two-stage local price-adjustment game.*

To summarize, and for a general quasi-exchangeable specification, Proposition 2 offers a unique prediction of stable and dispersed prices, while Proposition 4 shows that these prices are robust to local deviations in a two-stage local price-adjustment game. Proposition 3 establishes fuller robustness (in the sense that the prices are supported by pure-strategy equilibrium play in an unrestricted two-stage pricing game) under full exchangeability. Can that result be extended to situations in which the sizes of firms' captive audiences differ? We return to this question in Section 8, where we pin down (in a triopoly example) exact conditions under which the answer is positive. Nevertheless, we also find (in that section) conditions under which the efficient prices are not supported by the pure-strategy play of our two-stage pricing game.

6. Captives and Shoppers

The quasi-exchangeable specification encompasses the classic model of sales (Varian, 1980). However, Section 5 streamlines exposition by setting $I_m > 0$ for all m. In contrast, the true model of sales eliminates intermediate consideration sets: $I_m = 0$ for all $m \in \{2, \ldots, n-1\}$. Here we apply our quasi-exchangeability results to specifications that are arbitrarily close to the classic model. We then extend key results directly within an exact model-of-sales specification.

An Approximate Model of Sales. Fix a captive-and-shopper model of sales, and (for simplified exposition) index strictly asymmetric firms by their captive audience sizes: $\lambda_1 > \cdots > \lambda_n$.

Now consider a set of quasi-exchangeable model specifications indexed by $\varepsilon > 0$ where, using obvious notation, (i) $\lambda_i^\varepsilon = \lambda_i$; (ii) $\lambda_S^\varepsilon = \lambda_S$; and (iii) $0 < I_m^\varepsilon \leq \varepsilon$ for $m \in \{2, \ldots, n-1\}$.

Condition (iii) limits (via the bound ε) the mass of shoppers who compare only a strict subset of firms, and the specification converges to a pure model of sales as $\varepsilon \to 0$.

Recall from (7) that X_i is the mass of non-captive customers who buy from firm i. Inspecting that expression, notice that $\lim_{\varepsilon\to 0} X_i^\varepsilon = 0$ for all $i \in \{2, \dots, n-1\}$, and so in the model-of-sales limit ($\varepsilon \to 0$) only firm n serves non-captive customers. Similarly, an inspection of (9) from Proposition 2 shows that for efficient undercut-proof prices

$$\lim_{\varepsilon\to 0} \bar{p}_i^\varepsilon = v \quad \text{for} \quad i < n \quad \text{and} \quad \lim_{\varepsilon\to 0} \bar{p}_n^\varepsilon = p_{n-1}^\dagger \quad \text{where} \quad p_i^\dagger \equiv \frac{v\lambda_i}{\lambda_i + \lambda_s}. \tag{12}$$

Here $p_i^\dagger$ is the lowest undominated price for firm i in a model of sales: by setting this price and serving all shoppers, a firm earns the same as it does from exploiting its captive audience at the monopoly price of v. Equation (12) says that the n distinct prices collapse to two points.

Corollary (to Proposition 2). *In the model-of-sales limit of a quasi-exchangeable specification, the unique undercut-proof Pareto efficient price profile is for the smallest firm (in terms of captive audience), firm n, to charge the distinct lowest price, $p_{n-1}^\dagger < v$, and to serve all shoppers, while all other firms $i < n$ charge the monopoly price, v, and serve only their captives.*

We can apply Proposition 4 as we approach the limit: the efficient price profile that collapses to v and $p_{n-1}^\dagger$ is robust to local deviations. However, working directly with the actual (rather than approximate) model of sales we can make a stronger statement equivalent to Proposition 3.

An Exact Model of Sales. We now work with the actual captive-and-shopper model of sales by setting $I_m = 0$ for $m \in \{2, \dots, n-1\}$. Pairwise consideration sets are empty and so the "twoness" property does not hold. This means that claim (i) of Lemma 1 does not apply: strictly positive undercut-proof prices are not necessarily distinct.[23] Indeed, we will find (just as in the limit case considered above) that there can and will be a tie between $n-1$ prices.

We now consider what must be true of any maximal undercut-proof prices. The lowest price cannot be zero: such a firm (even if the there is more than one) could raise its price locally without violating no-undercutting constraints.[24] Given that the lowest price is strictly positive it must be charged by only a single firm (else a profitable undercut would be available). Firms who are not the cheapest, and so sell only to their captives, can raise their prices to the maximum, v, while maintaining undercut-proofness; a Pareto improvement. We conclude that there must be only two price points: a lowest (strictly positive) price and v.

We can focus on price profiles in which one firm i sets $\bar{p}_i < v$ while other firms set $\bar{p}_j = v$ for $j \neq i$. All such firms j earn $v\lambda_j$, and so to dissuade undercutting we need $\bar{p}_i \leq p_j^\dagger$, where $p_j^\dagger$ is

[23] We noted earlier that if a "k-ness" property holds, so that all consideration sets comprising $k > 1$ firms have positive mass, then there can be at most $k-1$ tied prices. A model of sales has this property only for $k = n$, which leaves open the possibility of $n-1$ tied prices. Ultimately, this is what we predict. Claim (ii) of Lemma 1 does not hold as stated. However, this is simply because we need to adjust our notation to deal with cases of tied prices. Finally, Lemma 2 continues to hold more generally even without the twoness property.

[24] Any higher-priced firm earns strictly positive profits, and would earn less by pricing close enough to zero.

from (12). Thus, the maximal profile when i is cheapest must satisfy

$$\bar{p}_i = \min_{j \neq i} p_j^\dagger = \begin{cases} p_{n-1}^\dagger & \text{if } i = n, \text{ or} \\ p_n^\dagger & \text{if } i \in \{1, \ldots, n-1\}. \end{cases} \tag{13}$$

We have found n maximal price profiles, which vary according to the identity of the shopper-capturing cheapest firm. All firms $j \neq i$ earn $v\lambda_j$. If $i < n$, firm i earns strictly less than $v\lambda_i$; but if $i = n$ then firm i earns strictly more. From this we conclude that $i = n$ (the firm with fewest captives is cheapest) pins down the unique Pareto efficient undercut-proof profile. Firm n commits to the highest price such that it wins all the shoppers without a fight.

Proposition 5 (Stable Prices in the Captive-Shopper Setting). *In the captive-shopper setting, there is a unique Pareto efficient profile of undercut-proof prices in which firm n, with the fewest captives, sets $\bar{p}_n = p_{n-1}^\dagger$ and other firms set the monopoly price v. Each firm $i < n$ earns its "captive only" profit $v\lambda_i$, while firm n earns strictly more: $p_{n-1}^\dagger(\lambda_n + \lambda_S) > v\lambda_n$.*

Proposition 5 suggests stable prices for the captive-shopper setting: there is no profitable downward deviation from $p_i = \bar{p}_i$, and there is no better profile of initial prices.

We now consider our two-stage pricing game. Consider a strategy profile in which firms choose the efficient undercut-proof prices of Proposition 5 as initial prices, and maintain those prices on the equilibrium path in the second stage. Standard arguments used already show that the only candidate for a profitable deviation for a firm is to deviate upwards at the first stage. In this case (with $n-1$ firms pricing at v) the only firm that can do so is firm n.

A deviation by firm n to $\hat{p}_n \in (\bar{p}_{n-1}^\dagger, v]$ leads to a subgame in which there is no pure-strategy Nash equilibrium. Lemma 7 reports the profits in a mixed-strategy equilibrium.

Lemma 7 (A Captive-Shopper Subgame). *In the captive-shopper setting, consider the subgame following $\hat{p}_n > p_{n-1}^\dagger$ and $\bar{p}_i = v$ for $i < n$. There is a unique Nash equilibrium of that subgame in which firm n earns $p_{n-1}^\dagger(\lambda_n + \lambda_S)$ and each firm i earns profit $v\lambda_i$.*

For example, if firm n deviates to a first-stage price $\hat{p}_n \in (p_{n-1}^\dagger, p_{n-2}^\dagger)$, then in the subgame's Nash equilibrium each $i < n-1$ sets its first-stage price ($p_i = \bar{p}_i = v$), while firms $n-1$ and n mix continuously over the interval $[p_{n-1}^\dagger, \hat{p}_n)$ with distribution functions

$$F_{n-1}(p) = \frac{(p - p_{n-1}^\dagger)(\lambda_S + \lambda_n)}{p\lambda_S} \quad \text{and} \quad F_n(p) = \frac{(p - p_{n-1}^\dagger)(\lambda_S + \lambda_{n-1})}{p\lambda_S}, \tag{14}$$

with n and $n-1$ placing remaining mass at $\hat{p}_n$ and $\bar{p}_{n-1} = v$, respectively. The profit earned by firm n is the same as reported in Proposition 5, making the upward deviation in its initial price non-profitable. Finally, within (off-path) subgames following any other choices of first-stage prices, any equilibrium may be played. Together, this leads to Proposition 6.

Proposition 6 (Stable Prices in a Two-Stage Captive-Shopper Game). *In the captive-shopper setting, the undercut-proof Pareto efficient profile of prices reported in Proposition 5, in which the firm with the smallest captive audience serves all shoppers, is the unique price profile supported by the equilibrium play of pure strategies in the two-stage game. Any other profile of maximal undercut-proof prices is robust only to local deviations.*

This also establishes that the efficient profile is the only one supported by the equilibrium play of pure strategies. Other candidate profiles (which are maximal given the order of firms) involve a different firm $i < n$ adopting the low-price shopper-capturing position. However, such a firm $i < n$ earns strictly less than $v\lambda_i$ and can profitably deviate to $\hat{p}_i = v$.

Nevertheless, those other candidate profiles (where the low-price firm is some $i < n$) are robust to local deviations. Consider the profile in which firm $i < n$ charges $p_n^\dagger$ and all other firms set $\bar{p}_j = v$. If firm i deviates to $\hat{p}_i > p_n^\dagger$ then, so long as that deviation is not too large, there is a unique equilibrium of the subgame in which firms i and n mix according to

$$F_i(p) = \frac{(p - p_n^\dagger)(\lambda_S + \lambda_n)}{p\lambda_S} \quad \text{and} \quad F_n(p) = \frac{(p - p_n^\dagger)(\lambda_S + \lambda_i)}{p\lambda_S}, \tag{15}$$

with remaining mass in these distributions placed at $\hat{p}_i$ and $\bar{p}_n = v$, respectively.

Sales. Returning to the established and classic single-stage model-of-sales pricing game, we recall that the mixed strategies of Varian (1980) were intended to capture "sales". Realized prices may be either close or far apart, and the identity of the cheapest firm is uncertain. With asymmetric captive shares, this is true for the two firms who "tango", while $n - 2$ others maintain the "regular" monopoly price (Baye, Kovenock, and de Vries, 1992, Section V).[25]

In contrast, we predict $n - 1$ regular prices alongside one starkly-lower "on-sale" price. Of course, over time there can be changes in which firm has the fewest captives, which would flip the identity of the on-sale firm. Such shifts can be more frequent if the sizes of captive audiences are relatively close, and so their order can change more easily. In that sense, we retain the spirit of sales, while providing new insights and pricing predictions.

7. Independent Awareness

The exchangeability setting allows arbitrary variation in the mass of customers with differently sized (with respect to the number of firms) consideration sets. It also allows consideration to be correlated. For example, in the captive-shopper case, a customer who sees more than one firm sees them all. Nevertheless, quasi-exchangeability allows asymmetry only in captive masses.

[25] In particular, $i \in \{1, \ldots, n-2\}$ charge $p_i = v$ while n and $n-1$ mix over the interval $[p_{n-1}^\dagger, v)$ with firm $n-1$ placing an atom at v. When marginal costs and captive shares are asymmetric, analysis is substantially complicated. There, we found there is (generically) a unique Nash equilibrium in which the (cost and captive) parameters dictate how many firms mix, which can be anything between 2 and n (Myatt and Ronayne, 2023b).

Here we restrict the correlation of firms in consideration sets, but allow for much greater generality in asymmetries. We build upon prior work including Butters (1977), Grossman and Shapiro (1984), Ireland (1993), McAfee (1994), and Eaton, MacDonald, and Meriluoto (2010): each price is exposed to an independent (but asymmetric) fraction of potential customers.

Consideration Sets with Independent Awareness. On the demand side, an independent fraction $\alpha_i \in (0,1)$ of customers is aware of firm i.[26] Using our notation $B \subseteq \{1,\ldots,n\}$ for a consideration set, the mass of customers who consider these firms is

$$\lambda(B) = \left(\prod\nolimits_{i\in B}\alpha_i\right)\left(\prod\nolimits_{i\notin B}(1-\alpha_i)\right) = \prod_{i=1}^{n}\alpha_i^{B_i}(1-\alpha_i)^{1-B_i}. \tag{16}$$

We say that firm i is larger than firm j if it enjoys greater awareness, so that $\alpha_i \geq \alpha_j$.[27]

Maximal and Efficient Undercut-Proof Prices. Here, the "twoness" property holds because $\lambda(\{i,j\}) = \alpha_i\alpha_j\prod_{k\notin\{i,j\}}(1-\alpha_k) > 0$, which means that maximal undercut-proof prices are distinct. We label firms so that $\bar{p}_1 > \cdots > \bar{p}_n$ and apply Lemma 3, to produce Lemma 8.

Lemma 8 (Maximal Prices for Independent Awareness). *Under independent awareness, maximal undercut-proof prices are $\bar{p}_1 = v$ and $\bar{p}_i = (1-\alpha_i)\bar{p}_{i-1} = v\prod_{j=2}^{i}(1-\alpha_j)$ for $i > 1$. Under these prices every firm j is just indifferent to undercutting any other firm $i > j$. The profit of each firm $i \in \{1,\ldots,n\}$ from this price profile is $\pi_i = v\alpha_i\prod_{j=2}^{n}(1-\alpha_j)$.*

The second claim says that all no-undercutting constraints bind simultaneously.[28] To see why, consider the profit of firm j from maintaining its price $\bar{p}_j$. It receives that price from customers aware of it, true with probability α_j, and who are aware of no cheaper firm, true with probability $\prod_{k>j}(1-\alpha_k)$, giving profit $\bar{p}_j\alpha_j\prod_{k>j}(1-\alpha_k)$. If j undercuts $i > j$ then it gets $\bar{p}_i$ (or arbitrarily close to it) from customers who consider j and no firm cheaper than i. That gives j a profit of (or arbitrarily close to) $\bar{p}_i\alpha_j\prod_{k>i}(1-\alpha_k)$. The no-undercutting constraint is then

$$\bar{p}_i\alpha_j\prod_{k>i}(1-\alpha_k) \leq \bar{p}_j\alpha_j\prod_{k>j}(1-\alpha_k) \quad\Leftrightarrow\quad \bar{p}_i \leq \bar{p}_j\prod_{k=j+1}^{i}(1-\alpha_k). \tag{17}$$

A special case is the local constraint: $\bar{p}_i \leq \bar{p}_{i-1}(1-\alpha_i)$, which does not involve the awareness of firm j, α_j. This means that undercutting constraints do not depend on the type of the firm that considers the undercut. Examining $\bar{p}_i \leq \bar{p}_{i-1}(1-\alpha_i)$, it is the awareness of the firm being undercut (firm i) that is relevant. This contrasts with our exchangeability specification under which it is the type of the firm contemplating the undercut that matters.[29]

[26] If two or more firms enjoy complete awareness, $\alpha_i = 1$, then the Bertrand (zero profit) outcome follows. Allowing (at most) one firm to be known to all customers does not affect our results (and is relevant to some results under endogenous advertising). But for smoother exposition, we carry $\alpha_i \in (0,1)\,\forall i$ forward in the text.

[27] The special case of symmetry ($\alpha_i = \alpha_j$ for all i, j) falls within the full exchangeability specification of Section 5.

[28] The feature is useful to construct the equilibrium strategies in the off-path subgames (to prove Lemma 9) when we consider whether the stable and efficient prices we will derive can be supported in our two-stage game. The same feature enabled the analogous construction to prove Lemma 6, under full exchangeability.

[29] In the quasi-exchangeability specification, the efficient profile orders firms by the size of their captive audiences. For that order of firms, the local no-undercutting constraints take the form $\bar{p}_i \leq \bar{p}_{i-1}(\lambda_{i-1}+X_{i-1})/(\lambda_{i-1}+X_i)$. Note that this depends on the type (λ_{i-1}) of firm $i-1$, which is the firm that contemplates the undercut.

The profit expressions reported in Lemma 8 are also of interest. Note that

$$\pi_i = v\alpha_i \prod_{j=2}^{n}(1-\alpha_j) = \frac{v\alpha_i}{1-\alpha_1}\prod_{j=1}^{n}(1-\alpha_j). \tag{18}$$

A firm's profit depends on the monopoly price v and its own awareness α_i in a natural way. The product term in the second expression ranges over all firms, and so does not depend on the order of them. That order influences firm i's profit only via the denominator term $1-\alpha_1$, which depends upon the awareness of the firm at the top of the pricing ladder. The profit of firm i (and of every firm) is increasing in α_1; and indeed this is the only way in which the order of firms influences the profits obtained from maximal undercut-proof prices, giving Proposition 7.

Proposition 7 (Stable Prices under Independent Awareness). *Under independent awareness, the Pareto efficient undercut-proof price profiles are all of those that order firms so that the largest firm charges the monopoly price, v. Other firms' prices, for any order of these firms, are given by the expressions of Lemma 8. All generate the same profits for each firm.*

We can contrast this result with Proposition 2 from our exchangeability specification. There, we found (at least for strictly asymmetric firms) a unique Pareto efficient undercut-proof profile of prices. Here, however, we identify $(n-1)!$ such profiles (all of which are profit equivalent).

Two-Stage Play under Independent Awareness. For any profile identified by Proposition 7 and price $\bar{p}_i$ for $i > 1$, we know the no-undercutting constraint of $i-1$ binds. Proposition 1 applies: an efficient undercut-proof profile here is robust to local deviations.

However, it is instructive to look at equilibrium strategies following such a (small) deviation. Suppose that firm $i > 1$ deviates upward by $\Delta_i < \bar{p}_{i-1} - \bar{p}_i$. In the associated subgame, we construct an equilibrium in which firms $j \notin \{i-1, i\}$ continue with their on-path play by choosing $p_j = \bar{p}_j$, while firms $i-1$ and i continuously mix over $[\bar{p}_i, \bar{p}_i + \Delta_i)$ with distributions

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_i}{p}\right) \quad \text{for } j \in \{i-1, i\}, \tag{19}$$

and place remaining mass at $\bar{p}_{i-1}$ and $\bar{p}_i + \Delta_i$ respectively, earning their on-path profits, π_j.

This argument is enough to establish that for a small enough deviation (so that Δ_i is not too large) we can construct an equilibrium which removes the incentive for any firm to deviate. In other words: any profile of maximal undercut-proof prices is robust to local deviations.

For larger deviations, this "tango" between firms $i-1$ and i can fail. To see why, suppose that firm i deviates all of the way up to the initial price of firm $i-1$. (This is an upward deviation of $\Delta_i = \bar{p}_{i-1} - \bar{p}_i$.) For our distributions to be valid, we need

$$\max_{j\in\{i-1,i\}} F_j(\bar{p}_{i-1}) = \frac{1}{\min\{\alpha_{i-1}, \alpha_i\}}\left(1 - \frac{\bar{p}_i}{\bar{p}_{i-1}}\right) = \frac{\alpha_i}{\min\{\alpha_{i-1}, \alpha_i\}} \leq 1 \quad \Leftrightarrow \quad \alpha_i \leq \alpha_{i-1}. \tag{20}$$

This says that the two firms must be in awareness order, with the larger firm at the higher price position. If they are out of order, so that $\alpha_i > \alpha_{i-1}$, then this construction fails.

To resolve the problem of deviations to higher prices, we bring another firm on to the "dance floor". For example, in this case, and if $i > 2$, we can construct an equilibrium in which firms $i-2$, $i-1$, and i mix. The construction is quite complex, and becomes more so for higher initial prices by the deviant firm i that move above the initial prices of other (lower indexed) firms. What is crucial for the construction is to find some more expensive firm that is larger than the deviant firm. For example, if $i = 2$ then we must have $\alpha_1 \geq \alpha_2$ if the construction of the equilibrium is to work. This holds for all possible deviations by all possible deviants if we place the largest firm at the top of the price sequence, so that $\alpha_1 \geq \alpha_i$ for all $i \in \{2, \dots, n\}$.

