A Theory of Protest Voting

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Originally: August 2014. Revised: January 2015, June 2015, August 2015.[1]

Abstract. The supporters of a mainstream candidate contemplate voting for a special-issue minority party in order to influence mainstream policy. However, such protest voting may open the door to a disliked opponent. In equilibrium, there is an offset effect: protest voting reacts negatively to voters' expectations about others' enthusiasm for the protest issue, and so an increased desire for a successful protest, relative to the desire to elect the mainstream candidate, does not fully translate into more protest voting. If the candidate learns from the size of the protest vote and responds endogenously (by accepting the protesters' demands when sufficiently enthusiastic backing for the protest issue is inferred) then this offset effect is strengthened further, and can be enough to overwhelm the direct effect. This implies that the electoral support for single-issue protest parties can be negatively related to the true underlying enthusiasm for their causes. The expected size of the protest and the risk that the candidate loses office are maximized when the expected enthusiasm for the protest issue makes the candidate indifferent ex ante to accepting the protesters' demands.

In a two-horse-race election, a voter's incentives seem straightforward: he should vote for his favorite candidate. Nevertheless, protest votes are sometimes cast for a leading opponent or for a single-issue minority party, and voters sometimes spoil their ballots.

Here I use a theoretical model to study how protest voting responds to the electoral environment, to beliefs about protest issues, and to voters' anticipation of a policymaker's reaction to a protest. I find that any increase in the expected strength of the enthusiasm for a successful protest is offset by a reduced willingness (other things equal) to engage in protest voting. Furthermore, if a winning candidate responds endogenously to the election result (she infers voters' preferences, and then decides whether to change policy) then the offset effect can be stronger then the direct effect. This means that the number of votes cast for a protest party is non-monotonically related to the true enthusiasm for

[1] I thank Jean-Pierre Benoît, Chris Wallace, seminar audience members, and Martin Cripps (the Editor), for their comments. Special thanks go to Alessandra Casella and Andrew Little for their outstanding conference discussions, and to the anonymous referees for detailed and constructive reports.

the protest issue. The expected support for the protest (and so the probability that the protest causes the candidate to lose) is highest when the expected enthusiasm for the protest issue makes the candidate indifferent ex ante to a change in policy.

Recent years have seen the success of protest parties and candidates. In Italy, the comedian Giuseppe Piero "Beppe" Grillo led the Five Star Movement to become a significant electoral force: in February 2013 his party captured over a quarter of the vote in the Italian Chamber of Deputies. This reflected dissatisfaction with other major parties, but Grillo did not (at least at that stage) offer a full solution. His success sent a message, but arguably his supporters might not have sought his capture of complete power.

The United Kingdom has seen the rise of the UK Independence Party (UKIP) who campaign for EU exit under their charismatic leader (and Member of the European Parliament) Nigel Farage. Remarkably, in 2014 they won a third of the UK's seats in the European Parliament, and in 2015 they went on to capture more than one in eight votes in the UK General Election. They do not have a full portfolio of policies and personnel, and complete government control would not be desired by most supporters. Indeed, their significant popular vote share in the General Election did not translate into parliamentary power: they won only one seat. Nevertheless, the potential electoral threat may have pushed the ruling Conservative Party toward eurosceptic candidates and a promised (to be fulfilled in 2016) referendum on EU membership.

Empirically, protest voting seems to be quantitatively significant. Many supporters of trailing candidates in plurality-rule elections switch away from their first choices, even when debates about measurement are recognized (Niemi, Whitten, and Franklin, 1992, 1993; Evans and Heath, 1993; Fisher, 2004); this is consistent with the usual strategic voting logic. However, many who prefer a leading candidate also switch; this is consistent with protest voting. For the 1987 British general election, Franklin, Niemi, and Whitten (1994) found that voters who switched away from their first choice were roughly evenly split between "instrumental" types who abandoned a trailing contender and what they called "expressive" voters who abandon one of the two leaders.

The British context involves multi-party competition. However, votes for minority candidates also occur in the bipartite environment of the United States. The presidential elections of 1968, 1980, and 1992 featured the independent candidates Wallace, Anderson, and Perot (Abramson, Aldrich, Paolino, and Rohde, 1995). In 1992, Ross Perot received more votes than the winning margin (Lacy and Burden, 1999). Potentially Perot voters could have switched the outcome, given that some (Alvarez and Nagler, 1995) have suggested that he drew more votes from Bush than from Clinton. Furthermore, it has been argued (Gold, 1995) that the similarity of major party candidates can open up opportunities for dissatisfied voters to switch to a candidate such as Perot.[2] Even if a third candidate is absent, voters may dissent by spoiling their ballots. Notably, Rosenthal and Sen (1973) documented such blank ballots in the context of the French Fifth Republic.