More generally, if we do not place the largest firm at the highest price position then this firm will have a profitable deviation in the first stage. To see why, suppose that $\alpha_i > \alpha_1$ so that firm i is strictly larger than the highest-priced firm, using Lemma 8, i's profit is

$$\frac{v\alpha_i \prod_{j=1}^{n}(1-\alpha_j)}{1-\alpha_1} < \frac{v\alpha_i \prod_{j=1}^{n}(1-\alpha_j)}{1-\alpha_i} = v\alpha_i \prod_{j \neq i}(1-\alpha_j). \tag{21}$$

This last expression is the profit that i achieves by setting $p_i = \bar{p}_i = v$ and selling only to captive customers. From this, we conclude that we must order firms with the largest awareness at the top. If we order firms completely (from the largest to the smallest as we move down the sequence of prices) then we can construct an equilibrium with on-path payoffs in any subgame when a firm deviates upward by any amount in the first stage.

Lemma 9 (Second-Stage Subgames: Independent Awareness). *In the independent awareness setting, consider the subgame following maximal (and efficient) prices when firms are ordered by size except that some firm k deviates to a higher initial price. There is a Nash equilibrium of that subgame in which each firm i earns its on-path profit, $\pi_i = v\alpha_i \prod_{j=2}^{n}(1-\alpha_j)$.*

Note that this lemma asks the firms to be completely ordered by size: $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. In the next section we obtain the same result in a triopoly where $\alpha_1 \geq \alpha_3 > \alpha_2$, so that firms are not completely ordered (but nevertheless the largest firm is at the top).

We assemble our observations into the following proposition.

Proposition 8 (Stable Prices in a Two-Stage Game: Independent Awareness). *If a profile of prices is supported by the equilibrium play of pure strategies then those undercut-proof prices are Pareto efficient, and so the most expensive firm is a largest firm: $\alpha_1 \geq \alpha_i$ for $i > 1$.*

The undercut-proof Pareto efficient price profile for firms ordered by size, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, is supported by the equilibrium play of pure strategies.

Any other maximal undercut-proof price profile is (at least) robust to local deviations.

This leaves open the possibility that multiple efficient profiles are supported by the equilibrium play of pure strategies. However, the expressions for firms' expected profits are the same for all such profiles, and also match those reported by Ireland (1993) and McAfee (1994). They considered a single stage of pricing and characterized mixed-strategy Nash equilibria. A message here is that we can establish stable price dispersion (entirely distinct prices chosen as pure strategies) without impacting firms' expected profits. Moreover, we do not need to take a stand on which efficient profile will occur in order to evaluate the profit outcome for the firms.

8. Limitations

Our central specifications (exchangeability and independent awareness) allow for considerable breadth. We do not, however, offer unambiguous results (of the kind reported in Propositions 3, 5 and 8) for all consideration-set specifications. This is because such results (as we now show) are not always possible. Here we use a trio of triopoly specifications which illustrate the limitations to our results and demonstrate the environment's complexities.

We show that (i) efficient undercut-proof prices are not always supported by the equilibrium play of pure strategies in our two-stage game; (ii) sometimes multiple efficient profiles can be supported; and (iii) there can be a unique efficient profile, supported by equilibrium play, in which the order of firms' prices does not correspond to a natural ordering of their sizes.

Two-Stage Play under Quasi-Exchangeability. Under quasi-exchangeability (Section 5) we know that the unique profile of efficient undercut-proof prices is robust to local deviations in a two-stage game (Proposition 4). However, these prices are not always robust to larger deviations: here we find a counter-example of efficient prices that are not supported by the equilibrium play of our two-stage pricing game for the whole parameter space.

To proceed, we take a quasi-exchangeability case in which $\lambda_1 > \lambda_2 > \lambda_3 > 0$. From Proposition 2, the unique Pareto efficient undercut-proof prices are

$$\bar{p}_1 = v, \quad \bar{p}_2 = \frac{v\lambda_1}{\lambda_1 + X_2}, \quad \text{and} \quad \bar{p}_3 = \frac{v\lambda_1}{\lambda_1 + X_2}\frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{22}$$

We can handle any upward deviation in the first-stage price of firm 2 (where only two firms mix). Similarly, if 3 deviates upward to $\hat{p}_3 \in (\bar{p}_3, \bar{p}_2]$ then, just as we did for local deviations, we can construct the following mixed-strategy profile for firms 2 and 3:

$$F_2(p) = \frac{(\lambda_3 + X_3)(p - \bar{p}_3)}{p(X_3 - X_2)} \quad \text{and} \quad F_3(p) = \frac{(\lambda_2 + X_3)(p - \bar{p}_3)}{p(X_3 - X_2)}, \tag{23}$$

which generates on-path expected profits for both firms across this range. These functions are increasing from $F_2(\bar{p}_3) = F_3(\bar{p}_3) = 0$, and satisfy $F_3(p) > F_2(p)$ for higher prices. Furthermore,

$$F_3(p) \leq F_3(\bar{p}_2) = \frac{(\lambda_2 + X_3)(\bar{p}_2 - \bar{p}_3)}{\bar{p}_2(X_3 - X_2)} = 1, \tag{24}$$

and so we have constructed valid distribution functions, which are completed by specifying that both firms place any remaining mass at their first-stage prices.

Now suppose instead that firm 3 deviates up to $\hat{p}_3 \in (\bar{p}_2, \bar{p}_1]$. We can use the same construction above to build an equilibrium in which firms 2 and 3 mix over the interval $[\bar{p}_3, \bar{p}_2)$, and where firm 2 places remaining mass at $\bar{p}_2$. We have already noted, however, that firm 3's distribution satisfies $F_3(\bar{p}_2) = 1$. Nevertheless, the higher initial price means that firm 3 is able to price higher than $\bar{p}_2$. Indeed, if it chooses $p > \bar{p}_2$ (so sacrificing the capture of the atom played by firm 2) then it will move its price all the way up $p = \hat{p}_3$. It earns this price on λ_3 captive customers and X_2 customers with consideration set $\{1, 3\}$, and so earns an expected profit of $\hat{p}_3(\lambda_3 + X_2)$. For our construction to work, this must be less than $\bar{p}_3(\lambda_3 + X_3)$. The deviant first-stage price can be as high as $\bar{p}_1 = v$, and so we need

$$v(\lambda_3 + X_2) \leq \bar{p}_3(\lambda_3 + X_3) \quad \Leftrightarrow \quad \frac{\lambda_3 + X_2}{\lambda_3 + X_3} \leq \frac{\lambda_1}{\lambda_1 + X_2} \frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{25}$$

This holds for some parameter values and fails for others. For example, it holds if X_2 is sufficiently small. If X_2 approaches zero then this quasi-exchangeable triopoly approaches the classic captive-shopper specification. We are assured, therefore, that if we are close to the captive-shopper setting then the efficient price profile is supported by equilibrium two-stage play. On the other hand, the inequality above fails if λ_2 and λ_3 are sufficiently close. (It strictly fails if $\lambda_2 = \lambda_3$ for example.) This means that we can find circumstances (e.g., $\lambda_1 > \lambda_2 \approx \lambda_3$) in which the efficient profile is not supported by the equilibrium play of pure strategies.

Our discussion relies upon a "tango" danced by firms 2 and 3. In our analysis of the fully exchangeable specification, we dealt with higher initial price deviations by constructing equilibria in which higher priced firms joined in. In a triopoly this is when all three firms mix beginning from the lowest initial price, $\bar{p}_3$. Here, however, this cannot work. That is because firm 1 (the largest firm, in terms of captive audience) is strictly unwilling to choose such a low price. The proof of the result that follows confirms the details: we show that if the inequality in (25) fails then we cannot construct a suitable equilibrium of the subgame, and we construct another equilibrium (of the subgame) which yields a strictly higher expected profit for the deviant.

Proposition 9 (A Strictly Quasi-Exchangeable Triopoly). *In a triopoly with strictly quasi-exchangeable consideration sets, the unique Pareto efficient undercut-proof prices are supported by the equilibrium play of pure strategies in our two-stage game if and only if*

$$\frac{\lambda_3 + X_2}{\lambda_3 + X_3} \leq \frac{\lambda_1}{\lambda_1 + X_2} \frac{\lambda_2 + X_2}{\lambda_2 + X_3}. \tag{26}$$

This holds if the mass of customers who conduct pairwise comparisons is sufficiently small, but it fails if the masses of captive customers for the second and third firms are sufficiently similar.

Multiple Equilibrium Profiles under Independent Awareness. For independent awareness (Section 7), a profile of efficient prices places the largest (most widely known) firm at the highest price ($\bar{p}_1 = v$). But this does not pin down the ordering of other firms. There are $(n-1)!$ such orderings for which we can construct (in the usual way) the maximal undercut-proof prices. These (different) price profiles generate the same profits. All firms are indifferent between them, and so all such profiles are efficient (Proposition 7).

Proposition 8 reported that the profile of prices in which firms are ordered by size (so that firms with greater awareness charge higher prices) is supported by the equilibrium play of pure strategies. We left open the possibility that other efficient profiles are supported.

We take up that issue here. Consider three firms with independent-awareness parameters $\alpha_i \in (0,1)$: $\alpha_1 > \alpha_3 > \alpha_2$. We order the firms so that $\bar{p}_1 > \bar{p}_2 > \bar{p}_3$. This means that the second and third firms are not in size order. The associated maximal (and efficient) prices are

$$\bar{p}_1 = v,\ \bar{p}_2 = v(1-\alpha_2),\ \text{and}\ \bar{p}_3 = v(1-\alpha_2)(1-\alpha_3). \tag{27}$$

Now consider our two-stage pricing game and an upward deviation by firm 3. Our previous approach was to build an equilibrium in the subgame in which firms 2 and 3 mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p}\right) \quad \text{for } j \in \{2,3\}. \tag{28}$$

The problem with this is that for p sufficiently high, $F_2(p)$ is not a valid distribution:

$$F_2(\bar{p}_2) = \frac{1}{\alpha_2}\left(1 - \frac{\bar{p}_3}{\bar{p}_2}\right) = \frac{\alpha_3}{\alpha_2} > 1. \tag{29}$$

This means that if firm 3 deviates upward (even while remaining below $\bar{p}_2$) then, in the subgame, we need all three firms to mix. In the proof Proposition 10 we build an equilibrium in which all three firms mix up until a specific price at which firm 1 places all remaining mass at its initial price while the remaining two firms continue to "tango". The construction is conceptually straightforward, but nevertheless detailed and delicate, and underpins our next result.

Proposition 10 (Two-Stage Play of an Awareness Triopoly). *In a triopoly under (strictly asymmetric) independent awareness, there are two efficient undercut-proof price profiles, both of which assign the highest price to the largest firm, but which differ by the order of the other two firms. Both profiles are supported by the equilibrium play of pure strategies.*

A Prominence Setting. Here we consider a specification for consideration sets that places one of three firms in a special or "prominent" position relative to the others. We find two different efficient profiles, which have different profit outcomes. Only one is supported by the equilibrium play of pure strategies, and that profile does not place firms in "size" order.

Firm $i = 1$ is "prominent" in the sense that it is known to all customers. This means that it is contained within any positive-mass consideration set. Customers see at most one of the other two firms $i \in \{2,3\}$, but never see all three. Summarizing, the three positive-mass consideration sets are $\{1\}$, $\{1,2\}$, and $\{1,3\}$. We use the following notation:

$$\phi_1 = \lambda(\{1\}),\ \phi_2 = \lambda(\{1,2\}),\ \text{and}\ \phi_3 = \lambda(\{1,3\}). \tag{30}$$

An interpretation is that the prominent firm is a national sales channel, whereas the other firms are local suppliers. Each local firm $i \in \{2,3\}$ has access to a base of customers who see i's price. Additionally, all potential customers are informed of firm 1's price.[30] We let $\phi_2 \geq \phi_3$.

[30] Inderst (2002) considered a related single-stage model, but did not fully characterize equilibrium; we characterize an equilibrium for one of his cases in Appendix B. Armstrong and Vickers (2022, Section 4) solved the single-stage game in a closely related setting, interpreting firms as a chain store with local rivals. In their setting any comparison involving a local firm involves the chain store, but local rivals also have captive audiences.

Under this specification only a customer with consideration set $\{1\}$ is truly captive: we have $\lambda_1 = \phi_1 > 0$ but $\lambda_2 = \lambda_3 = 0$. This specification also does not fit the regularity condition (in Section 4) which says that all firms have captive customers. Similarly, "twoness" does not hold because there are no pairwise comparisons of firms 2 and 3.

Nevertheless, we can readily identify both maximal and efficient undercut-proof prices. If prices are undercut-proof and strictly positive, then they must place the prominent firm at the top. (If the prominent firm charges strictly less than a local firm, then that local firm makes no sales and would then undercut the prominent firm.) If prices are maximal, then of course $\bar{p}_1 = v$.

Turning to no-undercutting constraints, we need only to check the prominent firm does not wish to undercut either of the local firms. If local firms are ordered so that $\bar{p}_2 \geq \bar{p}_3$, then the relevant constraints are $v\phi_1 \geq \bar{p}_2(\phi_1+\phi_2)$ and $v\phi_1 \geq \bar{p}_3(\phi_1+\phi_2+\phi_3)$. For prices to be maximal, these bind. Let $j > 1$ set the lowest price, $\bar{p}_j$. It is undercut-proof if

$$\bar{p}_j \leq v\phi_1/(\phi_1 + \phi_2 + \phi_3). \tag{31}$$

For efficiency (31) must bind, telling us the lowest price is independent of which firm sets it. Notice also that if both local firms set this price, then either could raise it slightly without provoking an undercut by firm 1, so such prices are not maximal. Thus, firms set distinct prices. Let $i \neq j$ be the local firm that sets the higher price. Firm 1 does not undercut $\bar{p}_i$ if

$$\bar{p}_i \leq v\phi_1/(\phi_1 + \phi_i). \tag{32}$$

For efficiency (32) must bind: $\bar{p}_i$ depends on how many customers consider i's price, unlike $\bar{p}_j$. Given $\bar{p}_1 = v$, each local supplier prefers that they charge $\bar{p}_i$ and the other charges $\bar{p}_j$, than vice versa. Thus, there are two Pareto efficient undercut-proof profiles: one in which $i = 2$ and $j = 3$, and one in which $i = 3$ and $j = 2$ (these coincide in the special case of $\phi_2 = \phi_3$, in which case there is exactly one Pareto efficient undercut-proof profile), summarized in Proposition 11.

We proceed to check whether the profiles are robust to unilateral deviations in the two-stage game. It is straightforward to construct a Nash equilibrium in the subgame following unilateral deviations upward by a non-prominent firm that preserve the order of firms' initial prices: the deviator and the prominent firm mix in the interval between the pre- and post-deviation initial price, while the other firm maintains its initial price. As usual, because the pre-deviation price is the lower bound of the support, the deviator's profit is unchanged.

It remains to consider a deviation by the cheapest firm j to a first-stage price $\hat{p}_j \in (\bar{p}_i, \bar{p}_1]$. In Appendix C we show that when $\phi_2 > \phi_3$ and $j = 3$, such a deviation leads to a subgame in which any Nash equilibrium gives firm 3 strictly greater profit. The reason is that the larger non-prominent firm 2 charges a low intermediate price to prevent the prominent firm undercutting it ($\bar{p}_2 = v\phi_1/(\phi_1 + \phi_2)$). This leaves an interval of prices, $(\bar{p}_2, v\phi_1/(\phi_1 + \phi_3))$, which are dominated for the prominent firm (by v), and so are safe for firm 3 to deviate to and yield it a profit strictly greater than $\bar{p}_3\phi_3$. (If $j = 2$, then a deviation of this sort is unavailable.)

Proposition 11 reports these results along with the unique efficient undercut-proof profile for an n-firm oligopoly with symmetrically-sized firms, which has a particularly concise form.

Proposition 11 (Stable Prices in a Prominence Setting). *In the prominence triopoly, there are two undercut-proof Pareto efficient profiles, described by:*

$$\bar{p}_1 = v, \quad \bar{p}_i = \frac{v\phi_1}{\phi_1 + \phi_i}, \quad \textit{and} \quad \bar{p}_j = \frac{v\phi_1}{\phi_1 + \phi_2 + \phi_3} \quad \textit{for} \quad i, j \in \{2, 3\} \quad \textit{and} \quad i \neq j, \tag{33}$$

Each firm $k \in \{1, 2, 3\}$ makes a profit equal to $\bar{p}_k\phi_k$. Both of these profiles are robust to local deviations. There is a unique profile supported by the equilibrium play of pure strategies in our two-stage game, in which the larger non-prominent firm is the cheapest, i.e., $i = 3$ and $j = 2$.

With n symmetrically-sized firms ($\phi_i = \phi$ for all $i \in \{1, \ldots, n\}$), there is a unique undercut-proof profile of prices in which a firm's price declines inversely to its position in the sequence:

$$\bar{p}_i = \frac{v}{i} \quad \textit{for all} \quad i \in \{1, \ldots, n\}. \tag{34}$$

This profile is supported by the equilibrium play of pure strategies in our two-stage game.

A (non-prominent) firm with a larger audience is cheaper. It remains the case that the profit of a non-prominent firm is increasing in its own size. However, the larger non-prominent firm can make a smaller profit than the other. This is true whenever their sizes are sufficiently close.

As the prominent firm's position strengthens (greater ϕ_1) price cuts hurt it more and so its rivals can set higher prices without being undercut. This implies non-prominent firms' prices and profits are increasing in ϕ_1 and that customers are worse off with a larger prominent firm.[31]

9. Stable Prices as Components of Deeper Models

Earlier sections of the paper are built upon exogenous consideration sets. Of course, consideration sets themselves might respond to the actions of firms and customers. For example, a firm's advertising choices might increase the mass of consideration sets which include its price. Similarly, increased price discovery by customers might shift mass from smaller to larger consideration sets as such customers obtain more price quotations. An investigation of such consideration-influencing activities is important, and in the context of this paper is a check that such actions do not unwind the predictions of stable price dispersion that we have made.

Naturally, a full consideration of both firms' advertising and customers' price discovery is beyond the scope of (at least the main body of) this paper. However, here we use a simple duopoly framework to illustrate the likely pattern of firms' and customers' actions on the consideration-set environment. Constructively, this exercise also demonstrates how our pricing framework is naturally incorporated into richer settings.

[31] In Appendix B we develop this model of prominence by adding an earlier stage to consider the incentives of a "prominence provider" which brings one of many local firms to national prominence. We find that this provider makes a prominence offer (which is accepted) to the local firm with the largest local customer base, which is the worst choice for customers because it amplifies the asymmetry between firms.

Duopoly, Revisited. The duopoly specification of Section 3 fits within all of the frameworks (exchangeability, captive-shopper, and independent awareness) considered in Sections 5 to 7. Recall that λ_i customers are captive to firm $i \in \{1,2\}$ and $\lambda_S \equiv \lambda(\{1,2\})$ customers compare both prices. For $\lambda_1 > \lambda_2$ the efficient undercut-proof prices and corresponding profits are

$$\bar{p}_1 = v \quad \text{and} \quad \bar{p}_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S} \quad \implies \quad \pi_1 = v\lambda_1 \quad \text{and} \quad \pi_2 = \frac{v\lambda_1(\lambda_2 + \lambda_S)}{\lambda_1 + \lambda_S}. \tag{35}$$

The profits match those from single-stage pricing models. Notice that (the larger, in terms of awareness) firm 1 cares solely about expanding its captive audience. The incentives of (the smaller) firm 2 are nuanced. For example, firm 2 benefits from an expansion in firm 1's captives.

Advertising. We now build upon the independent awareness model of Section 7, where the awareness α_i of a firm is a consequence of its advertising activities. This maps to the general duopoly model via $\lambda_1 = \alpha_1(1-\alpha_2)$, $\lambda_2 = \alpha_2(1-\alpha_1)$, and finally $\lambda_S = \alpha_1\alpha_2$. For $\alpha_1 > \alpha_2$,

$$\pi_1 = v\alpha_1(1-\alpha_2) \quad \text{and} \quad \pi_2 = v\alpha_2(1-\alpha_2). \tag{36}$$

We see that the two firms have very different incentives. The larger and so more expensive firm sets a price of $\bar{p}_1 = v$ that is not limited by any "no undercutting" constraint. For a given $\alpha_2 < 1$, its profits are linearly increasing in its advertising intensity.