I join Kselman and Niou (2011) in thinking of a protest vote as "a targeted signal of dissatisfaction to one's most-preferred political party." It can make sense when a voter has concerns beyond the identity of the winner. I study a stylized situation in which a voter ("he") would like his favorite mainstream candidate ("she") to win, but he also has a secondary objective which may be achieved by switching his vote.

One scenario in which this is true is when a voter prefers to avoid a large winning margin for the mainstream candidate. He may, for example, wish to (Franklin, Niemi, and Whitten, 1994, p. 552) "humble a party that is poised to win by an overwhelming margin." In the United Kingdom system, a government that wins a very large working majority in the House of Commons, such as the Thatcher government of the 1980s, faces few constraints and so is free to pursue extreme ideological policies. A more moderate voter may prefer such a government to enjoy a smaller working majority (this is the situation of the UK's current Cameron-led government) or perhaps even to be the largest party within a coalition (corresponding to the UK's preceding Cameron-Clegg administration). In other systems, there are formally specified majority requirements for different decisions. For example, in Hungary constitutional changes require a two-thirds parliamentary super-majority. A supporter of the ruling majority may nevertheless wish to prevent constitutional changes. In fact, the right-wing coalition led by Prime Minister Viktor Orbán fell short of this super-majority when it lost a by-election in February 2015. The by-election winner, Zoltán Kész, described his victory as "a yellow card to the government."[3]

A second scenario is when a voter wishes to send a message to policymakers. For example, a re-elected incumbent may choose to drop an unpopular policy if that policy's perceived popularity is low, or a winning candidate may respond on a particular policy dimension if support for the relevant single-issue party is large. As Franklin, Niemi, and Whitten (1994, p. 552) observed, "a voter might expressively vote for a small party in order to show support for the policies espoused by that party in the hopes that the voter's preferred party might be induced to adopt them." Similarly, Rosenstone, Behr, and Lazarus (1996) suggested that citizens cast a third-party ballot "to advance the same policy goals

[2]Such third-party opportunities arise elsewhere. For example Bowler and Lanoue (1992) studied the implications for Canadian voting behavior. Shifts in votes away from established leading parties are also a feature beyond national boundaries. For example, elections to the European Parliament do not determine the identity of a ruling government, and this can enable voters "to express their opposition to a particular government" or "to signal their preferences on a particular policy issue they care about which the main parties are ignoring, such as the environment, or immigration" (Hix and Marsh, 2011, p. 5).

[3]See, for example, the BBC News report: `http://www.bbc.co.uk/news/world-europe-31576491`.

they were precluded from achieving from within the major parties." An example of the "unpopular policy" situation might be the Poll Tax (officially, the Community Charge) of Margaret Thatcher's Conservative administration in the 1980s. An example of the "single issue" protest vote is the aforementioned support for anti-EU UKIP candidates in the United Kingdom. Here, the desired mainstream policy response might be exit from the EU, restructuring of the UK's EU membership, or the (already conceded) agreement to an exit referendum.[4] In these situations a policymaker responds to the protesters' demands if she believes that voters' feelings are sufficiently strong. Her perception of these feelings is determined by the election result. A natural strategy, and one which emerges from a formal signal-jamming model, is one in which the mainstream candidate responds (by dropping an unpopular policy, or by moving forward on the relevant policy issue) if the size of the protest exceeds a critical threshold.

In these scenarios, a vote can be pivotal in two ways: it may (if cast for the mainstream candidate) tip the balance to enable the candidate to win; however, a protest vote (against the candidate) may successfully constrain her power (the first scenario) or induce a policy change (the second scenario). A voter contemplates the relative likelihood of these events. This computation is related to classic strategic-voting scenarios. In a plurality-rule strategic-voting situation, a voter compares the probability that a sincere vote enables his favorite to win to the probability that a strategic vote for a less-preferred candidate defeats a disliked third opponent. The fundamental force is one of strategic complements: if others vote strategically, then this (heuristically) enhances the incentive to join them. In contrast, the force in a protest-voting scenario is one of strategic substitutes: if others engage in protest voting, then (again heuristically) a voter becomes more concerned with ensuring that the mainstream candidate does not lose.[5]

The most interesting findings that emerge concern the effect of the perceived strength of enthusiasm for the protest issue. The direct effect of an increase in such expected strength is (naturally) to increase protest voting. However, the expectation of greater protest voting from others reduces the incentive to protest, and so offsets the direct effect. The offset becomes exact as voters' beliefs about the electoral situation become precise. Fixing the response of a policymaker to the election outcome, this means that the extent of protest voting is unrelated to voters' true concerns about the protest issue.