The smaller and cheaper firm, however, sets $\bar{p}_2 = v(1-\alpha_2)$, preventing an undercut by firm 1. The more it advertises, the more attractive such an undercut becomes, and so the lower its price must be to keep firm 1 at bay. This leads to a trade off for firm 2 when choosing how much to advertise, reflected by the non-monotonicity of π_2 in α_2. In particular (and putting aside advertising costs, for now) firm 2 (if it smaller) will always prefer $\alpha_2 \leq \frac{1}{2}$.

We can readily embed the profits of (36) into an advertising-choice game. (More fully, we can imagine a three-stage game in which firms choose awareness parameters, then choose initial prices, and finally choose retail prices.) For example, if advertising is free and awareness is chosen from $\alpha_i \in [0, \bar{\alpha}]$ for some $\bar{\alpha} \in \left(\frac{1}{2}, 1\right)$, then a pure-strategy Nash equilibrium of an advertising game will take the form $\alpha_1 = \bar{\alpha}$ and $\alpha_2 = \frac{1}{2}$. The associated (stable) prices

$$\bar{p}_1 = v \quad \text{and} \quad \bar{p}_2 = \frac{v\lambda_1}{\lambda_1 + \lambda_S} = v(1-\alpha_2) = \frac{v}{2} \tag{37}$$

are dispersed. One firm maximizes its exposure to customers and sets the monopoly price, while the other limits its exposure to half of customers and charges half the monopoly price.

In Appendix B we provide a full treatment with n firms and asymmetric advertising cost functions.[32] One firm charges v and advertises distinctly more than all the others, who each advertise to a minority of customers and set mutually distinct lower prices. When advertising is costless, adding extra competitors adds additional lower prices (while retaining existing price positions) and increases the range of dispersed prices. With costly adverting, a fall in costs increases the awareness of each firm and the dispersed prices of the firms become further apart.

[32] Our costly awareness specification encompasses those used in the single-stage pricing models of Ireland (1993) and McAfee (1994). We generate the same firm profits, and so we obtain closely related predictions.

Search. We now study how customers may influence consideration sets and so prices.

Retaining the duopoly framework, suppose that a potential customer uses fixed-sample search technology à la Burdett and Judd (1983): this customer can seek out (without replacement) either zero, one, or two price quotations. Searching once finds each firm with equal probability. If the quotation is from the high-price firm v then there is no benefit; but if the low-price firm is found (with probability $\frac{1}{2}$) then the customer gains $v - \bar{p}_2$. A second search is guaranteed to find the cheaper firm, but this is beneficial only if the first search did not already do so. As such, the second search also generates a gain of $v - \bar{p}_2$ with probability $\frac{1}{2}$. Summarizing,

$$\text{E[benefit of 1st search]} = \text{E[benefit of 2nd search]} = \frac{v - \bar{p}_2}{2} = \frac{v\lambda_S}{2(\lambda_1 + \lambda_S)}. \tag{38}$$

Now adopt the classic constant-returns search technology so that gathering each quotation costs $\kappa \in \left(0, \frac{v}{2}\right)$. A customer finds it strictly optimal to obtain two quotations if and only if

$$\kappa < \frac{v\lambda_S}{2(\lambda_1 + \lambda_S)} \quad \Leftrightarrow \quad \lambda_S > \frac{2\kappa\lambda_1}{v - 2\kappa}, \tag{39}$$

will not search at all if the opposite strict inequality holds, and will be indifferent between all search strategies if there is an equality. This inequality reveals a strategic complementarity: if many others seek out both price quotations (so that the mass of shoppers λ_S is large) then there is greater price dispersion, and this increases the incentive of a customer to search.

This strategic complementarity suggests there there may be multiple equilibria with endogenous search.[33] To sketch a model of this, let us suppose that a mass $\bar{\lambda}_i$ of customers are exogenously captive to firm i, a mass $\bar{\lambda}_S$ are exogenously shoppers, and mass μ decide whether to search once, twice, or not at all. Writing μ_L and μ_H for the masses of low and high searchers (we choose these subscripts to avoid confusion with firm labels) we have $\mu_L + \mu_H \leq \mu$, and

$$\lambda_i = \bar{\lambda}_i + \mu_L \quad \text{for } i \in \{1, 2\} \text{ and } \quad \lambda_S = \bar{\lambda}_S + \mu_H. \tag{40}$$

If the various parameters here satisfy

$$\bar{\lambda}_S + \mu > \frac{2\kappa\bar{\lambda}_1}{v - 2\kappa} > \bar{\lambda}_S, \tag{41}$$

then there are multiple equilibria. We can construct a "high search" equilibrium (of a fully defined game, with an appropriate solution concept) in which all endogenous searchers seek out two quotations ($\mu_L = \mu$) and become shoppers. In this equilibrium prices are more disperse:

$$\bar{p}_1 = v \quad \text{and} \quad \bar{p}_2 = \frac{v\bar{\lambda}_1}{\bar{\lambda}_1 + \bar{\lambda}_S + \mu} \tag{42}$$

In a second "low search" equilibrium endogenous searchers stay home ($\mu_L = \mu_H = 0$). In this equilibrium the two price points are closer together and total activity is limited to $\bar{\lambda}_1 + \bar{\lambda}_2 + \bar{\lambda}_S$.[34]

[33] We provide and examine a full treatment in our related companion paper (Myatt and Ronayne, 2023d).

[34] There can also be interior equilibria which satisfy $\bar{\lambda}_S + \mu_H = \kappa(2\bar{\lambda}_1 + \mu_L)/(v - 2\kappa)$. Such an equilibrium is unstable in the well-known sense discussed in the context of fixed-sample search models by Fershtman and Fishman (1992): if we shift a small extra mass of customers to search twice (and let firms' prices adjust accordingly) the benefit of a second search increases and so all customers wish to search twice.

10. Concluding Discussion

A price-competition environment in which potential customers consider different subsets of suppliers is of fundamental economic interest and naturally occurs within many settings. This is evidenced by the application of the single-stage model to many research areas and interpretations, including and in addition to some of those cited earlier: price discrimination (e.g., Armstrong and Vickers, 2019; Fabra and Reguant, 2020); product substitutability (e.g., Inderst, 2002); consumer search (e.g., Stahl, 1989); strategic clearing-houses such as comparison websites (e.g., Baye and Morgan, 2001, 2009; Moraga-González and Wildenbeest, 2012; Ronayne, 2021; Shelegia and Wilson, 2021); and boundedly-rational consumers (e.g., Carlin, 2009; Chioveanu and Zhou, 2013; Heidhues, Johnen, and Kőszegi, 2021; Inderst and Obradovits, 2020; Piccione and Spiegler, 2012). Our work shows that predictions of prices that are stable and dispersed, in line with the empirical evidence, can be recovered within this environment.

Key to doing so is our assumption that firms find it difficult to raise a price, but not to lower it. Despite the inherent instability of mixed-strategy equilibria, single-stage models could still be appropriate to explain long price durations if firms face substantial rigidities to both upwards and downwards price revisions. Viewed through that lens, our approach contributes by explaining stable and dispersed prices while requiring rigidities only to upward price movements.

Of course, the classic approach found in the literature (the play of mixed strategies in a single-stage game) can be recovered if initial prices are (exogenously) set at the monopoly level ($\bar{p}_i = v$ for all i, in our notation) and firms go on to play a second-stage pricing game free of constraints. In fact, for each major setting we covered for which we know of a corresponding analysis in the literature, firms' profits match the expected profits earned in the Nash equilibria of the single-stage game.[35] This means that researchers analysing settings with a single-stage model in a subgame (e.g., our study of endogenous advertising expenditures in Section 7) can supplement or replace it with our two-stage-pricing approach. The profit equivalence means there is no disruption to earlier stages of their game, but stronger price predictions can be obtained.

We recognize, however, that the equality of expected profits can raise a concern with our two-stage game. In an equilibrium (with the on-path play of pure strategies) of our game, a firm is indifferent to raising its initial price. For example, in the captive-shopper model of Section 6, the low-price shopper-capturing firm n is happy (indifferent) to deviate from $\bar{p}_n = p^{\dagger}_{n-1}$ to the monopoly price, i.e., $\bar{p}_n = v$. In other words, first-stage initial-price choices are weak, rather than strict, best replies. Nevertheless, there are additional assumptions which make equilibria strict in the sense that an upward deviation in an initial price strictly harms the deviator.

[35] In the captive-shopper setting, the profits from Proposition 5 match those reported by Baye, Kovenock, and de Vries (1992); for full exchangeability, profits from Proposition 2 match those reported by Johnen and Ronayne (2021); for independent awareness, the profits from Proposition 7 match those of Ireland (1993); McAfee (1994). One exception that we document here is the case of the prominence triopoly in Section 8. Firms' profits from the pure-strategy play of our two-stage game strictly exceed, for one firm, the expected profit earned from an equilibrium of a classic single-stage game. We characterize that equilibrium in Appendix B.

A natural case is when a decision-maker is averse to risk. In our equilibrium with pure strategies, a firm's profit is fully determined when the equilibrium path is followed. An upward deviation leads to the same expected profit, but of course makes the eventual realization of profit uncertain (from the perspective of the initial price choice). To see this clearly, consider a world in which the initial pricing decision within a firm is made by a risk-averse manager who seeks to maximize the expected utility of profit, but where the final price is selected by a risk-neutral operational pricing agent who seeks to maximize expected profit.[36] The manager now has a strict incentive not to increase the firm's initial price away from an efficient undercut-proof profile and trigger a mixed-strategy equilibrium in the ensuing subgame.

As a last point of discussion, we offer an alternative route to obtain the stable prices that we predict in this paper. In essence, in the first of our two stages (the choice of initial prices) each firm has the possibility of taking a Stackelberg leadership position. Consider, for example, the captive-and-shopper setting and an appropriately defined game.[37] There, the most aggressive firm n acts as a Stackelberg leader, setting a price just low enough to ensure that no other firm wishes to undercut it in the second stage. The same effect occurs as the equilibrium outcome of a fully specified two-stage Stackelberg game in which all firms are given a full commitment opportunity in the first stage.[38]

Appendix A. Omitted Proofs

Proof of Lemma 1. The claims (i) and (ii) follow from arguments given in the main text. (For completeness, we present fuller proofs in our online supplemental Appendix C.) □

Proof of Lemma 2. It is without loss of generality to focus on profiles comprising strictly positive prices. (Any firm with $\bar{p}_i = 0$ has no choice to make, and no other firm with a positive initial price wishes to match this zero price. This means we simply ignore any zero-initial-price firms in what follows.) For undercut-proof prices $\bar{p}_1 > \cdots > \bar{p}_n > 0$ write $\pi_i = \bar{p}_i \sum_{B \subseteq \{1,\ldots,i\}} B_i \, \lambda(B)$ for the profit of firm i if all firms charge their initial prices, so that $p_j = \bar{p}_j$ for all $j \in \{1, \ldots, n\}$.

By charging $p_1 = \bar{p}_1$, firm 1 achieves the profit π_1, independent of the choices of other firms. For any other price it may charge, $p_1 < \bar{p}_1$, its profit is highest if $p_j = \bar{p}_j$ for $j \neq 1$, so that all others maintain their initial prices. The profile of initial prices is undercut-proof, and firm 1 earns strictly less than π_1 by strictly undercutting any other firm. If it matches another firm, then (given that ties are broken in an interior way) it also earns strictly less. Therefore, all $p_1 < \bar{p}_1$ are strictly dominated for firm 1. We conclude (as an induction basis) that firm 1 must charge $p_1 = \bar{p}_1$ (a pure strategy) in any Nash equilibrium.

For $i > 1$, suppose that $p_j = \bar{p}_j$ for all $j < i$ in any Nash equilibrium. Firm i can guarantee a profit π_i by charging $p_i = \bar{p}_i$. Recycling the argument above, even if others maintain their

[36]We sketch this out in Appendix B and examine it more in a companion paper (Myatt and Ronayne, 2023b).
[37]We describe this scenario in more detail in Appendix B.
[38]We thank Mark Armstrong and Juuso Välimäki for suggesting that we flesh out this connection.

initial prices (so maximizing the profit of firm i) then firm i earns strictly less from $p_i < \bar{p}_i$. We conclude that $p_i = \bar{p}_i$. By the principle of induction, this holds for all $i \in \{1, \ldots, n\}$. □

Proof of Lemma 3. This follows from the argument in the main text. □

Proof of Lemmas 4 and 5. Suppose that the no-undercutting constraint of firm i is slack, so that either $i = 1$ and $\bar{p}_1 < v$ or $i > 1$ and equation (3) holds as a strict inequality. Firm i can strictly raise $\bar{p}_i$ while maintaining undercut-proofness, and enter a subgame with (by Lemma 2) a unique Nash equilibrium which gives firm i a strictly higher expected profit. □

The following lemma is used in proofs, including Proposition 1.

Lemma A1. *Consider a strategy profile in a pricing game in which firms i and j mix (continuously) over an interval $[p_L, p_H]$ not intersecting the support of any other firm, and where i and j are both indifferent (as they are in mixed-strategy Nash equilibrium) across that interval. The expected profit of any other firm from deviating to a price $p \in [p_L, p_H]$ is convex in p.*

Proof. We note that $F_l(p)$ is constant for $p \in [p_L, p_H]$ and $l \notin \{i, j\}$; we write F_l for this constant. Varying the prices of i and j within the interval $[p_L, p_H]$ has no effect on their sales when there is no comparison between them. We write Y_i and Y_j for such sales:

$$Y_i = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_i (1 - B_j) \prod_{l \notin \{i,j\}} (1 - B_l F_l) \tag{A1}$$

$$Y_j = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_j (1 - B_i) \prod_{l \notin \{i,j\}} (1 - B_l F_l) \tag{A2}$$

We also write Z for the sales made by the cheaper of i and j when they are compared:

$$Z = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_i B_j \prod_{l \notin \{i,j\}} (1 - B_l F_l). \tag{A3}$$

With this notation in hand, the firms' expected profits from any price $p \in [p_L, p_H]$ are

$$\pi_i(p) = p \left(Y_i + Z(1 - F_j(p))\right) \quad \text{and} \quad \pi_j(p) = p \left(Y_j + Z(1 - F_i(p))\right). \tag{A4}$$

These profits are constant across this interval and so

$$1 - F_j(p) = \frac{\pi_i - pY_i}{pZ} \quad \text{and} \quad 1 - F_i(p) = \frac{\pi_j - pY_j}{pZ}. \tag{A5}$$

Now consider the profit of some firm $k \notin \{i, j\}$ deviating to a price in this interval. We write Y_k for the sales made when there is no comparison between k and either (or both) of i and j:

$$Y_k = \sum_{B \subseteq \{1,\ldots,n\}} \lambda(B) B_k (1 - B_i)(1 - B_j) \prod_{l \notin \{i,j,k\}} (1 - B_l F_l), \tag{A6}$$

where these expected sales are guaranteed for any price in $[p_L, p_H]$. Other possible sales involve comparisons of k with i, with j, or with both i and j. Possible sales for these three cases are

$$Z_{ik} = \sum_{B\subseteq\{1,\ldots,n\}} \lambda(B)B_kB_i(1-B_j)\prod_{l\notin\{i,j,k\}}(1-B_lF_l), \tag{A7}$$

$$Z_{jk} = \sum_{B\subseteq\{1,\ldots,n\}} \lambda(B)B_k(1-B_i)B_j\prod_{l\notin\{i,j,k\}}(1-B_lF_l), \tag{A8}$$

$$Z_{ijk} = \sum_{B\subseteq\{1,\ldots,n\}} \lambda(B)B_k(1-B_i)(1-B_j)\prod_{l\notin\{i,j,k\}}(1-B_lF_l). \tag{A9}$$

The expected profit of firm k from charging price $p\in[p_L,p_H]$ is

$$\begin{aligned}\pi_k(p) &= p\left[Y_k + Z_{ik}(1-F_i(p)) + Z_{jk}(1-F_j(p)) + Z_{ijk}(1-F_i(p))(1-F_j(p))\right]\\ &= p\left[Y_k + Z_{ik}\frac{\pi_j - pY_j}{pZ} + Z_{jk}\frac{\pi_i - pY_i}{pZ} + Z_{ijk}\frac{\pi_j - pY_j}{pZ}\frac{\pi_i - pY_i}{pZ}\right]\\ &= pY_k + Z_{ik}\frac{\pi_j - pY_j}{Z} + Z_{jk}\frac{\pi_i - pY_i}{Z} + \frac{Z_{ijk}}{Z^2}\left[\frac{\pi_i\pi_j}{p} + Y_iY_jp - (\pi_iY_j + \pi_jY_i)\right],\end{aligned} \tag{A10}$$

which by inspection is convex in p, and strictly so if $Z_{ijk} > 0$. □

Proof of Proposition 1. Fix the candidate price profile. Given (5), this satisfies $\bar{p}_1 = v$ and

$$\bar{p}_i = \bar{p}_{i-1}\frac{\sum_{B\subseteq\{1,\ldots,i-1\}} B_{i-1}\,\lambda(B)}{\sum_{B\subseteq\{1,\ldots,i\}} B_{i-1}\,\lambda(B)} \quad \text{for all } i\in\{2,\ldots,n\}. \tag{A11}$$

Consider a strategy profile of the two-stage local price-adjustment game in which all firms choose $\Delta_i = 0$ in the first stage, and choose $p_i = \bar{p}_i$ on the equilibrium path in the second stage. The prices we have selected are undercut-proof, and so $p_i = \bar{p}_i$ for all i is the unique outcome in the second stage, by Lemma 2. Firm i earns a profit $\bar{p}_i \sum_{B\subseteq\{1,\ldots,i\}} B_i\lambda(B)$.

Consider a first-stage deviation by $i > 1$ to $\Delta_i \in (0,\bar{\Delta}]$ with $\bar{\Delta} > 0$ sufficiently small such that $\bar{\Delta} \le \bar{p}_{i-1} - \bar{p}_i$, and each firm $j < i-1$ that strictly prefers $\bar{p}_j$ to undercutting $\bar{p}_i$ also strictly prefers $\bar{p}_j$ to undercutting $\bar{p}_i + \bar{\Delta}$. In the subgame construct a strategy profile in which $j \notin \{i-1, i\}$ choose $p_j = \bar{p}_j$, while $j\in\{i-1,i\}$ mix over $[\bar{p}_i, \bar{p}_i + \Delta_i)$ with distributions

$$F_j(p) = \frac{(p-\bar{p}_i)\sum_{B\subseteq\{1,\ldots,i\}} B_k\,\lambda(B)}{p\sum_{B\subseteq\{1,\ldots,i\}} B_iB_{i-1}\,\lambda(B)} \quad \text{for } j,k\in\{i-1,i\}, j\neq k, \tag{A12}$$

and then place remaining mass at $\bar{p}_{i-1}$ and $\bar{p}_i + \Delta_i$ respectively. These are valid CDFs which continuously increase from $F_j(\bar{p}_i) = 0$ and satisfy $F_j(\bar{p}_i + \Delta_i) \le 1$ for $j \in \{i-1, i\}$ if Δ_i is sufficiently small. Moreover, prices within this interval give the firms $j\in\{i-1,i\}$ their on-path expected profits. To see why, note that for $j\in\{i-1,i\}$ and $k\in\{i-1,i\}$ for $k\neq j$,

$$\underbrace{(p-\bar{p}_i)\sum_{B\subseteq\{1,\ldots,i\}} B_j\,\lambda(B)}_{\text{gain from lifting price}} = \underbrace{pF_k(p)\sum_{B\subseteq\{1,\ldots,i\}} B_iB_{i-1}\,\lambda(B)}_{\text{lost sales to } k\neq j}. \tag{A13}$$

The left-hand side is the gain to j from charging a price higher than $\bar{p}_i$. (The summation represents sales from being the cheapest of $\{1,\ldots,i\}$.) The right-hand side is then the value of sales lost to the competitor k, which incorporates the probability that k prices below p.

The condition (5) says that $i-1$ is one of the firms $j<i$ that is indifferent to undercutting i. We chose $\bar{\Delta}$ such that any firm that strictly prefers not to undercut $\bar{p}_i$ also strictly prefers not to undercut $\bar{p}_i+\Delta_i$ and so we are assured that such a firm prefers not to join the "tango" between $i-1$ and i. It remains to check that any firm $j<i-1$ (i.e., other than $i-1$) that is indifferent to undercutting $\bar{p}_i$ is unwilling to join the dance (i.e., set some $p\in[\bar{p}_i,\bar{p}_i+\Delta_i)$). By Lemma A1 we only need to check that j does at least as well with $\bar{p}_j$ than both (i) $\bar{p}_i$, and (ii) (undercutting) $\bar{p}_i+\Delta_i$. As for (i), we know j is indifferent between $\bar{p}_j$ and $\bar{p}_i$. For (ii), recall that we chose Δ_i to be sufficiently small such that $F_j(\bar{p}_i+\Delta_i)\le 1$ for $j\in\{i-1,i\}$. In fact, using (A12) we find $F_i^{-1}(1)=\bar{p}_{i-1}$ (i.e., the function F_i reaches 1 at exactly $\bar{p}_{i-1}$). Therefore, j gets a strictly lower profit from undercutting $\bar{p}_{i-1}$ than charging $\bar{p}_j$ (i has no mass at $\bar{p}_{i-1}$, so no matter the mass $i-1$ places there, j would not undercut $\bar{p}_{i-1}$ because the initial price profile is undercut-proof). It follows that undercutting $\bar{p}_i+\Delta_i(\le\bar{p}_{i-1})$ gets j an even lower expected profit than if it were to undercut $\bar{p}_{i-1}$, and so j prefers $\bar{p}_j$.

We have constructed an equilibrium of the subgame in which all firms receive their equilibrium-path profits. This means that the deviation in initial price by firm i was not profitable. To complete our description of a subgame-perfect equilibrium, we need only specify strategy profiles in other subgames that are not reached via a single-firm deviation in the first stage. For those subgames, we pick any equilibrium. (An equilibrium exists in all pricing subgames, because they satisfy the conditions of Theorem 5 of Dasgupta and Maskin (1986, p.14).) $\square$

Proof of Proposition 2. Fix a profile of maximal undercut-proof prices. We find the first $k\in\{1,\ldots,n-1\}$ such that $\lambda_k<\lambda_{k+1}$. We claim that for all $i\in\{2,\ldots,k+1\}$

$$\bar{p}_i=\bar{p}_{i-1}\frac{\lambda_{i-1}+X_{i-1}}{\lambda_{i-1}+X_i}\quad\text{and so}\quad \bar{p}_i\equiv v\prod_{j=2}^{i}\frac{\lambda_{j-1}+X_{j-1}}{\lambda_{j-1}+X_j}. \tag{A14}$$

The first equality says that the local no-undercutting constraint binds at each step; setting $\bar{p}_1=v$ and repeated substitution gives the second equality. To prove this, we note that this must be true for $i=2$ (forming an induction basis) because there is only one no-undercutting constraint that applies, and it must bind because prices are maximal. Now suppose that the claim holds (as an induction hypothesis) for all $j\in\{2,\ldots,i-1\}$. For firm i,

$$\begin{aligned}
\bar{p}_i&=\min_{j<i}\left\{\bar{p}_j\frac{\lambda_j+X_j}{\lambda_j+X_i}\right\}=\min_{j<i}\left\{\bar{p}_j\frac{\lambda_j+X_j}{\lambda_j+X_i}\left(\prod_{k=j+1}^{i}\frac{\lambda_{k-1}+X_k}{\lambda_{k-1}+X_{k-1}}\right)\left(\prod_{k=j+1}^{i}\frac{\lambda_{k-1}+X_{k-1}}{\lambda_{k-1}+X_k}\right)\right\}\\
&=v\prod_{j=2}^{i}\frac{\lambda_{j-1}+X_{j-1}}{\lambda_{j-1}+X_j}\min_{j<i}\left\{\frac{\lambda_j+X_j}{\lambda_j+X_i}\left(\prod_{k=j+1}^{i}\frac{\lambda_{k-1}+X_k}{\lambda_{k-1}+X_{k-1}}\right)\right\}\\
&=v\prod_{j=2}^{i}\frac{\lambda_{j-1}+X_{j-1}}{\lambda_{j-1}+X_j}\left\{1,\min_{j<i-1}\left\{\frac{\lambda_j+X_j}{\lambda_j+X_i}\left(\prod_{k=j+1}^{i}\frac{\lambda_{k-1}+X_k}{\lambda_{k-1}+X_{k-1}}\right)\right\}\right\}\\
&=v\prod_{j=2}^{i}\frac{\lambda_{j-1}+X_{j-1}}{\lambda_{j-1}+X_j}=\bar{p}_{i-1}\frac{\lambda_{i-1}+X_{i-1}}{\lambda_{i-1}+X_i}.
\end{aligned}\tag{A15}$$

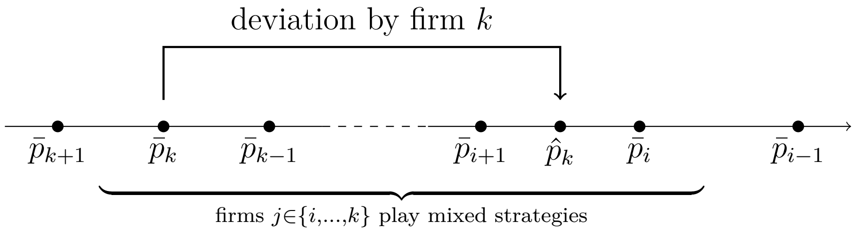

Figure 1. An Upward Deviation in Initial Price

The first four lines use algebraic re-arrangement. The final line holds because for each $j < i-1$,

$$\frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \leq \frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_j + X_k}{\lambda_j + X_{k-1}} = \frac{\lambda_j + X_j}{\lambda_j + X_i} \frac{\lambda_j + X_i}{\lambda_j + X_j} = 1. \tag{A16}$$

The inequality in the chain holds because $X_k \geq X_{k-1}$ in each of the ratio terms, which means that such terms are each decreasing in λ_{k-1}. An upper bound for each term is obtained by replacing λ_{k-1} with $\lambda_j \leq \lambda_{k-1}$, where this inequality holds because $j \leq k-1 \leq i-1$ and (by assumption) firms below i are in size order. The claim holds by the principle of induction.

Now consider firms k and $k+1$; the first out-of-order pair.

$$\frac{\bar{p}_{k+1}}{\bar{p}_k} = \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}} < \frac{\lambda_{k+1} + X_k}{\lambda_{k+1} + X_{k+1}}. \tag{A17}$$

The equality is the (binding) no undercutting constraint. The inequality holds because $X_k < X_{k+1}$ and $\lambda_{k+1} > \lambda_k$. This means that if we switch the positions of k and $k+1$ in the profile of prices (while maintaining the actual prices) then the no-undercutting holds strictly. Firm k is now lower in the price order than before, but with the same profit, and so (as before) does not want to undercut any lower-priced firms. Firms $j < k$ face the same no-undercutting opportunities as before, and so their no-undercutting constraints still hold. Firm $k+1$ is now strictly better off. We have constructed a Pareto-superior undercut-proof profile. □

Proof of Lemma 6. Suppose that firm k (where necessarily $k > 1$) deviates upward to $\hat{p}_k > \bar{p}_k$. There is some $i < k$ such that $\bar{p}_{i+1} < \hat{p}_k \leq \bar{p}_i$. For example, one case is where $i = k-1$, so that firm k deviates upward without crossing the initial price of another firm. Another case is when $i = 1$ and $\hat{p}_k = \bar{p}_1 = v$, which means that k removes any restriction on its final retail price. We build a mixed-strategy equilibrium (illustrated in Figure 1) in which all firms earn their (common) on-path equilibrium expected profits, $v\lambda$. Firms $j \in \{i, \ldots, k\}$ mix (with atoms and gaps) over the interval $[\bar{p}_k, \bar{p}_i]$. Others set their initial prices: $p_j = \bar{p}_j$ for $j \notin \{i, \ldots, k\}$.

Given that firms $\{1, \ldots, k\}$ price (by construction below) at or above $\bar{p}_k$, any firm $l \in \{k+1, \ldots, n\}$ has no profitable deviation downward, and is constrained upward. Firms $j \in \{i, \ldots, k\}$ will (again by construction below) earn their on-path equilibrium profits, and this implies that firms $\{1, \ldots, i-1\}$ cannot profitably deviate to within $[\bar{p}_k, \bar{p}_i]$. This is because an upper bound to the expected profit a lower-indexed firm can achieve by doing so is that from "throwing some

$j \in \{i \ldots, k\}$ off the dance floor" and charging one of the prices j used to. Given the symmetry of firms, this gives the deviator the same expected profit j had before their ejection, $v\lambda$.

We now build the strategies used by the actively mixing firms $\{i, \ldots, k\}$. This group consists of the deviant firm k and all lower-indexed firms up to the firm i with the lowest initial price that weakly exceeds the deviant's new initial price. We consider three cases.

Case (i): a deviation that does not cross another first-stage price.

If $i = k-1$, so that $\hat{p}_k \in (\bar{p}_k, \bar{p}_{k-1}]$, then firms k and $k-1$ mix continuously over the single interval of prices $[\bar{p}_k, \hat{p}_k)$, and then place atoms (these are strictly positive if and only if $\hat{p}_k < \bar{p}_{k-1}$) at their respective initial prices $\hat{p}_k$ and $\bar{p}_{k-1}$. They mix using the same distribution $F(p)$. Taking the indifference condition for firm k (the same condition holds for firm $k-1$), $F(p)$ satisfies

$$\begin{aligned}\lambda v &= p \sum_{B \subseteq \{1,\ldots,k-2\}} [\lambda(B \cup \{k\}) + (1 - F(p))\lambda(B \cup \{k, k-1\})] \\ &= p \sum_{x=0}^{k-2} \binom{k-2}{x} \left[\frac{I_{1+x}}{\binom{n}{1+x}} + \frac{[1 - F(p)]I_{2+x}}{\binom{n}{2+x}} \right], \end{aligned} \tag{A18}$$

where we define $I_1 \equiv n\lambda$.[39] The left-hand side is the expected profit of firm k. The right-hand side is the price p multiplied by the probability that firm k wins any comparisons. Firm k wins from comparisons which group it with any subset of $\{1, \ldots, k-2\}$ (these are the higher priced firms). Additionally, it wins comparisons that also include $k-1$ so long as $k-1$ prices above p, which happens with probability $1 - F(p)$. The second line computes the sizes of the relevant comparison sets. The summation over x ranges over the possible sizes of $B \subseteq \{1, \ldots, k-2\}$, noting that for each x there are $\binom{k-2}{x}$ relevant sets. Bringing in firm k, these comparison sets are of size $1+x$. The total mass of comparison sets of this size is I_{1+x}, and there are $\binom{n}{1+x}$ such sets. Hence $I_{1+x}/\binom{n}{1+x}$ is the size of each comparison set. Similar calculations apply when firm $k-1$ is added, where this time the combined mass of the relevant comparison sets is multiplied by $1 - F(p)$. The solution for $F(p)$ is strictly increasing in p, $F(\bar{p}_k) = 0$, and $F(\bar{p}_{k-1}) = 1$.

Case (ii): a deviation into the upper part of a higher price interval.

A second case is when the deviant initial price of firm k crosses the initial price of at least one other firm, so that $i < k-1$ or equivalently $\hat{p}_k > \bar{p}_{k-1}$, and when that deviant price is sufficiently high in the interval $(\bar{p}_{i+1}, \bar{p}_i]$. Specifically, we consider a case with $\hat{p}_k \in [\tilde{p}_i, \bar{p}_i]$ where $\tilde{p}_i \in (\bar{p}_{i+1}, \bar{p}_i)$ is a threshold to be determined below.

We build an equilibrium mixed-strategy profile in which there is a threshold $\tilde{p}_j \in (\bar{p}_{j+1}, \bar{p}_j)$ for each $j \in \{i, \ldots, k-2\}$ such that the interval $(\bar{p}_{j+1}, \tilde{p}_j)$, which is the lower part of the interval between the initial prices of firms $j+1$ and j, is a gap in the mixing distributions of all firms. (This gap must exist because, for any price in that interval, a firm would prefer to undercut the price $\bar{p}_{j+1}$ in order to capture an atom which will be played by firm $j+1$.) For $j > i$, over the upper part of the interval $[\tilde{p}_j, \bar{p}_j)$ firms in $\{i, \ldots, j\} \cup \{k\}$ will mix continuously. Firm j will

[39] Note that captive customers are included in the first line, with $\lambda(\emptyset \cup \{k\})$.

then place an atom at $\bar{p}_j$. Turning to the top interval between the prices of $i+1$ and i, firms i and k will mix continuously over $[\widetilde{p}_i, \hat{p}_k)$ and then will place remaining mass at their respective initial prices. Across the lowest interval $[\bar{p}_k, \bar{p}_{k-1})$ all firms mix continuously, with firm $k-1$ placing an atom at its initial price $\bar{p}_{k-1}$. For $k=4$ and $i=1$ the basic plan of the equilibrium support of the firms' mixed strategies is illustrated in Figure 2.

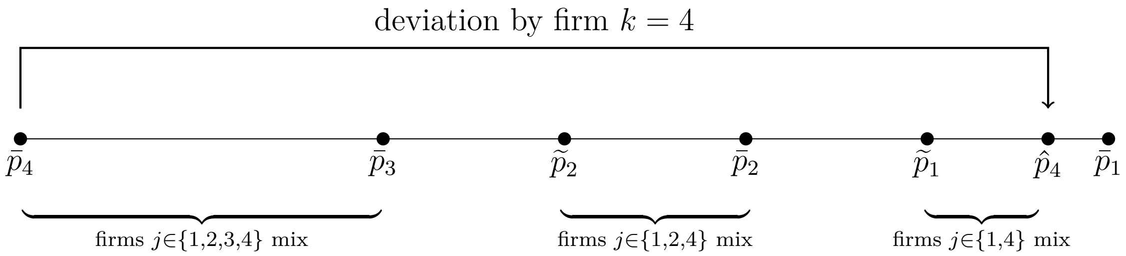

Figure 2. Mixing Supports for an Equilibrium of Type Case (ii)

For each $j \in \{i, \ldots, k-1\}$ (these are firms tempted to undercut following k's deviant first-stage choice), consider the interval of prices $[\bar{p}_{j+1}, \bar{p}_j)$. Firms $\{i, \ldots, j\} \cup \{k\}$ will actively used mixed strategies within this interval, where this is a strict subset for $j < k-1$. Note that there are $j-(i-1)+1$ such firms. Specify the cumulative distribution function, $F_j(p)$, to satisfy

$$\begin{aligned}\lambda v &= p \sum_{B\subseteq\{1,\ldots,i-1\}} \sum_{\widetilde{B}\subseteq\{i,\ldots,j\}} \lambda\left(B \cup \widetilde{B} \cup \{k\}\right) [1-F_j(p)]^{|\widetilde{B}|} \\ &= p \sum_{x=0}^{i-1} \sum_{y=0}^{j-i+1} \binom{i-1}{x} \binom{j-i+1}{y} \frac{I_{1+x+y}}{\binom{n}{1+x+y}} [1-F_j(p)]^y \end{aligned} \tag{A19}$$

The left-hand side is the (common) equilibrium expected profit of each firm. The right-hand side is k's expected profit when, at price p, all firms in $\{i, \ldots, j\}$ mix according to the distribution $F_j(p)$. The first summation collects together subsets of lower-indexed firms who always lose any comparisons with price p. The second summation deals with those who are actively mixing. For any set $\widetilde{B}$ there are $|\widetilde{B}|$ such firms, and so the price p wins comparisons against them all with probability $[1-F_j(p)]^{|\widetilde{B}|}$. The second line follows from the various masses of consideration sets. This is an indifference condition for firm k. However, the same indifference condition also holds for other firms in $\{i, \ldots, j\}$. The solution for $F_j(p)$ satisfies $F_j(\bar{p}_{j+1}) = 0$ and is strictly increasing. Defining $F_j(\bar{p}_j) = \lim_{p\uparrow\bar{p}_j} F_j(p)$, the $k-i$ solutions satisfy

$$F_{k-1}(\bar{p}_{k-1}) < F_{k-2}(\bar{p}_{k-2}) < \cdots < F_i(\bar{p}_i) = 1. \tag{A20}$$

Looking across the whole interval $[\bar{p}_k, \bar{p}_i)$, we might aim to join the $k-1$ functions to form a single distribution. However, such a function would jump downward at each initial price (to zero), and so would not be a valid distribution function. We "smooth out" these jumps as follows. For each $j \in \{i, \ldots, k-2\}$ we define $\widetilde{p}_j \in (\bar{p}_{j+1}, \bar{p}_j)$ to be the unique solution to

$$F_{j+1}(\bar{p}_{j+1}) = F_j(\widetilde{p}_j). \tag{A21}$$

We now stitch together a full cumulative distribution as follows. First, we define $F(p) = F_{k-1}(p)$ for $p \in [\bar{p}_k, \bar{p}_{k-1}]$. For all other $j \in \{i, \ldots, k-2\}$ we define

$$F(p) = \begin{cases} F_{j+1}(\bar{p}_{j+1}) & p \in (\bar{p}_{j+1}, \widetilde{p}_j] \\ F_j(p) & p \in (\widetilde{p}_j, \bar{p}_j] \end{cases} \tag{A22}$$

By construction, this is a continuously increasing cumulative distribution function, which satisfies $F(\bar{p}_k) = 0$ and increases to $F(\bar{p}_i) = 1$. It is constant for each interval $[\bar{p}_{j+1}, \widetilde{p}_j]$ for each $j \in \{i, \ldots, k-2\}$, but otherwise is strictly increasing.

We are finally ready to build our strategy profile for the firms. Firm k (the deviant firm) mixes according to $F(p)$ across $p \in [\bar{p}_k, \hat{p}_k)$ and places any remaining mass (if $\hat{p}_k < \bar{p}_i$) at its first-stage price, and so plays an atom of size $1 - F(\hat{p}_k)$ at $\hat{p}_k$. Firm i also mixes according to $F(p)$ for $p \in [\bar{p}_k, \hat{p}_k)$ and then places its remaining mass $1 - F(\hat{p}_k)$ at $\bar{p}_i$. (This means that firms i and k behave symmetrically save for the location of their atoms.) A firm $j \in \{i+1, \ldots, k-1\}$ mixes according to $F(p)$ across $p \in [\bar{p}_k, \bar{p}_j)$ and then places its remaining mass $1 - F(\bar{p}_j)$ at $\bar{p}_j$. This construction yields a mixed-strategy Nash equilibrium profile so long as the deviant initial price satisfies $\hat{p}_k \geq \widetilde{p}_i$.

We note that the constructed distribution function $F(p)$ is used by all firms below their respective initial prices. At any point in the support of a firm's strategy (so that $F(p)$ is strictly increasing) the function is constructed so that each mixing firm earns the on-path equilibrium expected profit, $v\lambda$. Any price within a gap (where $F(p)$ is constant) generates an expected profit strictly below $v\lambda$. (At such prices a firm performs strictly better by undercutting the next initial price below and so capturing the atom of another firm.)

The strategy profile constructed above requires firm k to place an atom at its deviant initial price $\hat{p}_k$. If $\hat{p}_k \in (\bar{p}_{i+1}, \widetilde{p}_i)$, however, the deviant initial price lies strictly within an interval across which $F(p)$ is constant and so generates an expected profit strictly below $v\lambda$. We adapt our construction to cover that case next.

Case (iii): a deviation into the lower part of a higher price interval.

We now consider $\hat{p}_k \in (\bar{p}_{i+1}, \widetilde{p}_i)$. What we do here is to construct an equilibrium in which firms follow the previous strategy profile up to some critical price $p^\star$, at which point firm i ceases to participate (in essence, this firm "leaves the dance floor") and places remaining mass at its initial price. Specifically, we define $p^\star$ to be the lowest price which satisfies

$$F(p^\star) = F_i(\hat{p}_k). \tag{A23}$$

Necessarily this critical price satisfies $p^\star < \bar{p}_{i+1}$. We retain our definition of $F(p)$ for $p \leq p^\star$.

We now change firm i's strategy so that it mixes according to $F(p)$ for $p \in [\bar{p}_k, p^\star]$ but then places remaining mass at its initial price, so that it has an atom at $\bar{p}_i$ of size $1 - F(p^\star) = 1 - F_i(\hat{p}_k)$.