[4]A related example emerges from votes cast for the Green Party in the United Kingdom. Sufficient electoral support would allow them to participate (and so promote environmental issues) in General Election television events. Ofcom (the broadcast regulator in the UK) judged the Green Party to have fallen below the threshold required for major party status in the 2015 election.

[5]In the context of my model, I confirm this. However, when I allow (in Section 5) voters to receive private signals of electorate-wide preferences, I find that the situation is more nuanced: a greater response by others to their private signals (a stronger tendency to cast a protest vote when a signal reveals others are less enthusiastic) can induce a voter to respond more strongly to his own private signal.

The offset effect becomes even stronger once the endogenous response of a policymaking politician is considered. The ballot-box support for a single-issue protest party is an informative signal of concern for that party's issue. In this context, a protest vote is an act of signal jamming. A politician understands this, and accounts for the endogenous presence of signal-jamming protest votes when she makes inferences. If she expects (fixing a voter's preferences) greater willingness to engage in protest voting, then she optimally discounts (and so reacts less to) any protest. The feedback effect from this signal-jamming logic can be so strong that stronger enthusiasm (both expected and realized) for a protest issue can actually reduce protest voting overall. Concretely, imagine a situation in which a politician reacts if the size of the protest crosses a line in the sand. The direct effect of stronger enthusiasm for the protest issue is to increase protest voting. However, the strategic-substitutes logic feeds back into a reduced tendency to engage in protest voting. Voters are now less willing to protest, and so even a limited protest strongly indicates popular disquiet. This endogenously moves the line in the sand: a smaller protest vote is needed to induce a reaction. This further lessens the incentive for protest voting. Following this logic to its equilibrium conclusion, the overall effect can be a net loss in ballot-box support for the single-issue protest party.

Emerging from this logic is a key finding: the number of votes cast for single-issue protest parties can be negatively related to the true strength of the enthusiasm for their causes. Equivalently, the recent increase in the vote share for a party such as UKIP in the United Kingdom does not necessarily imply that anti-European sentiments have strengthened throughout the electorate. This, then, is a central take-home message.

Beyond this applied message, the paper offers a further theoretical contribution via an extended model in which voters receive additional private signals (beyond the information contained in their own preference realizations) about the popular enthusiasm for the protest issue. Here I find (perhaps counter-intuitively) that the strength of voters' reactions to their private information exhibits the properties of a strategic complement: if other voters react more to their private signals, then this reinforces an individual's incentive to do so. Interestingly, the results for this case (protest voting with private signals) are mirror images of those obtained in closely related models of strategic voting.

The paper follows a conventional structure: following a discussion of related literature (Section 1) I analyse a model in which the candidate's behavior is exogenous (Sections 2–3). I then extend the model to allow for an endogenous candidate response (Section 4) and to consider a richer information structure in which voters receive additional private signals of the electoral situation (Section 5) before offering concluding remarks (Section 6). Further extensions are in Appendix A and proofs are in Appendix B.

1. Related Literature

I have noted that there is strong empirical evidence that a quantitatively significant fraction of voters switch away from leading candidates in the United Kingdom (Franklin, Niemi, and Whitten, 1994) and the United States (Burden, 2005). There is further evidence for many other countries including Austria, Denmark, Italy, Norway, Spain, Sweden, and others (van der Brug, Fennema, and Tillie, 2000; Bergh, 2004; Erlingsson and Persson, 2011; Campante, Durante, and Sobbrio, 2013; Superti, 2014).