This construction means that firm k earns its on-path equilibrium expected profit, $v\lambda$, from playing the price $\hat{p}_k$. For $p > p^\star$ firm i no longer actively mixes, and so we modify the behavior of other firms to maintain appropriate indifferences for each $j \in \{i+1, k-1\}$, and prices in the interval $[\bar{p}_{j+1}, \bar{p}_j)$ that are at or above $p^\star$ we specify $F_j^\star(p)$ to satisfy

$$\begin{aligned}\lambda v &= p \sum_{B \subseteq \{1,\ldots,i-1\}} \sum_{\widetilde{B} \subseteq \{i+1,\ldots,j\}} \left[1 - F_j^\star(p)\right]^{|\widetilde{B}|} \left[\lambda\left(B \cup \widetilde{B} \cup \{k\}\right) + (1 - F(p^\star))\lambda\left(B \cup \widetilde{B} \cup \{i,k\}\right)\right] \\ &= p \sum_{x=0}^{i-1} \sum_{y=0}^{j-i} \binom{i-1}{x}\binom{j-i}{y}[1 - F_j^\star(p)]^y \left[\frac{I_{1+x+y}}{\binom{n}{1+x+y}} + \frac{I_{2+x+y}[1 - F(p^\star)]}{\binom{n}{2+x+y}}\right]. \end{aligned} \tag{A24}$$

This is an indifference condition for k, but also applies to other relevant firms. It adjusts (A19) to treat i separately, as i prices above p with (constant) probability $1 - F(p^\star)$ for $p \in (p^\star, \bar{p}_i)$. The solution satisfies $F_j^\star(p) > F_j(p)$ for $p > p^\star$ ($F_j^\star(p) = F_j(p)$ for $p = p^\star$). To proceed, we replace $F_j(p)$ with $F_j^\star(p)$ for $p > p^\star$. We then redefine $F(p)$ and the thresholds $\widetilde{p}_j$ appropriately. This modification ensures k is indifferent between $\hat{p}_k$ and slightly undercutting $\bar{p}_{i+1}$. □

Proof of Proposition 3. There are no profitable downward deviations on the equilibrium path. For subgames following any single-firm upward deviation, we apply Lemma 6. □

Proof of Proposition 4. This follows from the argument given in the main text. □

Proof of corollary to Proposition 2. This follows from the argument given in the main text. □

Proof of Proposition 5. This follows from the argument given in the main text. □

Proof of Lemma 7. This result is covered by Proposition 5 of Myatt and Ronayne (2023b). In the current paper, marginal costs are symmetric and captive shares are strictly asymmetric, which implies the lowest dominated price of each firm, as defined in (25), is distinct, i.e., $p_i^\dagger \neq p_j^\dagger$ for $i \neq j$. As shown in Myatt and Ronayne (2023b), this removes the instances that can give multiple equilibria, and leaves us with a unique Nash equilibrium. □

Proof of Proposition 6. Because the prices in Proposition 5 are undercut-proof, we established $p_i = \bar{p}_i$ at the second stage, and no firm wishes to deviate down in the first stage. It remains to check upward deviations by firm n in the first stage. From Lemma 7, there is an equilibrium of the subgame following such a deviation that gives each firm (including n) its on-path expected profit. This means that firm n has no incentive to deviate at the first stage. (As before, for any other subgames—these are reached only if at least two firms deviate at the first stage—we can specify any equilibrium.) We have constructed a subgame-perfect equilibrium as claimed.

Consider any *other* profile of maximal undercut-proof prices, i.e., one in which some firm $i \in \{1, \ldots, n-1\}$ chooses $\bar{p}_i = p_n^\dagger$ while each $j \neq i$ chooses $\bar{p}_j = v$. Suppose that firm i deviates upward at the first stage to an initial price $p_n^\dagger + \Delta_i < p_{n-1}^\dagger$. Over the interval $[p_n^\dagger, p_n^\dagger + \Delta_i)$, let firms i and n mix according to (15), and place remaining mass at their respective initial prices. The distributions reported are continuously increasing from $F_i(p_n^\dagger) = F_n(p_n^\dagger) = 0$, and satisfy $F_i(p) < F_n(p) \leq 1$ if p is not too large (guaranteed by choosing Δ_i sufficiently small). By construction, firms i and n earn their on-path expected profits, and no other firm has an incentive to deviate (the prices across which i and n mix are dominated for other firms). $\square$

Proof of Lemma 8. We know that $\bar{p}_1 = v$, from Lemma 3. The text following the lemma confirms $\bar{p}_i \leq \bar{p}_{i-1}(1-\alpha_i)$ must hold for every $i \in \{2, \dots, n\}$. Choosing maximal prices so that these constraints all bind generates the solutions stated in the lemma. More generally, there is a no-undercutting constraint $\bar{p}_j \leq \prod_{i \geq k > j}(1-\alpha_k)$ for every $i < j$, as derived in the text, and these constraints are all satisfied by the stated solutions for maximal undercut-proof prices. $\square$

Proof of Proposition 7. This follows from the argument given in the main text. $\square$

Proof of Lemma 9. The proof follows a similar structure to the proof of Lemma 6. Due also to its length, it is relegated to our online supplemental Appendix C. $\square$

Proof of Proposition 8. This follows from the argument given in the main text. $\square$

The proofs of Propositions 9 to 11 are contained in our online supplemental Appendix C.

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Supplemental Appendices for Online Reference

Appendix B. Extensions

Here we sketch several extensions to elaborate upon tangential points from the main text. Some of these draw upon our related work: a consideration or product prominence (Myatt and Ronayne, 2023c), endogenous advertising (Myatt and Ronayne, 2023a), two-stage pricing with risk aversion, and a sequential-move captive-and-shopper game (Myatt and Ronayne, 2023b).

Prominence. In Section 8 we described a triopoly in which one firm is prominently considered. One efficient profile of undercut-proof prices is supported by the equilibrium play of pure strategies (Proposition 11). Profits for the firms (which we order so that $\phi_2 \geq \phi_3$) are

$$\pi_1 = v\phi_1, \quad \pi_2 = \bar{p}_2\phi_2 = \frac{v\phi_1\phi_2}{\phi_1+\phi_2+\phi_3}, \quad \text{and} \quad \pi_3 = \bar{p}_3\phi_3 = \frac{v\phi_1\phi_3}{\phi_1+\phi_3}. \tag{B1}$$

Here we describe a Nash equilibrium from the play of the standard single-stage pricing game. (This is an equilibrium in a subgame of our two-stage game following $\bar{p}_1 = \bar{p}_2 = \bar{p}_3 = v$ at $t = 1$.) In this equilibrium, firm 2 mixes over the interval $[\bar{p}_2, \bar{p}_3]$ according to the distribution

$$F_2(p) = \frac{\phi_1+\phi_2+\phi_3}{\phi_2} - \frac{v\phi_1}{\phi_2 p}, \tag{B2}$$

where $\bar{p}_2 = v\phi_1/(\phi_1+\phi_2+\phi_3)$ and $\bar{p}_3 = v\phi_1/(\phi_1+\phi_3)$ are the equilibrium-supported initial prices from Proposition 11. Firm 3 then mixes over the interval $[\bar{p}_3, v]$ according to

$$F_3(p) = \frac{\phi_1+\phi_3}{\phi_3} - \frac{v\phi_1}{\phi_3 p}. \tag{B3}$$

Finally, the prominent firm 1 mixes over the entire interval $[\bar{p}_2, v)$ with the distribution

$$F_1(p) = 1 - \frac{v\phi_1}{p(\phi_1+\phi_2+\phi_3)}, \tag{B4}$$

with remaining mass as an atom of size $\phi_1/(\phi_1+\phi_2+\phi_3)$ at v. It is straightforward to confirm that all firms are indifferent across all $p_i \in [\bar{p}_2, v)$. In this equilibrium firms 1 and 2 earn the expected profits reported above in (B1). However, the expected profit of firm 3 is

$$\tilde{\pi}_3 = \frac{v\phi_1\phi_3}{\phi_1+\phi_2+\phi_3} < \frac{v\phi_1\phi_3}{\phi_1+\phi_3} = \pi_3 \tag{B5}$$

and so firm 3 is strictly worse off than it would be on the equilibrium path of our two-stage game. As noted in footnote 35, this is an example of a setting in which our profit predictions do not coincide with those from a Nash equilibrium of the corresponding single-stage game.

We note that the single-stage game studied here is one studied by Inderst (2002, Section 3). We obtain an equivalence by setting $\phi_2 = \phi_3$ (so that the non-prominent firms are symmetric) and $\delta = 0$ in his paper (which eliminates any captives for non-prominent firms). Lemma 3 of

Inderst (2002) suggests that the non-prominent firms must mix over the same support, whereas we have an equilibrium in which their supports are non-overlapping.[40]

This setting can further illustrate how our pricing framework can be a component of a deeper model. Inspired by papers in which suppliers pay for prominence (Armstrong and Zhou, 2011; Chen and He, 2011), we introduce a prominence provider that sells that position to firms.

Suppose that all three firms begin with exclusive local customer bases, so that firm $i \in \{1, 2, 3\}$ would charge v to ϕ_i customers within its locality. A monopolist prominence provider, M, offers, in a preliminary (pre-pricing) stage, to bring one firm to national prominence. For example, a provider may be a department store that chooses a product to display in the window, or a website that shows a product on its home page or highlights it at the top of search results.

Specifically, M makes a take it or leave it offer to one firm, and commits to make a specified competitor prominent if the offer is refused. We assume firms have differently sized bases and label them so that $\phi_1 > \phi_2 > \phi_3$. Following the allocation of prominence, we assume that firms set prices that are supported by the equilibrium play of pure strategies (in which the larger non-prominent firm is cheapest, as per Proposition 11).

Because firms' profits are increasing in the size of the prominent firm's base, and the largest firm is the cheapest when it is not prominent, M maximizes its fee (which is accepted in equilibrium) by offering prominence to the firm with the largest base, firm 1, while threatening to make their rival with the smallest base, firm 3, prominent if it refuses.

In essence, a small non-prominent firm has a threatening lean and hungry look, which strengthens the ability of M to extract a fee from a large firm. As such, in equilibrium, M bestows prominence upon firm 1. The prominence provider profits by exploiting the asymmetries between the largest and smallest firm. It then compounds this asymmetry by making firm 1 prominent. This is to the detriment of customers, for whom firm 1 is the worst choice.

Advertising. In Section 9 we sketched an extension to the independent awareness specification in which endogenous adverting decisions by firms influence consideration sets. Here we flesh out that extension, noting that details are reported elsewhere (Myatt and Ronayne, 2023a).

Specifically, we now think of firms that play the following three-stage perfect-information game:

($t = 1$) firms simultaneously choose their awareness parameters $\alpha_i \in [0, 1]$; and then
($t = 2$) firms simultaneously choose their initial price positions $\bar{p}_i \in [0, v]$; and last
($t = 3$) firms simultaneously choose their final retail prices $p_i \in [0, \bar{p}_i]$.

A firm's payoff is its operating profit minus the cost of advertising, where that advertising determines the awareness of the firm. Firm i's advertising cost $C_i(\alpha_i)$ is smoothly increasing,

[40] It is possible that the source of the difference in predictions might lie within derivation of the second displayed equation of the proof of Lemma 3 in the appendix of Inderst (2002).

convex, $C_i(0) = 0$, and $C_i'(0) < v$. When firms are asymmetric we index them so that $C_1'(\alpha) < \cdots < C_n'(\alpha)$ for all $\alpha \in (0, 1]$. This differs from McAfee (1994) by allowing for asymmetric firms, while in Ireland (1993) firms face no costs of advertising.[41]

We seek subgame-perfect equilibria with the play of pure strategies (for advertising choices, initial prices, and final retail prices) along the equilibrium path, and we also look for the play of pure strategies following any first-stage deviations in advertising choices.

Following any first-stage advertising choices, we know that any subgame-perfect equilibrium involves the on-path play of pure strategies (in prices) only if the associated prices are undercut-proof and Pareto efficient (Proposition 8). The profits of firms (before the deduction of advertising costs) are uniquely defined in such a case. This means that we can simply refer to an equilibrium of the advertising game (with pure on-path strategies) with payoffs $\pi_i - C_i(\alpha_i)$.

Definition (Equilibrium with Endogenous Advertising). *A profile of advertising strategies is* supported by the equilibrium play of pure strategies *if there is a subgame-perfect equilibrium in which pure strategies are played, both on the equilibrium path and on any path beginning within any second-stage subgame. Such a profile is a pure strategy Nash equilibrium of a simultaneous-move advertising game in which firms' payoffs are $\pi_i - C_i(\alpha_i)$ where π_i is the profit of firm i from any Pareto efficient profile of undercut-proof prices.*

Given that firms are not yet ordered by their (now endogenous) choice of advertising exposure, we can write these expected sales revenues as

$$\pi_i = \begin{cases} v\alpha_i \prod_{j \neq i}(1 - \alpha_j) & \alpha_i > \max_{j \neq i}\{\alpha_j\} \text{ and} \\ v\alpha_i(1 - \alpha_i) \prod_{j \notin \{i,k\}}(1 - \alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j \neq i}\{\alpha_j\}, \end{cases} \tag{B6}$$

and where both expressions apply when firm i ties to be the largest firm.

A firm's sales revenue reacts differently to its advertising reach depending on whether that firm is the largest. The largest firm sets the highest (monopoly) price and so does not worry about another firm undercutting them. Therefore for the largest firm, an increase in α_i increases its expected revenue linearly. In contrast, smaller firms' prices must be set to deter undercutting by larger firms. For them, there are two competing effects: fixing second-period prices, an increase in α_i scales up sales; however, it also forces its second-period price down (and that of any smaller firms because of the recursive nature of prices). In fact,

$$\frac{\partial \pi_i}{\partial \alpha_i} = \begin{cases} v \prod_{j \neq i}(1 - \alpha_j) & \alpha_i > \max_{j \neq i}\{\alpha_j\} \text{ and} \\ v(1 - 2\alpha_i) \prod_{j \notin \{i,k\}}(1 - \alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j \neq i}\{\alpha_j\}. \end{cases} \tag{B7}$$

For a smaller firm, revenue is decreasing in advertising exposure when a firm reaches a majority of customers, that is, when $\alpha_i > 1/2$. If not, then this revenue kinks upward as α_i passes through

[41] McAfee (1994) also related his paper to that of Robert and Stahl (1993), who specified the simultaneous (rather than sequential) choice of advertising exposure and price.

the maximum advertising exposure of competing firms. Specifically,

$$\frac{\lim_{\alpha_i \downarrow \max_{j\neq i} \alpha_j} \partial\pi_i/\partial\alpha_i}{\lim_{\alpha_i \uparrow \max_{j\neq i} \alpha_j} \partial\pi_i/\partial\alpha_i} = \frac{1-\max_{j\neq i}\alpha_j}{1-2\max_{j\neq i}\alpha_j} > 1, \tag{B8}$$

where the inequality is strict because (once dominated strategies have been eliminated) every firm chooses positive exposure. This implies that no firm chooses its advertising reach to be exactly equal to the maximum of others, and so there is a unique largest firm.

For smaller firms, advertising increases sales revenue only if $\alpha_i \leq 1/2$. This implies firms other than the largest restrict awareness to a minority of potential customers (no matter the cost).

The proofs of Lemma B1, and Propositions B1 and B2 can be found in Appendix C.

Lemma B1 (Properties of Advertising Choices). *In any profile of advertising choices supported by the equilibrium play of pure strategies there is a unique largest firm, and all other firms advertise to a minority of customers.*

On the revenue side, the largest firm always faces an incentive to increase its exposure. Labeling this firm as k, it is straightforward to confirm that, in equilibrium, $\partial\pi_k/\partial\alpha_k \geq 1/2^{n-1}$. Hence, if $C'(1) < 1/2^{n-1}$ then firm k chooses $\alpha_k = 1$ and advertises to everyone.

An advertising equilibrium is characterized by the specification of a leading (and largest) firm k, and n advertising choices which satisfy the n first-order conditions

$$\frac{C_k'(\alpha_k)}{v} = \prod_{j\neq k}(1-\alpha_j) \quad \text{and} \quad \frac{C_i'(\alpha_i)}{v} = (1-2\alpha_i)\prod_{j\notin\{i,k\}}(1-\alpha_j)\ \forall i \neq k. \tag{B9}$$

Because payoffs can be written to rely on a product of all firms' advertising choices, we can (and do, in the proof of Proposition B1) treat this as an aggregative game and solve accordingly (see, e.g., Anderson, Erkal, and Piccinin, 2020; Nocke and Schutz, 2018).

To fully characterize an equilibrium we also need to check for any non-local deviations. For example, one of the smaller firms $i \neq k$ has the option to deviate and choose $\alpha_i > \alpha_k$, and become the largest firm. The proof of Proposition B1 checks such remaining details.

Proposition B1 (Pure Strategies on Path: Endogenous Advertising). *There is at least one profile of advertising choices supported by the equilibrium play of pure strategies.*

In any such equilibrium, one firm chooses a strictly higher advertising level than all the others, sets a price equal to the monopoly price, and only sells to customers who are uniquely aware of its product. Other firms advertise to at most half of potential customers and set lower prices.

In equilibrium, one leading firm advertises distinctly more than others. Proposition B1 does not identify this firm. If the advertising cost functions are not too different then any firm can play this role.[42] If they are different then the leading firm is one with relatively low advertising

[42]This is true for the specifications of Ireland (1993) and McAfee (1994), under which costs are symmetric.

costs.[43] The other minority-audience firms can, however, be ordered given the structure of the advertising cost functions. For example, if $k = 1$ then advertising choices satisfy $\alpha_1 > \cdots > \alpha_n$.

If firms are symmetric ($C_i(\alpha_i) = C(\alpha_i)$ for all i) then the first-order conditions simplify appreciably. Writing α for the common advertising choice of the smaller firms,[44]

$$\frac{C'(\alpha_k)}{v} = (1-\alpha)^{n-1} \quad \text{and} \quad \frac{C'(\alpha)}{v} = (1-2\alpha)(1-\alpha)^{n-2}. \tag{B10}$$

A special case is when advertising is free (Ireland, 1993), where there is a pathological equilibrium in which multiple firms choose $\alpha_i = 1$ and profits are subsequently driven to zero. Putting this aside (or by allowing costs to be close to free) the "free advertising" case yields $\alpha = 1/2$ for $n-1$ firms, and complete coverage, $\alpha_k = 1$, for one firm.

Another case of interest is the cost specification derived from the random mailbox postings technology suggested by Butters (1977).[45] Equivalently, this is what McAfee (1994) called constant returns to scale in the availability of a firm's price.[46] This is obtained by setting $C(\alpha) = \gamma \log[1/(1-\alpha)]$, so that the marginal cost of increased advertising satisfies $C'(\alpha) = \gamma/(1-\alpha)$. Setting $\gamma = 1$ without loss of generality (this cost coefficient only matters relative to the valuation v of customers for the product) and requiring $v > 1$ (otherwise all firms choose zero advertising) the relevant first-order conditions become

$$\frac{1}{v(1-\alpha_k)} = (1-\alpha)^{n-1} \quad \text{and} \quad \frac{1}{v} = (1-2\alpha)(1-\alpha)^{n-1}. \tag{B11}$$

These equations solve recursively. Substituting the second into the first, we find that $\alpha_k = 2\alpha$: no matter what the level of cost, the large firm reaches twice as many customers as each smaller firm. The solution for α satisfies the natural comparative-static property that α is increasing in the product valuation v, and so is decreasing in the advertising cost parameter γ.

Proposition B2 (Equilibrium with Symmetric Advertising Costs). *If advertising is free, as it is under the specification of Ireland (1993), then, in an equilibrium in which firms earn positive profits, the largest firm chooses maximum advertising exposure, while others advertise to half of potential customers. The largest firm earns twice the profit of each smaller firm.*

If the cost of advertising reach is determined by a random mailbox postings technology, as it is under the constant returns case of McAfee (1994), so that $C(\alpha) = -\gamma \log(1-\alpha)$, then the largest firm chooses advertising awareness equal to double that of the competing small firms. Advertising is increasing in customers' willingness to pay.

In both cases, with firms' labels chosen appropriately, prices satisfy $\bar{p}_i = v/2^{i-1}$.