Nevertheless, there are relatively few theoretical models of protest voting. This paper contributes by providing an equilibrium analysis of protest voting, and (amongst other comparative-statics) by highlighting the possibility of a negative relationship between the size of protests and the true preference for them. It is related to work which considers the signal-jamming role of election results, including work by Piketty (2000), Castanheira (2003), Razin (2003), and Meirowitz and Shotts (2009).[6] The modeling technology uses techniques from analyses of strategic voting in the presence of aggregate uncertainty; the relevant papers are those by Myatt (2007) and Dewan and Myatt (2007).[7]

Recent contributions to a broader theory of voting have identified different routes via which a vote may be instrumental. For example, Castanheira (2003) observed that a vote may be "outcome pivotal" (it determines the election result) but also "communication pivotal" (it changes others' future behavior by influencing how they learn about the world). Piketty (2000) identified three channels for communicative voting: firstly, voters may wish to induce policy shifts by mainstream parties; secondly, they may wish to learn about candidates in order to assist the coordination of votes in future elections; and, thirdly, voters may wish to use their votes to influence others' opinions and so others' future votes. He concentrated on the third of these channels. My model is focused on the first channel, and other recent contributions to the literature share that focus.

Shotts (2006) studied a two-election model in which office-motivated candidates infer the preferred policy of the median voter from a first election, and move to that policy ready for the second election; some voters face an incentive to engage in signal-jamming in the first election. He described an equilibrium in which moderate voters abstain in the first-election. Such abstention was ruled out by Meirowitz and Shotts (2009). Furthermore, using the Shotts (2006) model, Hummel (2011) demonstrated that abstention vanishes

[6]Beyond those discussed here, there are other related recent contributions. Kselman and Niou (2011) extended their work on strategic voting (Kselman and Niou, 2010) and described the situations in which a protest vote might make sense, but they did not conduct a game-theoretic analysis; Kang (2004) discussed the application of "exit and voice" ideas (Hirschman, 1970) to protest voting; Smirnov and Fowler (2007) considered the influence of margins of victory on candidates' future positions; and Smith and Bueno de Mesquita (2012) considered how voters may vote for a disliked incumbent to obtain a district-specific prize.
[7]Other recent papers which emphasize the importance of aggregate uncertainty include models of voter turnout (Myatt, 2012; Evren, 2012) and associated party campaigns (Mandler, 2013).

in a large election. Meirowitz and Shotts (2009) found that the long-run signal-jamming incentive (to influence candidates' future policies) dominates the short-run instrumental incentive (to elect the favored candidate in the first election) when the electorate is large. The authors of these papers specified models in which there is no aggregate uncertainty; voters' types are independent draws from a known distribution. My paper shares with these the feature that voters may engage in signal jamming to influence policy. However, unlike these papers I use a model in which there is aggregate uncertainty.[8]

Aggregate uncertainty features in the model of Razin (2003). In his common-value world, centrist voters may be hit with a shock that moves their (common) position. Candidates respond (when they win) to the inferred shock but also incorporate their own biases into their policy choices. Voters receive private binary signals of the shock. Razin (2003) identified the tension between the signaling (moving the policy) and pivotal (choosing the right winner) motivations for vote choices. A distinction between his work and mine is that Razin (2003) considered a common-value environment whereas I consider a private-value world. There are extensive modeling differences too.[9]

Aggregate uncertainty is also present in the model of Castanheira (2003). Actors learn about the median voter by observing an election, and this influences subsequent policy positions. However, the model specification means that (Castanheira, 2003, p. 1208) "observing the vote results of only two parties is not sufficient to learn where the median voter stands" and so "the vote share of losers thus reveals additional information." This generates votes for extremists (via voters who pursue a communication objective) and the anticipation of this can influence the positions of mainstream candidates.

One interpretation of my paper is that it confirms results that emerge from simpler models in which voters' types are independent draws from a known distribution. For example, if those types are known and homogeneous then the ratio of pivotal probabilities (between the probability of determining the candidate's election and the probability of enabling a successful protest) must be equal to a constant. This can happen only when the probability a vote is equidistant (in an appropriate sense) between the thresholds needed for candidate and protest successes, which implies that vote shares are invariant to popular enthusiasm for the protest. The paper's early results show that this is robust to slight aggregate uncertainty and to heterogeneous voters. However, the paper goes further. Aggregate uncertainty opens up the possibility of signal-jamming behavior; it enables

[8]Relatedly, Meirowitz and Tucker (2007) proposed a three-voter model in which a poor showing for a candidate induces her to increase her effort rather than change her policy position. Other more distantly related contributions to this strand of the literature include the analysis of voters' strategic responses to polls (Meirowitz, 2005b) and the analysis of voting and candidate behavior in primaries when those primaries reveal information relevant to a subsequent general election (Meirowitz, 2005a).