[43] Formally: there is some $k^\star$ such that there is an equilibrium in which any $k \in \{1, \ldots, k^\star\}$ leads the industry.

[44] The expressions in (B10) are precisely the equilibrium conditions stated by McAfee (1994).

[45] Suppose that customers are divided into $1/\Delta$ segments each of size Δ. Each segment corresponds to a mailbox. An advertisement costs $\gamma_i \Delta$ for firm i, and randomly hits one of the segments. Hence, with a total spend of $C_i(\alpha_i)$, a firm is able to distribute $C_i(\alpha_i)/(\gamma_i \Delta)$ advertisements. It follows that $\alpha_i = 1 - (1-\Delta)^{C_i(\alpha_i)/(\gamma_i \Delta)}$. Taking the limit as $\Delta \to 0$, we observe that $(1-\Delta)^{C_i(\alpha_i)/(\gamma_i \Delta)} \to \exp(-C(\alpha_i)/\gamma_i)$. Solving suggests a cost specification $C_i(\alpha_i) = \gamma_i \log[1/(1-\alpha_i)]$ where (for asymmetric firms) we assume that $\gamma_n > \cdots > \gamma_1 > 0$.

[46] Under "constant returns" two merging firms do not save advertising costs. The probability that a customer considers firm i or j is $1-(1-\alpha_i)(1-\alpha_j)$. There are constant returns if $C(\alpha_i)+C(\alpha_j) = C(1-(1-\alpha_i)(1-\alpha_j))$.

The "independent awareness" advertising technology and its endogenous selection are not new to this paper: Ireland (1993) and McAfee (1994) both report that the leading firm is twice the size (in terms of advertising reach) and earns twice the profit of other firms. Other authors have, more recently, studied versions of the single-stage model but with a pre-pricing stage in which firms determine their captive shares and have also found asymmetric equilibrium advertising outlays (Chioveanu, 2008; Ronayne and Taylor, 2020). In contrast to those papers, our result maintains the prediction of asymmetric advertising intensities while allowing for the on-path play of pure strategies. We identify (as the final claim of Proposition B2) an interesting pricing sequence: the margin of each firm in the pricing ladder is half that of the firm above.

Costly Search. In Section 9 we also sketched a model of customer search. In a related paper (Myatt and Ronayne, 2023d) we consider more fully that search model. We build upon the fixed-sample search technology of Burdett and Judd (1983) and Janssen and Moraga-González (2004): customers choose how many (costly) price quotations to request and then select the best available price. Firms set prices using our two-stage approach in which initial prices are set non-cooperatively, and then firms have a second-stage opportunity to discount those prices. Again we predict that firms choose entirely distinct prices. Search behavior and the comparative-static properties differ from those of Janssen and Moraga-González (2004) including the number of quotations customers obtain and the relationship between expected price and how many firms are in the industry.

Two-Stage Pricing with Risk Aversion. In our concluding remarks we observe that in our two-stage pricing game firms are typically indifferent to raising their initial prices. If a firm deviates by doing so, then (for each setting—see the proofs of Lemmas 6, 7 and 9 and Propositions 9 and 11) we constructed a mixed-strategy equilibrium for the ensuing subgame which generates the same expected profit for the deviating firm. This means that there is only a weak incentive for each firm to maintain its initial price.

This is all underpinned by the assumption (otherwise maintained throughout) that firms are risk neutral. Suppose instead that we split each firm into two players: a manager, and an operational pricing agent. We define a game (of perfect information) with $2n$ players in which

($t = 1$) the firms' managers simultaneously choose initial price positions $\bar{p}_i \in [0, v]$; and then
($t = 2$) the firms' agents simultaneously choose their firms' retail prices $p_i \in [0, \bar{p}_i]$.

Agents' payoffs are simply profits, and so they are assumed to be risk neutral and maximize expected profit. The manager of firm i, however, has payoff $u_i(\pi_i)$, which is a smoothly increasing and concave function of the firm's profit. (The more general and key assumption here is that the manager is more risk averse than the pricing agent.) Equilibrium play in any subgame is unaffected by the move to this "$2n$ player" environment. If firms' managers choose the initial prices $\bar{p}_i$ described in our results, then they obtain payoffs $u_i(\pi_i)$ where π_i is a the corresponding profit of firm i under the relevant price profile. Any upward deviation leads to a subgame with the same expected profit, but a lower expected utility. This means that manager i's choice of $\bar{p}_i$ is the unique best reply to the initial prices, $\bar{p}_j$, of managers $j \neq i$.

Moreover, in the captive-shopper setting (Section 6) we find (see Myatt and Ronayne, 2023b, Proposition 8) conditions under which the prices reported in Proposition 6 are the unique subgame-perfect equilibrium of this two-stage manager-agent game.

A Stackelberg Version of the Captive-and-Shopper Game. Also in our concluding remarks, we mentioned a Stackelberg interpretation. Keeping (for simplicity of discussion) to the captive-shopper setting of Section 6 with firms ordered by their masses of captive customers, $\lambda_1 > \cdots > \lambda_n > 0$, suppose that a choice of initial price in the first stage is a commitment to a final retail price (a firm that does so becomes, endogenously, a Stackelberg leader) that can be neither raised nor lowered, but that every firm has the option to remain unconstrained, i.e., such a commitment is optional. In the second stage, all unconstrained firms proceed (as endogenous Stackelberg followers) to select their final retail prices. Just as before, we look for an equilibrium in which pure strategies are chosen along the equilibrium path.[47]

There is a subgame-perfect equilibrium in which firm n commits (as the unique Stackelberg leader) to $\bar{p}_n = p^\dagger_{n-1}$ in the first stage, while other (follower) firms remain unconstrained. In the second stage, firms $i < n$ charge $p_i = v$ and sell to captives, while firm n serves the shoppers.

It is easy to see that no firm $i < n$ has a profitable first-stage or second-stage deviation (to capture shoppers requires a dominated price) and that firm n loses strictly with a lower first-stage price choice (this firm already serves all shoppers with $p^\dagger_{n-1}$). If firm n deviates to a higher price in the first stage, then it loses the shoppers (some of the time) to $n-1$ in the second stage. If firm n deviates to remain unconstrained in the first stage then we revert to a subgame in which Lemma 7 applies, and so firm n (once again) does not gain from this deviation.

This example motivates a richer exercise in which firms can choose to commit to (e.g., advertise) a price at any point of a T-stage game, where firms face a flow of customers, who arrive each period. In that setting we show (see Myatt and Ronayne, 2023b, Proposition 10) that so long as T is not too small, then in any subgame-perfect equilibrium firm n commits to the distinctly low price $p^\dagger_{n-1}$ in the first period and sells to the shoppers, while all other firms $i < n$ charge the monopoly price and sell only to their captives.

[47]For technical reasons, we retain a free choice of how to break ties when two firms charge the same price.

Appendix C. Other Omitted Profs

Proof of Lemma 1. We first prove claim (i). If strictly positive prices tie, then a firm in that tie recognizes there is a positive mass of customers who compare them to another firm in that tie (and no other). Such a firm strictly improves by undercutting. Thus, any strictly positive prices within a profile are distinct. A special case is claim (i), when all prices are strictly positive.

Turning to claim (ii), if prices are undercut-proof, then no higher priced firm $j < i$ wishes to undercut a cheaper competitor i. If $\bar{p}_i = 0$ then this is trivially true. If $\bar{p}_i > 0$, then the strictly positive prices $\bar{p}_i$ and $\bar{p}_j$ are distinct and so firm j earns $\bar{p}_j$ from any comparisons that exclude any higher indexed (strictly cheaper) firms. These are all the comparison sets $B \subseteq \{1, \ldots, j\}$, each of which has mass $\lambda(B)$, that include firm j, which is incorporated by the use of the indicator $B_j \in \{0, 1\}$. Hence $\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)$ is the profit of j. The same logic says that j can achieve (arbitrarily close to) a profit $\bar{p}_i \sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)$ by undercutting $\bar{p}_i$ and so winning any comparisons amongst the first i firms. Thus, the no-undercutting constraint is

$$\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B) \geq \bar{p}_i \sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B) \quad \Leftrightarrow \quad \bar{p}_i \leq \frac{\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)}. \tag{C1}$$

This must hold for all $j < i$, giving condition (3) in the lemma. Now suppose that we have a price profile that satisfies (3). The inequality (C1) holds for any pair $j < i$. This inequality is the correct no-undercutting constraint so long as the positive prices involved are distinct. However, the inequalities holding imply that the prices are distinct. To see this, note that

$$\bar{p}_i \leq \frac{\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\sum_{B \subseteq \{1,\ldots,i\}} B_j \, \lambda(B)} \leq \frac{\bar{p}_j \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)}{\lambda(\{i, j\}) + \sum_{B \subseteq \{1,\ldots,j\}} B_j \, \lambda(B)} < \bar{p}_j, \tag{C2}$$

where the final strict inequality follows from our maintained "twoness" assumption. □

Proof of Lemma 9. The proof follows a similar structure to the proof of Lemma 6.

If firm k (where necessarily $k > 1$) deviates upward to $\hat{p}_k > \bar{p}_k$, then there is some $i < k$ such that $\hat{p}_k \in (\bar{p}_{i+1}, \bar{p}_i]$. Just as in the proof of Lemma 6, we build a mixed-strategy equilibrium (illustrated in Figure 1) in which all firms $j \in \{i, \ldots, k\}$ mix (with atoms and gaps) over the interval $[\bar{p}_k, \bar{p}_i]$. Other firms maintain their initial prices: $p_j = \bar{p}_j$ for $j \notin \{i, \ldots, k\}$.

Just as before, $l \in \{k + 1, \ldots, n\}$ has no profitable deviation, for the usual reasons. A firm $l \in \{1, \ldots, i-1\}$ cannot profitably deviate to within $[\bar{p}_k, \bar{p}_i]$. An upper bound on its profit from doing so is what it would get by "stealing" the price position of some $j \in \{i \ \ldots, k\}$. Specifically, suppose l sets a price in $[\bar{p}_k, \bar{p}_i]$ and could arrange for j to price above it. Under independent awareness, l's expected profit from a price position in competition with other firms is the same as it was for j, save for the fact that their expected profits are scaled by α_l and α_j, respectively. However, those scalings also apply to the on-path equilibrium expected profits. This means that l does not profitably gain by "stepping on to the dancefloor" with higher-indexed firms.

We now build the strategies used by the actively mixing firms $\{i, \ldots, k\}$.

Case (i): a local deviation upward to $\hat{p}_k \in (\bar{p}_k, \bar{p}_{k-1}]$.

Consider a strategy profile in which any firm $j \notin \{k-1, k\}$ maintains its initial price, while firms $j \in \{k-1, k\}$ continuously mix over $[\bar{p}_k, \hat{p}_k)$ according to distribution functions

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_k}{p}\right), \tag{C3}$$

and place remaining mass at their initial prices. These CDFs satisfy $F_j(\bar{p}_k) = 0$. Because $\alpha_k \le \alpha_{k-1}$ implies $F_{k-1}(p) \le F_k(p)$, we need only check that $F_k(p)$ is a valid CDF:

$$F_k(p) \le 1 \quad \Leftrightarrow \quad p \le \frac{\bar{p}_k}{1-\alpha_k} = \bar{p}_{k-1}, \tag{C4}$$

which holds because $\hat{p}_k \le \bar{p}_{k-1}$. Prices within this interval generate the expected profit

$$\pi_k(p) = p\alpha_k\left(1 - \alpha_{k-1}F_{k-1}(p)\right)\prod_{j>k}(1-\alpha_j) = \bar{p}_k\alpha_k\prod_{j>k}(1-\alpha_j), \tag{C5}$$

which is the on-path expected profit of firm k, π_k. A similar calculation holds for $k-1$.

For the remaining cases, firm k deviates to $\hat{p}_k \in (\bar{p}_{i+1}, \bar{p}_i]$ where $i < k-1$.

Case (ii): a deviation to the upper part of a higher price interval, so that $\hat{p}_k \in ((1-\alpha_k)^{1/2}\bar{p}_i, \bar{p}_i]$.

We write $F_j(p)$ for the mixed strategy of j. In the lowest interval of prices $[\bar{p}_k, \bar{p}_{k-1})$ we set

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \left(\frac{\bar{p}_k}{p}\right)^{1/(k-i)}\right), \tag{C6}$$

for each firm $j \in \{i, \ldots, k\}$. These are well-defined continuously increasing CDFs. Note that

$$\lim_{p \uparrow \bar{p}_{k-1}} F_j(p) = \frac{1}{\alpha_j}\left(1 - (1-\alpha_k)^{1/(k-i)}\right) \le \frac{1}{\alpha_k}\left(1 - (1-\alpha_k)^{1/(k-i)}\right) < 1, \tag{C7}$$

and so these solutions require $k-1$ (this firm faces the constraint $p_{k-1} \le \bar{p}_{k-1}$) to place an atom at its initial price $\bar{p}_{k-1}$. The expected profit for j from any price within this interval is,

$$p\alpha_j\prod_{l \ne j}(1 - \alpha_l F_l(p)) = \bar{p}_k\alpha_j\prod_{l>k}(1-\alpha_l) = v\alpha_j\prod_{l>1}(1-\alpha_l), \tag{C8}$$

which is the on-path equilibrium expected profit for firm j, π_j.

Next, for each $j \in \{i+1, \ldots, k-2\}$ consider the price interval $[\bar{p}_{j+1}, \bar{p}_j)$. This interval lies above the initial price of any firm $l \in \{j+1, \ldots, k-1, k+1, \ldots, n\}$, and so $F_l(p) = 1$ for all such firms. The firms $l \in \{i, \ldots, j\} \cup \{k\}$ (there are $j-i+2$ such firms) all actively mix via

$$F_l(p) = \begin{cases} \dfrac{1}{\alpha_l}\left(1 - (1-\alpha_k)^{1/(j-i+2)}\right) & p \in [\bar{p}_{j+1}, (1-\alpha_k)^{1/(j-i+2)}\,\bar{p}_j) \\[2ex] \dfrac{1}{\alpha_l}\left(1 - \left(\dfrac{\bar{p}_j(1-\alpha_k)}{p}\right)^{1/(j-i+1)}\right) & p \in [(1-\alpha_k)^{1/(j-i+2)}\,\bar{p}_j, \bar{p}_j). \end{cases} \tag{C9}$$

Equivalently,

$$F_l(p) = \max\left\{\lim_{\widetilde{p}\uparrow\bar{p}_{j+1}} F_l(\widetilde{p}), \frac{1}{\alpha_l}\left(1-\left(\frac{\bar{p}_j(1-\alpha_k)}{p}\right)^{1/(j-i+1)}\right)\right\}. \tag{C10}$$

This means that the CDF remains flat (there is a gap in the support) across the lower part of the interval $[\bar{p}_{j+1}, \bar{p}_j)$. For any price in such a gap, a firm would prefer to deviate and undercut the initial price $\bar{p}_{j+1}$ given that firm $j+1$ places an atom there. Indeed,

$$\lim_{p\uparrow\bar{p}_j} F_l(p) = \frac{1}{\alpha_l}\left(1-(1-\alpha_k)^{1/(j-i+1)}\right) \le \frac{1}{\alpha_k}\left(1-(1-\alpha_k)^{1/(j-i+1)}\right) < 1, \tag{C11}$$

and so firm j places an atom at its initial price position. Any price $p \in [(1-\alpha_k)^{1/(j-i+2)}\,\bar{p}_j, \bar{p}_j)$ generates the on-path expected profits for any mixing firm. For example, firm k gets

$$\begin{aligned}
&p\alpha_k\left[\prod_{h\in\{j+1,\dots,k-1,k+1,\dots,n\}}(1-\alpha_h)\right]\left[\prod_{l\in\{i,\dots,j\}}(1-\alpha_l F_l(p))\right]\\
&=\bar{p}_j\alpha_k\left[\prod_{h\in\{j+1,\dots,k-1,k+1,\dots,n\}}(1-\alpha_h)\right](1-\alpha_k)\\
&=\bar{p}_n\alpha_k = \pi_k.
\end{aligned} \tag{C12}$$

For the top price interval (this is for $j=i$), the same formulae apply up to $\hat{p}_k$. That is,

$$F_l(p) = \begin{cases} \dfrac{1}{\alpha_l}\left(1-(1-\alpha_k)^{1/2}\right) & p \in [\bar{p}_{i+1}, \bar{p}_i\,(1-\alpha_k)^{1/2}) \\ \dfrac{1}{\alpha_l}\left(1-\left(\dfrac{\bar{p}_i(1-\alpha_k)}{p}\right)\right) & p \in [\bar{p}_i\,(1-\alpha_k)^{1/2}, \bar{p}_i). \end{cases} \tag{C13}$$

The two firms $l \in \{i, k\}$ then place their remaining mass on their initial prices. (If $\hat{p}_k = \bar{p}_i$ then the CDFs described above specify $F_k(\bar{p}_i) = 1$ and so firm k has no atom.)

Case (iii): a deviation to the lower part of a higher price interval, so that $\hat{p}_k \in (\bar{p}_{i+1}, \bar{p}_i(1-\alpha_k)]$.

For this case we build the same strategy profile that we would use if $\hat{p}_k = \bar{p}_{i+1}$. There, firm i does not participate, and always chooses its initial price so that $p_i = \bar{p}_i$. If $i = k-2$ then we build the strategy profile described in case (i), and if $i < k-2$ then we use the strategy profile from case (ii). In both cases, for prices just below $\bar{p}_{i+1}$, the two firms k and $i+1$ mix. Specifically, for $p \in [\bar{p}_{i+1}\,(1-\alpha_k)^{1/2}, \bar{p}_{i+1})$ and $l \in \{k, i+1\}$,

$$F_l(p) = \frac{1}{\alpha_l}\left(1-\left(\frac{\bar{p}_{i+1}(1-\alpha_k)}{p}\right)\right). \tag{C14}$$

We know $\alpha_k \le \alpha_l$ and so $F_l(p) \le F_k(p)$. Moreover, $\lim_{p\uparrow\bar{p}_{i+1}} F_k(p) = 1$. This means that k places all mass continuously below $\bar{p}_{i+1}$, and so does not use any prices within $(\bar{p}_{i+1}, \hat{p}_k]$. However, for $\alpha_k < \alpha_l$, l places an atom at $\bar{p}_{i+1}$. Firms earn their equilibrium expected profits.

Notice that firm k places all mass below $\bar{p}_{i+1}$, which captures the atom of firm $i+1$. We need to check that k does not get more than its equilibrium expected profit, π_k, by charging $\hat{p}_k$:

$$\hat{p}_k \alpha_k \prod_{j \in \{i+1,\ldots,k-1,k+1,\ldots,n\}} (1-\alpha_j) = \frac{\hat{p}_k \alpha_k}{1-\alpha_k} \prod_{j=i+1}^{n} (1-\alpha_j) \leq \pi_k \quad \Leftrightarrow \quad \hat{p}_k \leq \bar{p}_i(1-\alpha_k), \tag{C15}$$

where this inequality holds by assumption in this case.

Case (iv): a deviation to an intermediate range, so that $\hat{p}_k \in (\bar{p}_i(1-\alpha_k), \bar{p}_i(1-\alpha_k)^{1/2}]$.

Case (iii) of Lemma 6's proof is similar in nature. For $p \in [\bar{p}_k, \bar{p}_{k-1})$, the lowest interval, define:

$$F_l^+(p) = \frac{1}{\alpha_l}\left(1 - \left(\frac{\bar{p}_k}{p}\right)^{1/(k-i)}\right) \quad l \in \{i, \ldots, k\} \tag{C16}$$

$$F_l^-(p) = \frac{1}{\alpha_l}\left(1 - \left(\frac{\bar{p}_k}{p}\frac{\hat{p}_k}{\bar{p}_i(1-\alpha_k)}\right)^{1/(k-i-1)}\right) \quad l \in \{i+1, \ldots, k\}. \tag{C17}$$

Next, for each $j \in \{i+1, \ldots, k-2\}$ and the corresponding price interval $[\bar{p}_{j+1}, \bar{p}_j)$, define

$$F_l^+(p) = \frac{1}{\alpha_l}\left(1 - \min\left\{(1-\alpha_k)^{\frac{1}{j-i+2}}, \left(\frac{\bar{p}_j(1-\alpha_k)}{p}\right)^{\frac{1}{j-i+1}}\right\}\right) \quad l \in \{i, \ldots, j\} \cup \{k\} \tag{C18}$$

$$F_l^-(p) = \frac{1}{\alpha_l}\left(1 - \min\left\{\left(\frac{\hat{p}_k}{\bar{p}_i}\right)^{\frac{1}{j-i+1}}, \left(\frac{\bar{p}_j}{p}\frac{\hat{p}_k}{\bar{p}_i}\right)^{\frac{1}{j-i}}\right\}\right) \quad l \in \{i+1, \ldots, j\} \cup \{k\}. \tag{C19}$$

For the largest-awareness firm i and $p \in [\bar{p}_k, \bar{p}_i)$ define

$$F_i(p) = \min\left\{F_i^+(p), \frac{1}{\alpha_i}\left(1 - \frac{\bar{p}_i(1-\alpha_k)}{\hat{p}_k}\right)\right\}, \tag{C20}$$

and let i place its remaining mass at the initial price $\bar{p}_i$.