[9]The common vs. private-value distinction arises in a comparison with work by McMurray (2014). He considered voters who receive private signals of an ideal policy, and candidates learn from the election. This relates to the "swing voter's curse" papers of Feddersen and Pesendorfer (1996, 1997, 1998).

preference intensity and the protest's popularity to influence (non-monotonically) vote shares and political outcomes.

The modeling technology exploits the relationship between protest voting and strategic voting. The model has anti-coordination features: the best outcome for voters is obtained if some but not all protest. This contrasts with a classic strategic-voting setting in which voters choose between two challengers when they wish to defeat a disliked opponent: coordination behind either challenger produces a good outcome, but a split allows the disliked opponent to sneak through. Older analyses of strategic voting (Palfrey, 1989; Myerson and Weber, 1993; Cox, 1994) omitted aggregate uncertainty and predicted (if unstable equilibria are put aside) full coordination. More recently, however, Myatt (2007) included aggregate uncertainty and predicted multi-candidate support. With a common-value specification, Dewan and Myatt (2007) used a related model to study the coordinating effects of party leadership. This paper uses the technical elements of these antecedents, but where payoffs reward anti-coordination rather than coordination.[10]

There are also many models of protests that are not focused on protest voting. Such models often involve an element of team-based collective action: an act of protest is costly, and the protest succeeds (generating a public good that is valued by the protesters) if and only if it is sufficiently large. For many models in this tradition, individual protesters view their own actions as instrumentally negligible. Instead, payoffs are structured such that a protester gains by joining a successful protest, and does not enjoy the fruits of the protest if he refrains from participation. Thus a protester asks himself "what is the chance that the protest succeeds?" rather than "how likely am I to be pivotal to the success of the protest?" In such models, the key strategic force is one of strategic complements rather than substitutes. Recent models of protest and rebellion have been inspired by earlier work connecting mass political action elite responses (Lohmann, 1994, 1993) and have drawn upon techniques of global games (Carlsson and van Damme, 1993; Morris and Shin, 1998, 2003; Hellwig, 2002; Angeletos, Hellwig, and Pavan, 2007) in which players receive signals of some variable that determines the nature of the game being played. Versions of such regime-change models have included opportunities for leaders to engage in signaling, signal-jamming, or other early-stage actions that influence the play of a coordination game (Bueno de Mesquita, 2010; Little, 2012; Edmond, 2013; Egorov and Sonin, 2014; Little, Tucker, and LaGatta, 2014).[11]

[10]Protest votes are (heuristically) strategic substitutes. This relates protest voting to the turnout decision. Recent theories of turnout exhibit an underdog effect that is related to the offset effect highlighted here (Krishna and Morgan, 2011, 2012; Myatt, 2012; Evren, 2012; Faravelli, Man, and Walsh, 2013; Herrera, Morelli, and Palfrey, 2014; Kartal, 2014; Faravelli and Sanchez-Pages, 2014).

[11]Specifically, the models of Edmond (2013) and Egorov and Sonin (2014) allow a leader to observe and then signal a relevant underlying state variable; the vanguard player in Bueno de Mesquita (2010) cannot observe the state variable but can jam that variable with costly effort; and in Little (2012) and Little, Tucker, and LaGatta (2014) an incumbent can control the release of a state-variable-relevant public signal. Other

2. A Model of Protest Voting

Players. There are n voters ("he") and a mainstream candidate for office ("she"). For now I fix exogenously the behavior of the candidate, and focus on a simultaneous-move game played by the voters. In Section 4 the candidate becomes active as a player.

Moves and Outcomes. Each voter either votes for the candidate, or he casts a protest vote. I write b for the number of protest ballots. There are three possible outcomes: (i) if b is small then the candidate wins but the protest fails; (ii) if b is moderately large then she also wins, but the protest is large enough to succeed; and (iii) if b is large enough then the loss in support for the mainstream candidate is enough for her to lose the election.

Formally, there are two thresholds p_L and p_H satisfying $0 < p_L < p_H < 1$ such that

$$\text{outcome} = \begin{cases} \text{protest fails \& candidate wins} & \text{if } \frac{b}{n} < p_L, \\ \text{protest succeeds \& candidate wins} & \text{if } p_L < \frac{b}{n} < p_H, \text{ and} \\ \text{candidate loses} & \text{if } p_H < \frac{b}{n}. \end{cases} \tag{1}$$

The parameter p_H is exogenous throughout. However, when the endogenous reaction of the candidate is permitted (in Section 4) the parameter p_L becomes endogenous.