For other firms $l \in \{i+1, \ldots, k\}$ and prices $p < \bar{p}_l$ define

$$F_l(p) = \begin{cases} F_l^+(p) & F_i(p) = F_i^+(p) \\ F_l^-(p) & \text{otherwise,} \end{cases} \tag{C21}$$

with remaining mass at the firm's initial price (so that $F_l(p) = 1$ for $p \geq \bar{p}_l$). □

Proof of Proposition 9. Consider the profile of efficient prices from (22). As usual we construct a strategy profile for our two-stage game in which firms charge those prices in the first stage, and maintain those prices in the second stage. The prices are undercut-proof and so there is no profitable second-stage deviation, nor any profitable downward first-stage deviation. It remains to consider upward deviations by either firm 2 or 3 in the first stage. (As usual, we can specify the play of any equilibrium in games that are further from the equilibrium path.)

If firm 2 deviates upward to $\hat{p}_2 > \bar{p}_2$ then we construct an equilibrium in which firm 3 charges $p_3 = \bar{p}_3$ (earning its on-path expected profit) while 1 and 2 mix using the distributions

$$F_2(p) = \frac{(\lambda_1 + X_2)(p - \bar{p}_2)}{pX_2} \quad \text{and} \quad F_1(p) = \frac{(\lambda_2 + X_2)(p - \bar{p}_2)}{pX_2} \tag{C22}$$

over the interval $[\bar{p}_2, \hat{p}_2)$ with (if $\hat{p}_2 < \bar{p}_1$) both firms placing remaining mass at their initial prices. If $\hat{p}_2 = \bar{p}_1 = v$ then the solutions above yield $F_2(\bar{p}_1) = F_2(v) = 1$ and so only firm 1 plays an atom at its initial price $\bar{p}_1 = v$. These strategies generate the on-path equilibrium expected payoffs for both firms across the support of their mixed strategies, and it is straightforward to confirm that they have no incentive to deviate elsewhere.

As noted in the text, the more difficult case involves firm 3 deviating upward to $\hat{p}_3 > \bar{p}_3$. We construct an equilibrium in which firm 1 sets $p_1 = \bar{p}_1$. We then (as explained in the main text) build mixed strategies for firms 2 and 3 over $[\bar{p}_3, \min\{\hat{p}_3, \bar{p}_2\})$ with distributions

$$F_2(p) = \frac{(\lambda_3 + X_3)(p - \bar{p}_3)}{p(X_3 - X_2)} \quad \text{and} \quad F_3(p) = \frac{(\lambda_2 + X_3)(p - \bar{p}_3)}{p(X_3 - X_2)}, \tag{C23}$$

where both firms place remaining mass at their initial prices. These distributions give both firms their on-path expected profits across the support. As noted in the text, $F_3(\bar{p}_2) = 1$. This means that if $\hat{p}_3 > \bar{p}_2$ then firm 3 cannot play any price $p_3 \in (\bar{p}_2, \hat{p}_3]$. We need to check that firm 3 does not wish to play such a price. By the argument in the text, that is true if and only if $\hat{p}_3(\lambda_3+X_2) \le \bar{p}_3(\lambda_3+X_3)$, which is satisfied for all $\hat{p}_3 \le v$ if and only if $v(\lambda_3+X_2) \le \bar{p}_3(\lambda_3+X_3)$. Rearranging this gives the inequality (26) stated in the proposition.

So far we have shown that, if (26) holds, there is a strategy profile in which firms 2 and 3 mix and obtain their on-path expected profits, and where they have no incentive to deviate anywhere else. However, we need to check that firm 1 does not wish to deviate from charging $p_1 = \bar{p}_1 = v$. If $\hat{p}_3 < \bar{p}_2$ then any deviation $p_1 \in (\hat{p}_3, \bar{p}_2)$ should be to just below $\bar{p}_2$ to capture the atom of firm 2. However, this is not profitable owing to the no-undercutting constraint. This means that we need to check firm 1's expected profit from deviating to some price $p_1 \in [\bar{p}_3, \min\{\hat{p}_3, \bar{p}_2\})$ which is (in essence) the "dance floor" across which firms 2 and 3 tango. By Lemma A1, that expected profit, $\pi_1(p_1)$, is quasi-convex in p_1 over the interval $[\bar{p}_3, \bar{p}_2)$, which means that

$$\begin{aligned}\pi_1(p_1) \le \max\left\{\pi_1(\bar{p}_3), \lim_{p_1 \uparrow \bar{p}_2} \pi_1(p_1)\right\} &= \max\left\{\bar{p}_3(\lambda_1 + X_3), \bar{p}_2(\lambda_1 + X_2(1 - \lim_{p_1 \uparrow \bar{p}_2} F_2(p_1))\right\} \\ &< \max\{v\lambda_1, \bar{p}_2(\lambda_1 + X_2)\} = v\lambda_1. \end{aligned} \tag{C24}$$

The strict inequality holds for both of the components over which the maximum is taken. Specifically, $\bar{p}_2(\lambda_1 + X_2(1 - \lim_{p_1 \uparrow \bar{p}_2} F_2(p_1))) < \bar{p}_2(\lambda_1 + X_2)$ because firm 2 places an atom at $\bar{p}_2$. Also $\bar{p}_3(\lambda_1 + X_3) < v\lambda_1$ because firm 1 strictly prefers not to undercut firm 3. Explicitly:

$$\begin{aligned}\bar{p}_3(\lambda_1 + X_3) =& v\lambda_1 \frac{\lambda_1 + X_3}{\lambda_1 + X_2}\frac{\lambda_2 + X_2}{\lambda_2 + X_3} \\ =& v\lambda_1 \frac{\lambda_1\lambda_2 + X_2X_3 + (\lambda_1 + \lambda_2)X_2 + \lambda_2(X_3 - X_2)}{\lambda_1\lambda_2 + X_2X_3 + (\lambda_1 + \lambda_2)X_2 + \lambda_1(X_3 - X_2)} < v\lambda_1. \end{aligned} \tag{C25}$$

From this we conclude that firm 1 does not wish to step onto the dance floor.

In summary, we have constructed an equilibrium in deviant subgames with expected profits equal to those on path so long as $\hat{p}_3(\lambda_3 + X_2) \le \bar{p}_3(\lambda_3 + X_3)$, which is necessarily true if the inequality (26) holds. Now suppose that this inequality fails, which means that a deviation

$\hat{p}_3(\lambda_3 + X_2) > \bar{p}_3(\lambda_3 + X_3)$ is possible. Our construction (such that 3 earns its on-path expected profit also in the deviant second-stage subgame) no longer works, as we now explain.

We know that firm 1 can achieve at least $v\lambda_1$ by charging $p_1 = v$. This means that the price $\bar{p}_3$, and prices just above it, are strictly dominated for 1. It follows that the support of any mixed strategy for firm 1 lies strictly above $\bar{p}_3$. If the support for the mixed strategy of firm 2 were to lie strictly above $\bar{p}_3$, then firm 3 could achieve strictly more than its on path expected profit. (There would be a price $p_3 > \bar{p}_3$ below the support of the competitors which would allow firm 3 to win all comparisons and so earn $p_3(\lambda_3 + X_3) > \bar{p}_3(\lambda_3 + X_3)$.)

We conclude that firm 2 must mix down to $\bar{p}_3$ or below. Suppose that $\bar{p}_3$ is indeed the lower bound. (We can make the same argument for a strictly lower lower bound.) Firms 2 and 3 must mix continuously as we move up from that lower bound. Given that their expected profits are determined by capturing all comparisons at the lower bound, we can solve for their mixed strategies with the solutions for F_2 and F_3 as before. As we move up the price range, we can evaluate $\pi_1(p)$ from firm 1 joining in at any price p. We have already showed that this is strictly less than $v\lambda_1$. We conclude that firm 1 never joins the dance floor as we move up through the prices. Eventually we reach the same conclusion that we did before: firm 3 has a strict incentive to set $p_3 = \hat{p}_3$, and our intended construction fails. $\square$

Proof of Proposition 10. Fix the profile of maximal undercut-proof prices stated in the text. As usual, we construct a strategy profile in which $p_i = \bar{p}_i$ is on the path of subgame-perfect equilibrium. There are no profitable downward deviations at the first stage, for the usual reason: the same deviation at the second stage does weakly better. We can, of course, specify any equilibrium in subgames that are not reached with a unilateral deviation in the first stage.

We now focus on upward deviations by either firm 2 or firm 3 in the first stage. Suppose that firm 2 raises its initial price to $\hat{p}_2$. In the subgame, firms $j \in \{1, 2\}$ mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_2}{p}\right) \quad \text{for} \quad p \in [\bar{p}_2, \hat{p}_2), \tag{C26}$$

and place remaining mass at their initial prices. Firm 3 plays $p_3 = \bar{p}_3$. This profile yields equilibrium expected profits. For example, for prices in this interval, $j, k \in \{1, 2\}$ and $j \neq k$,

$$\pi_k(p) = \alpha_k p(1 - \alpha_3)\left(1 - \alpha_j F_j(p)\right) = \bar{p}_2\alpha_k(1 - \alpha_3) = v\alpha_k(1 - \alpha_3)(1 - \alpha_2) = \pi_k. \tag{C27}$$

Moreover, $F_1(p) \leq F_2(p) \leq F_2(\bar{p}_1) = \frac{1}{\alpha_2}\left(1 - \frac{\bar{p}_2}{\bar{p}_1}\right) = 1$, and so these are valid CDFs.

Next consider deviations by firm 3 to $\hat{p}_3 > \bar{p}_3$. One possibility is $\bar{p}_3 < \hat{p}_3 \leq v(1 - \alpha_3) < \bar{p}_2$ (the last inequality holds because $\bar{p}_2 = v(1 - \alpha_2)$ and $\alpha_3 > \alpha_2$.) Suppose firms $j \in \{2, 3\}$ mix via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p}\right) \quad \text{for} \quad p \in [\bar{p}_3, \hat{p}_3), \tag{C28}$$

and place remaining mass at their first-stage prices. Firm 1 plays $\bar{p}_1$. Straightforwardly, prices by $j \in \{2, 3\}$ in $[\bar{p}_3, \hat{p}_3)$ yield equilibrium expected profits, and prices by firm 1 there earn it strictly less than in equilibrium. We need to check $F_j(p)$ are valid CDFs:

$$F_3(p) \leq F_2(p) \leq F_2(\hat{p}_3) = \frac{1}{\alpha_2}\left(1 - \frac{\bar{p}_3}{\hat{p}_3}\right) \leq 1 \quad \Leftrightarrow \quad \hat{p}_3 \leq \frac{\bar{p}_3}{1-\alpha_2} = v(1-\alpha_3), \tag{C29}$$

which holds by assumption in this case.

The remaining deviations by firm 3 are to $\hat{p}_3 > v(1-\alpha_3)$. The strategy profiles that we construct specify mixing by each $j \in \{1,2,3\}$ via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \left(\frac{\bar{p}_3}{p}\right)^{1/2}\right) \quad \text{for} \quad p \in [\bar{p}_3, \widetilde{p}), \tag{C30}$$

for some $\widetilde{p}$. It is simple to check that all firms earn equilibrium expected profits in this interval.

Different cases involve different choices for $\widetilde{p}$. First suppose $(1-\alpha_3)^{1/2} \leq 1-\alpha_2$. This is true if and only if $\frac{v(1-\alpha_3)}{1-\alpha_2} \leq \bar{p}_2$. For this parameter case, suppose $\hat{p}_3 < \frac{v(1-\alpha_3)}{1-\alpha_2}$ and set:

$$\widetilde{p} = \frac{(\hat{p}_3)^2(1-\alpha_2)}{v(1-\alpha_3)}. \tag{C31}$$

Firm 1 places remaining mass at its first-stage price. Firms $j \in \{2,3\}$ mix according to:

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p(1-\alpha_1 F_1(\widetilde{p}))}\right) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p}\left(\frac{\widetilde{p}}{\bar{p}_3}\right)^{1/2}\right) \quad \text{for} \quad p \in [\widetilde{p}, \hat{p}_3). \tag{C32}$$

Both earn equilibrium expected profits across this interval, the CDFs are continuous at $\widetilde{p}$ and $F_2(\hat{p}_3) = 1$. We complete the specification with 3 placing all remaining mass as an atom at $\hat{p}_3$.

If $\hat{p}_3 = \frac{v(1-\alpha_3)}{1-\alpha_2}$, there is no interval with exactly two firms mixing. Expressions (C30) and (C31) give the equilibrium strategies, and firms place any remaining mass at their first-stage prices.

Now suppose instead that $\hat{p}_3 > \frac{v(1-\alpha_3)}{1-\alpha_2}$. For this case we set $\widetilde{p} = \frac{v(1-\alpha_3)}{1-\alpha_2}$, and we note that the solution for the CDFs below $\widetilde{p}$ satisfies $F_2(\widetilde{p}) = 1$. Hence firm 2 prices only below $\widetilde{p}$, and does not use the ability to price in $(\widetilde{p}, \bar{p}_2]$. Firms $j \in \{1,3\}$ then mix according to

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p(1-\alpha_2)}\right) \quad \text{for} \quad p \in [\widetilde{p}, \hat{p}_3)\,. \tag{C33}$$

Both firms then place remaining atoms at their first-stage prices. (If $\hat{p}_3 = v$ then this formula specifies $F_3(v) = 1$, and so only firm 1 has an atom.)

It remains to consider parameters satisfying $(1-\alpha_3)^{1/2} > (1-\alpha_2)$, so that $\bar{p}_2 < \frac{v(1-\alpha_3)}{1-\alpha_2}$.

If $\hat{p}_3 \in (v(1-\alpha_3), \bar{p}_2]$, then we use the same approach as before by setting $\widetilde{p}$ as per (C31), and building an equilibrium in which firms 2 and 3 mix over $[\widetilde{p}, \hat{p}_3)$ which exhausts the CDF for firm 2 as $\hat{p}_3$ is reached, at which point firm 3 places an atom.

If $\hat{p}_3 \in \left(\bar{p}_2, v\left(1-\alpha_3\right)^{1/2}\right]$, we also set $\widetilde{p}$ as per (C31). All firms mix up to $\widetilde{p}$, firm 1 puts remaining mass on its first-stage price, and $j \in \{2,3\}$ mix via

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p(1-\alpha_1 F_1(\widetilde{p}))}\right) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p}\left(\frac{\widetilde{p}}{\bar{p}_3}\right)^{1/2}\right) \quad \text{for} \quad p \in [\widetilde{p}, \bar{p}_2), \tag{C34}$$

with 2 playing an atom at $\bar{p}_2$ and 3 at $\hat{p}_3$. The value of $\widetilde{p}$ ensures that 3 earns its equilibrium expected profit from $\hat{p}_3$, making it just indifferent to undercutting firm 2's atom at $\bar{p}_2$.

The final case is $\hat{p}_3 \in \left(v\left(1-\alpha_3\right)^{1/2}, v\right]$. We set $\widetilde{p} = \bar{p}_2$. This means that all three firms mix up $\bar{p}_2$, with firm 2 playing an atom at $\bar{p}_2$. The remaining firms $j \in \{1,3\}$ play

$$F_j(p) = \frac{1}{\alpha_j}\left(1 - \frac{\bar{p}_3}{p(1-\alpha_2)}\right) \quad \text{for} \quad p \in \left[v\left(1-\alpha_3\right)^{1/2}, \hat{p}_3\right), \tag{C35}$$

with their CDFs remaining constant for $p \in \left(\bar{p}_2, v\left(1-\alpha_3\right)^{1/2}\right)$. $\square$

Proof of Proposition 11. We now prove the claims without complete proofs in the main text, which we divide into three parts: Parts 1 and 2 address Nash equilibria in subgames following local and non-local deviations, respectively; Part 3 covers the n-firm symmetric-size case.

Part 1. Here, we provide a Nash equilibrium strategy for each firm in the subgame following local deviations from the profile of prices $\bar{p}_1, \bar{p}_i, \bar{p}_j$, where $\bar{p}_1 > \bar{p}_i > \bar{p}_j$, from Proposition 11.

The first class of local deviations has firm i setting some $\hat{p}_i \in (\bar{p}_i, \bar{p}_1]$. The following strategies constitute a Nash equilibrium of the ensuing subgame. Firms 1 and i mix over $[\bar{p}_i, \hat{p}_i)$ via

$$F_1 = 1 - \frac{\bar{p}_i}{p}, \quad F_i = 1 - \frac{(v-p)\phi_1}{p\phi_i}, \tag{C36}$$

with residual mass placed at $\bar{p}_1$ and $\hat{p}_i$ respectively. Firm j sets $p_j = \bar{p}_j$.

The second class of local deviations has firm j setting some $\hat{p}_j \in (\bar{p}_j, \bar{p}_i]$. The following strategies constitute a Nash equilibrium of the ensuing subgame. Firms 1 and j mix over $[\bar{p}_j, \hat{p}_j)$ via

$$F_1 = 1 - \frac{\bar{p}_j}{p}, \quad F_j = 1 - \frac{(v-p)\phi_1 - p\phi_i}{p\phi_j}, \tag{C37}$$

with residual mass placed at $\bar{p}_1$ and $\hat{p}_j$, respectively; firm i sets $p_i = \bar{p}_i$, earning $\bar{p}_i\phi_i(1-F_1(\hat{p}_j)) = \bar{p}_i\phi_i(\bar{p}_j/\hat{p}_j)$. We confirm i does not have an incentive to deviate to some $p \in [\bar{p}_j, \hat{p}_j)$:

$$p\phi_i(1-F_1(p)) = \bar{p}_j\phi_i \leq \bar{p}_j\phi_i(\bar{p}_i/\hat{p}_j). \tag{C38}$$

Part 2. Here we address "non-local" deviations. The first case is that when the smaller non-prominent firm is cheaper, i.e., $i = 2$ and $j = 3$, and $\phi_2 > \phi_3$, with $\bar{p}_1 > \bar{p}_2 > \bar{p}_3$ as stated in Proposition 11. We now prove that in any Nash equilibrium of the subgame following first-stage prices $\bar{p}_1$, $\bar{p}_2$, and $\hat{p}_3 \in (\bar{p}_2, v\phi_1/(\phi_1+\phi_3))$, firm 3 gets a strictly greater profit than $\bar{p}_3\phi_3$.

In any Nash equilibrium of such a subgame:

(i) No firm places an atom strictly below its first-stage price: if a firm did, then no competitor would ever price at or just above this atom, and so the firm could safely move the atom upward.

(ii) The prominent firm uses a mixed strategy: if pure, each firm's price equals their first-stage price, and the prominent firm would find it profitable to undercut $\bar{p}_2$ and capture all customers.

(iii) For the prominent firm, prices $p < \bar{p}_3$ are strictly dominated, as are $p \in (\bar{p}_2, v\phi_1/(\phi_1 + \phi_3))$. A firm $k \in \{2, 3\}$ can secure all the relevant customers by charging $\bar{p}_3$ and so can guarantee an expected profit of $\bar{p}_3\phi_k > 0$. Take the highest price charged by any non-prominent firm. This wins customers with strictly positive probability (as it must to generate a positive expected profit) only if the prominent firm prices above it with strictly positive probability. Thus the prominent firm places an atom at $\bar{p}_1 = v$, which implies its expected profit is $v\phi_1$.