Voters' Payoffs. A voter would like the candidate to win but for the protest to succeed. Formally, the payoff of voter i is determined solely by the election outcome, and

$$U_i^{\text{win}} > \max\left\{U_i^{\text{lose}}, U_i^{\text{fail}}\right\}, \tag{2}$$

where the notation should be clear following an inspection of Figure 1.

The desire to register a sufficiently large protest (this is $U_i^{\text{win}} - U_i^{\text{fail}}$) must be balanced against the desire to avoid an outright loss (this is $U_i^{\text{win}} - U_i^{\text{lose}}$). Taken together,

$$u_i \equiv \log\left[\frac{U_i^{\text{win}} - U_i^{\text{fail}}}{U_i^{\text{win}} - U_i^{\text{lose}}}\right] \tag{3}$$

is the preference type of voter i. When u_i is higher a voter cares more about a successful protest relative to ensuring a win for the candidate; hence, voters with higher preference types are more willing to cast a protest vote. Voters' preferences are heterogeneous. There is an average type θ, and individual types are normally distributed around it:

$$u_i \,|\, \theta \sim N(\theta, \sigma^2), \tag{4}$$

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researchers have considered noisy signaling models in which a leader's move is observed with noise (Hollyer, Rosendorff, and Vreeland, 2014, 2015), they have incorporated multiple mass-action groups such as citizens and elite members (Casper and Tyson, 2014), and they have included repression and punishment (Shadmehr and Bernhardt, 2011; Tyson and Smith, 2014).

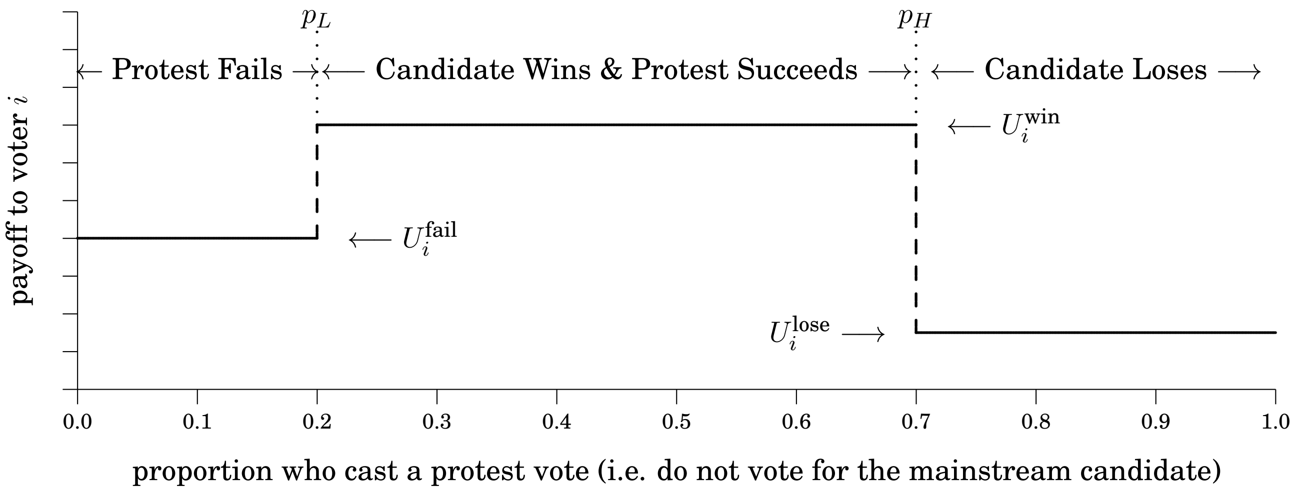

FIGURE 1. Outcomes and Voters' Payoffs

where types are conditionally independent, and so $\operatorname{cov}[u_i, u_{i'} \mid \theta] = 0$ for $i \neq i'$. The average θ is the true underlying strength of feeling about the protest issue, relative to the popularity of the candidate. θ is unknown: there is aggregate uncertainty about electorate-wide preferences, and so unconditionally voters' types are correlated.