(iv) Excluding the atom at $\bar{p}_1 = v$, consider the support of the prominent firm's (continuous) mixed strategy. This lies within the union of the competitors' supports: any other price can be safely raised (that is, without losing sales) which strictly raises profit. The support of any competitor lies within the support of the prominent firm, and for the same reason. It follows that the two supports (the prominent firm's, and the union of the competitors') coincide. At the lower bound of that support, the prominent firm sells to everyone, a mass $\phi_1 + \phi_2 + \phi_3$. This firm's profit is $v\phi_1$, and so that lower-bound price must equal $\bar{p}_3 = v\phi_1/(\phi_1 + \phi_2 + \phi_3)$.

(v) Consider the interval $[\bar{p}_3, \bar{p}_2)$. Price $\bar{p}_3$ is the lower bound of firm 1's support and therefore also for some $h \in \{2, 3\}$. There cannot be any gaps in the union of all firms' supports in $[\bar{p}_3, \bar{p}_2)$. For all other prices in that interval that h plays, h must be indifferent: $\bar{p}_3\phi_h = p\phi_h(1-F_1(p)) \Leftrightarrow F_1(p) = 1 - \bar{p}_3/p$. For any $p \in [\bar{p}_3, \bar{p}_2)$ charged by $k \neq h$, k must be indifferent to p and the infimum of those, x, implying k's expected profit is $x\phi_k(1 - F_1(x)) = \bar{p}_3\phi_k$ and so 1 must again price by the same CDF for p charged by k: $F_1(p) = 1 - \bar{p}_3/p$ over all $p \in [\bar{p}_3, \bar{p}_2)$.

(vi) No first-stage price is in $[\bar{p}_3, \bar{p}_2)$, and so there are no atoms. Within this interval there is no gap within the support of the prominent firm: if so, then there would be a gap in the support of the competitors' strategies, and so the prominent firm could safely (i.e., without losing sales) move a price from the bottom of the gap upward, and so strictly gain. Similarly, there is no gap with the union of opponents' supports. Given that, at least one $h \in \{2, 3\}$ is willing to set $\bar{p}_2$, earning an expected profit of least $\bar{p}_3\phi_h$. Because $F_1(p)$ does not depend on which firm has p in their support, the two non-prominent firms face the same expected profit when pricing against the prominent firm, and so 3 can guarantee at least $\bar{p}_3\phi_3$ by setting $\bar{p}_2$.

(vii) Firm 3 earns $\bar{p}_3\phi_3$ on the equilibrium path, and at least that much by playing $p_3 = \bar{p}_2$ in the deviant subgame. Recall that the prominent firm 1 never prices just above $\bar{p}_2$. Hence, prices p slightly above $\bar{p}_2$ earn $p\phi_3(1 - F_1(\bar{p}_2)) = \bar{p}_3\phi_3(p/\bar{p}_2) > \bar{p}_3\phi_3$. We conclude that any equilibrium in this subgame yields a profitable deviation, and that the profile of prices with firm 3 as the cheapest is not supported by the equilibrium play of pure strategies.

The second case is that when the larger non-prominent firm is cheaper, i.e., $i = 3$ and $j = 2$, with $\bar{p}_1 > \bar{p}_3 > \bar{p}_2$ as stated in Proposition 11. Consider the subgame following a deviation of firm 2 to some $\hat{p}_2 \in (\bar{p}_3, \bar{p}_1]$, then the following strategy profile constitutes a Nash equilibrium.

All firms mix: firm 1 over $[\bar{p}_2, \hat{p}_2)$, 2 over $[v\phi_1/(\phi_1+\phi_2), \hat{p}_2)$ and 3 over $[\bar{p}_2, v\phi_1/(\phi_1+\phi_2))$ via

$$F_1(p) = 1 - \frac{\bar{p}_2}{p}, \quad F_2 = 1 - \frac{(v-p)\phi_1}{p\phi_2}, \quad \text{and} \quad F_3 = 1 - \frac{(v-p)\phi_1 - p\phi_2}{p\phi_3}, \tag{C39}$$

with any residual mass for firms 1 and 2 placed at $\bar{p}_1$ and $\hat{p}_2$, respectively. Firm 2 earns $\phi_2\bar{p}_2$, the same as without the deviation. We conclude that the prices in the proposition with $i = 3$ and $j = 2$ are supported as the on-path strategies of a subgame-perfect equilibrium.

Part 3. The remaining claim concerns n firms and $\phi_1 = \cdots = \phi_n \equiv \phi$. Without loss of generality, label the firms inversely to price so that $\bar{p}_1 > \cdots > \bar{p}_n > 0$ where firm 1 is the prominent firm. As usual, $\bar{p}_1 = v$ in any efficient undercut-proof profile. Now consider $\bar{p}_i$ for $i > 1$. The prominent firm's no-undercutting constraints (one for each local firm) are

$$v\phi_1 \geq \bar{p}_i\left(\sum\nolimits_{j=1}^{i}\phi_j\right) \quad \Leftrightarrow \quad \bar{p}_i \leq \frac{v\phi_1}{\sum_{j=1}^{i}\phi_j} = \frac{v}{i}. \tag{C40}$$

For efficiency these bind, and so $\bar{p}_i = v/i$. As usual, no firm has an incentive to lower its first-stage price or undercut in the second stage. It remains to check upward first-stage deviations.

Suppose that firm $i > 1$ raises its first-stage price to $\hat{p}_i > \bar{p}_i$. Consider this strategy profile in the subgame. Firm 1 mixes over $[\bar{p}_i, \hat{p}_i)$ using the distribution function

$$F_1(p) = 1 - \frac{\bar{p}_i}{p}, \tag{C41}$$

and places all remaining mass at $\bar{p}_1 = v$. Any cheaper firm, $j > i$ sets $p_j = \bar{p}_j$ as a pure strategy. Any firm $j < i$, which satisfies $\bar{p}_j \geq \hat{p}_i$ also plays a pure strategy, $p_j = \bar{p}_j$. Any other firm $j \neq i$ satisfies $\bar{p}_i < \bar{p}_j < \hat{p}_i$. Such a firm mixes over $[\bar{p}_{j+1}, \bar{p}_j)$ using the distribution function

$$F_j(p) = 1 - \frac{v - jp}{p}. \tag{C42}$$

Finally, consider the deviant firm i. Take the lowest index k (and so highest first-stage price $\bar{p}_k$) which satisfies $\bar{p}_k < \hat{p}_i$. Firm i mixes over $[\bar{p}_k, \hat{p}_i)$ using the distribution function

$$F_i(p) = 1 - \frac{v - (k-1)p}{p}, \tag{C43}$$

and places all remaining mass at the deviant price $\hat{p}_i$.

These strategies generate a Nash equilibrium of the subgame. Firm 1 earns $v\phi_1$, the same as without the deviation, and so by undercut-proofness of the initial prices, firm 1 does not do better with a price outside $[\bar{p}_i, \hat{p}_i)$. For any local firm $j < i$, a price satisfying $\bar{p}_i \leq p \leq \hat{p}_i$ is in firm 1's support and yields an expected profit equal to $\bar{p}_i\phi$. If $\bar{p}_j \leq \hat{p}_i$ then a firm can do no better than this, and so optimally plays the prescribed strategy. If $\bar{p}_j > \hat{p}_i$ then j is strictly better off by setting $p_j = \bar{p}_j$, and so does so. $\square$

This next lemma is a more general version of Proposition 2.

Lemma C1. *For firms placed in size order, $\lambda_1 \geq \cdots \geq \lambda_n$, define the following prices:*

$$p_1^\ddagger = v \quad \textit{and} \quad p_i^\ddagger \equiv v \prod_{j=2}^{i} \frac{\lambda_{j-1} + X_{j-1}}{\lambda_{j-1} + X_j}. \tag{C44}$$

Next, for any *given order of firms, consider the set of maximal undercut-proof prices:*

(1) These prices satisfy $\bar{p}_i \leq p_i^\ddagger$ for all i.

(2) If $\lambda_1 \geq \cdots \geq \lambda_{i-1}$ (so that firms indexed below i are in size order) then $\bar{p}_i = p_i^\ddagger$.

(3) The i^{th} highest price is highest when firms are in size order, for all i.

(4) Placing firms in size order maximizes the industry profit.

(5) All firms would unanimously prefer to be placed in size order.

Proof. Claim (1) is straightforward. It holds trivially for $i = 1$. If it holds for all $j < i$ then

$$\bar{p}_i = \min_{j<i} \left\{ \bar{p}_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \leq \bar{p}_{i-1} \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i} \leq p_{i-1}^\ddagger \frac{\lambda_{i-1} + X_{i-1}}{\lambda_{i-1} + X_i} = p_i^\ddagger, \tag{C45}$$

and so it holds also for i, and, by the principle of induction, for all i.

Claim (2) can also be proved inductively. It holds for $i = 2$. If it holds for all $j < i$ then

$$\begin{aligned} \bar{p}_i = \min_{j<i} \left\{ \bar{p}_j \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} &= \min_{j<i} \left\{ p_j^\ddagger \frac{\lambda_j + X_j}{\lambda_j + X_i} \right\} \\ &= \min_{j<i} \left\{ p_j^\ddagger \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} \right) \right\} \\ &= p_i^\ddagger \min_{j<i} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \\ &= p_i^\ddagger \left\{ 1, \min_{j<i-1} \left\{ \frac{\lambda_j + X_j}{\lambda_j + X_i} \left(\prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \right) \right\} \right\} \\ &= p_i^\ddagger. \end{aligned} \tag{C46}$$

The first line holds by the inductive hypothesis. The second line introduces product terms which cancel each other. The third line recognises that the second product term multiplied by $p_j^\ddagger$ is $p_i^\ddagger$. The fourth line is obtained by separating out the first term for $j = i - 1$ and the remaining terms for $j < i - 1$. The final line is obtained by noting that for each $j < i - 1$,

$$\frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k-1}} \leq \frac{\lambda_j + X_j}{\lambda_j + X_i} \prod_{k=j+1}^{i} \frac{\lambda_j + X_k}{\lambda_j + X_{k-1}} = \frac{\lambda_j + X_j}{\lambda_j + X_i} \frac{\lambda_j + X_i}{\lambda_j + X_j} = 1. \tag{C47}$$

The inequality in the chain holds because $X_k \geq X_{k-1}$ in each of the ratio terms, which means that such terms are each decreasing in λ_{k-1}. An upper bound for each term is obtained by replacing λ_{k-1} with $\lambda_j \leq \lambda_{k-1}$, where this inequality holds because $j \leq k - 1 \leq i - 1$ and (by assumption) firms below i are in size order. The claim holds by the principle of induction.

For Claim (3), suppose that firms are not in size order. Consider the first firm k that is out of order: $\lambda_1 \geq \cdots \geq \lambda_{k-1}$ but $\lambda_k > \lambda_{k-1}$. We know that $\bar{p}_i = p_i^\ddagger$ for all $i \leq k$. This means that

$$\bar{p}_k = \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k}, \tag{C48}$$

which (given that $X_{k-1} < X_k$) is strictly increasing in λ_{k-1}. The next price is

$$\begin{aligned}
\bar{p}_{k+1} &= \min\left\{\bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}, \bar{p}_k\frac{\lambda_k + X_k}{\lambda_k + X_{k+1}}\right\} \\
&= \bar{p}_{k-1}\min\left\{\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}, \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k}\right\} \\
&= \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k}\min\left\{\frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}, \frac{\lambda_k + X_k}{\lambda_k + X_{k+1}}\right\} \\
&= \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}}.
\end{aligned} \tag{C49}$$

Suppose that we interchange the two firms; we swap λ_k and λ_{k-1}. Prices $\bar{p}_i$ for $i < k$ remain unchanged. We write $\tilde{p}_k$ for the remaining maximal undercut-proof prices. Clearly,

$$\tilde{p}_k = \bar{p}_{k-1}\frac{\lambda_k + X_{k-1}}{\lambda_k + X_k} > \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} = \bar{p}_k, \tag{C50}$$

where the inequality holds because $\lambda_{k-1} < \lambda_k$. Next,

$$\begin{aligned}
\tilde{p}_{k+1} &= \min\left\{\bar{p}_{k-1}\frac{\lambda_k + X_{k-1}}{\lambda_k + X_{k+1}}, \tilde{p}_k\frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}\right\} \\
&= \bar{p}_{k-1}\min\left\{\frac{\lambda_k + X_{k-1}}{\lambda_k + X_{k+1}}, \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}\frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}\right\} \\
&= \bar{p}_{k-1}\frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}\min\left\{\frac{\lambda_k + X_k}{\lambda_k + X_{k+1}}, \frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}}\right\} \\
&= \bar{p}_{k-1}\frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}\frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}} \\
&> \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k}\frac{\lambda_{k-1} + X_k}{\lambda_{k-1} + X_{k+1}} \\
&= \bar{p}_{k-1}\frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_{k+1}} \\
&= \bar{p}_{k+1}.
\end{aligned} \tag{C51}$$

Straightforwardly all other prices for $i > k + 1$ must also (at least weakly) rise.

For Claim (4), from Claim (3) prices are highest by placing firms in order. This maximizes the industry profit $\sum_{i=1}^n \bar{p}_i X_i$ from comparator customers. The profit from captives is maximized when the largest firms charge the highest prices, and this is so when firms are in size order.

This claim also holds when we correct a misstep in the first group of firms: if $\lambda_1 \geq \cdots \geq \lambda_{k-1}$ but $\lambda_k < \lambda_{k-1}$, then switching $k - 1$ and k raises the profits earned by the first k firms.

For Claim (5), consider again the procedure above of switching $k - 1$ and k into the correct order. Clearly, firm $k - 1$ benefits: this firm was previously indifferent to charging $\bar{p}_{k-1}$ and charging $\bar{p}_k$, but now gains strictly because $\widetilde{p}_k > \bar{p}_k$. Firm k benefits from this switch if

$$\bar{p}_k(\lambda_k + X_k) \geq \bar{p}_{k-1}(\lambda_k + X_{k-1}) \quad \Leftrightarrow \quad \frac{\lambda_{k-1} + X_{k-1}}{\lambda_{k-1} + X_k} > \frac{\lambda_k + X_{k-1}}{\lambda_k + X_k}, \tag{C52}$$

where this last inequality holds because $\lambda_k < \lambda_{k-1}$. This means that the first pair of misordered firms both gain by "correcting" their order, as well as raising the profits of all firms $i > k$. □

Proof of Lemma B1. The first claim follows from the argument in the text. The second claim holds because if a firm is not the largest then the derivative of its profit π_i (ignoring awareness costs) with respect to α_i has the same sign as $1 - 2\alpha_i$, which is strictly negative if $\alpha_i > \frac{1}{2}$. □

Proof of Proposition B1. We seek an equilibrium of the advertising game, where firm i's payoff is $\pi_i - C_i(\alpha_i)$. We write $\alpha_i^\star$ for the (pure strategy) equilibrium choice of firm i. Recalling that we ordered firms according to $C_i'(\cdot)$, we will show there is an equilibrium in which firm 1 (the firm with the lowest (marginal) cost of advertising) chooses $\alpha_1^\star > \max_{i \neq 1}\{\alpha_i^\star\}$. Advertising choices (for $\alpha_i \in (0,1)$) satisfy the first-order conditions (B9). With $k = 1$, those become

$$\frac{C_1'(\alpha_1)}{v} = \prod_{j>1}(1 - \alpha_j) \quad \text{and} \quad \frac{C_i'(\alpha_i)}{v} = \frac{1 - 2\alpha_i}{1 - \alpha_i}\prod_{j>1}(1 - \alpha_j) \ \forall i > 1, \tag{C53}$$

where for the set of $n - 1$ first-order conditions for $i > 1$ we have divided by $1 - \alpha_i$ knowing that the equilibrium must satisfy $\alpha_i^\star \leq 1/2$. Define $R = \prod_{j>1}(1 - \alpha_j)$. If $Rv > C_i'(0)$, then we define $A_i(R)$ to be the $\alpha_i \in (0, 1/2)$ that satisfies firm i's first-order condition. That is,

$$\frac{C_i'(A_i(R))}{v} = \frac{1 - 2A_i(R)}{1 - A_i(R)}R. \tag{C54}$$

This is uniquely defined because the left-hand side is continuously increasing in $A_i(R)$ and the right-hand side is continuously decreasing (beginning from R and decreasing to zero at $A_i(R) = 1/2$). Furthermore, this solution is strictly increasing in R. If $Rv \leq C_i'(0)$ (so that the right-hand side lies everywhere below the left-hand side) then we set $A_i(R) = 0$. To find R we seek a solution to $R = \prod_{j>1}(1 - A_j(R))$. The right-hand side lies within $[0, 1]$, begins above zero, and is decreasing in R, and so we can find a unique solution $R^\star$. We then set $\alpha_i^\star = A_i(R^\star)$ for $i > 1$. Finally, we can find $\alpha_1^\star$, where $\alpha_1^\star = 1$ if $C_1'(1) < vR^\star$, but otherwise $\alpha_1^\star$ is the unique positive solution (and one which satisfies $\alpha_1^\star > \alpha_i^\star$ for $i > 1$) to the condition $C_1'(\alpha_1) = vR^\star$.

The remaining deviation checks are non-local:

(i) 1 deviates to $\hat{\alpha}_1 \leq \alpha_j^\star$ where $j\colon \alpha_j^\star = \max_{i>1}\{\alpha_i^\star\}$. Firm j satisfies a first-order condition at $\alpha_j^\star$. Therefore, the best such deviation for 1 is to $\hat{\alpha}_1 = \alpha_j^\star$ (1's revenue (cost) curve is the same (flatter) for $\hat{\alpha}_1 \in [0, \alpha_j^\star]$ than j's over the same interval when $\alpha_i = \alpha_i^\star$ for $i \neq j$). By continuity and 1's first-order condition, 1's profit at any $\hat{\alpha}_1 \geq \alpha_j^\star$ is less than at $\alpha_1^\star$.

(ii) $i > 1$ deviates to $\hat{\alpha}_i \geq \alpha_1^\star$. Firm 1 satisfies their first-order condition at $\alpha_1^\star$. Therefore, the best such deviation for $i > 1$ is to $\hat{\alpha}_i = \alpha_1^\star$ (i's revenue (cost) curve is flatter (steeper) for $\hat{\alpha}_i \in [\alpha_1^\star, 1]$ than 1's over the same interval when $\alpha_i = \alpha_i^\star$ for $i > 1$). But by continuity and i's first-order condition, i's profit at any $\hat{\alpha}_i \leq \alpha_1^\star$ is less than at $\alpha_i^\star$.

The other claims of the proposition follow from Lemma B1. □

Proof of Proposition B2. When all firms have zero costs, we put aside trivial equilibria where more than one firm chooses $\alpha_i = 1$ which lead to zero profit outcomes. From symmetry it follows that although the profile of equilibrium advertising choices we report is unique, the assignment of firms is not. Subject to this disclaimer, the main text explains that one firm will advertise with the outright highest intensity, and we label this firm 1.

By (B6), the profit of firm 1 is strictly (and linearly) increasing in α_1 for any $\alpha_i < 1$ for $i > 1$, hence $\alpha_1^\star = 1$. Given $\alpha_1^\star = 1$, (B6) shows that the profit of the non-largest firms is maximized at $\alpha_i = 1/2$ for any $\alpha_j < 1$ where $j \neq 1, i$, hence $\alpha_i^\star = 1/2$ for $i > 1$.

For positive costs, firms $i, j > 1$ must satisfy their first-order conditions given in (B9) but with $C_i = C$. Taking the ratio of i's and j's condition yields

$$\frac{C'(\alpha_i)}{C'(\alpha_j)} = \frac{(1-2\alpha_i)(1-\alpha_j)}{(1-2\alpha_j)(1-\alpha_i)}. \tag{C55}$$

If $\alpha_i > (<)\alpha_j$ the left-hand side $> 1(< 1)$ but the right-hand side $< 1(> 1)$. However, if $\alpha_i = \alpha_j$, (C55) is satisfied. Hence $\alpha_i^\star = \alpha_j^\star$. Letting $C(\alpha) = -\log(1-\alpha)$ gives (B11), the solution to which gives the values of $\alpha_1^\star$ and $\alpha_i^\star$ for $i > 1$, and that $\alpha_1^\star = 2\alpha_i^\star$. Similar reasoning to that in the proof of Proposition B1 rules out profitable non-local deviations. □