Note that there is aggregate uncertainty about the median preference intensity, but the fraction of the electorate who support the protest issue is fixed. Nevertheless, the model can be modified to allow for uncertainty over this fraction.[12]

Information. Prior to the realization of their types, voters commonly believe that

$$\theta \sim N\left(\mu, \frac{\sigma^2}{\psi}\right), \tag{5}$$

and so ψ measures the accuracy of beliefs about underlying preferences. A voter's type u_i is a signal of θ with precision $1/\sigma^2$.[13] Updating beliefs introspectively,

$$\theta \mid u_i \sim N\left(\frac{\psi\mu + u_i}{\psi + 1}, \frac{\sigma^2}{\psi + 1}\right). \tag{6}$$

A voter's expectation of others' preferences is related to his own preference. Also:

$$\mathrm{E}[u_j \mid u_i] = \frac{\psi\mu + u_i}{\psi + 1}, \quad \operatorname{var}[u_j \mid u_i] = \frac{(\psi + 2)\sigma^2}{\psi + 1}, \quad \text{and} \quad \operatorname{cov}[u_j, u_{j'} \mid u_i] = \frac{\sigma^2}{\psi + 1}. \tag{7}$$

(In Section 5 I extend the model to allow for additional informative signals about θ.)

[12]One way to do this is to set $u_i \equiv (U_i^{\text{win}} - U_i^{\text{fail}})/(U_i^{\text{win}} - U_i^{\text{lose}})$, so that a voter wishes the protest to succeed if and only if $u_i > 0$, and where the fraction of protest-supporters is $\Phi(\theta/\sigma)$. This is discussed in Appendix A. An issue that arises is that (for certain specifications) very high type voters switch away from casting a protest vote because the force of strategic substitutes becomes overwhelmingly strong for them.

[13]Another version of the model (described in Appendix A) is obtained when beliefs about θ are independent of voters' types. For such a "common beliefs" specification a unique cutpoint equilibrium always exists; if voters update their beliefs introspectively then (as I explain in Section 3) a restriction on σ is required.

Solution Concept. A voter's strategy is the probability that he protests conditional on his type. A cutpoint strategy is very natural: for some cutpoint $u^\star$, voter i protests if $u_i > u^\star$ but votes for the candidate if $u_i < u^\star$. Type-symmetric equilibria in which the probability of a protest vote is increasing in a voter's type always take this form.[14]

The usual solution concept would be a type-symmetric (Bayesian) Nash equilibrium. Here, however, the primary focus is on behavior in large electorates. One reason for this is to ensure that uncertainties over idiosyncratic type realizations do not drive things. A second (pragmatic) reason is that the solutions simplify when n is large. A standard approach is to find an equilibrium cutpoint $u^\star_n$ for each n and then examine $\lim_{n\to\infty} u^\star_n$. Here, however, I follow earlier work (Myatt, 2007; Dewan and Myatt, 2007) by defining a single solution concept over the sequence of voting games indexed by the electorate's size. Specifically, I seek a single cutpoint $u^\star$ with the property that each voter type finds himself playing a strict best response if the electorate is sufficiently large.[15]

Definition. *A cutpoint $u^\star$ yields a* voting equilibrium *if, when all others use this cutpoint, each voter type $u_i \neq u^\star$ strictly prefers to use the cutpoint if n is sufficiently large.*

This solution concept is a shortcut: it defines directly the equivalent of a limiting equilibrium cutpoint without resorting to the computation of exact equilibria for each finite n. Happily, an orthodox approach generates the same answer: a sequence of Bayesian Nash equilibrium cutpoints converges to the voting equilibrium as $n \to \infty$ (Appendix A).[16]

Interpretation. Here I interpret the specification in the context of a specific case: the anti-EU protest votes cast for UKIP in the United Kingdom.

In this context, the players might be supporters of the (right-of-centre) Conservative Party who are sympathetic to the exit of the UK from the European Union. If a sufficiently large fraction (more than p_H) cast protest votes, then the Conservatives lose power. In a two-candidate election where the Conservatives and UKIP are candidates, and where all voters are anti-EU but pro-Conservative, an appropriate specification might be $p_H = \frac{1}{2}$. If the players are a subset of Conservatives (for instance, when pro-EU Conservative supporters always vote for the Conservative candidate) then $p_H > \frac{1}{2}$ might be appropriate: in this situation a greater fraction of anti-EU voters need to protest in order to risk a Conservative loss. If there are other competitors, such as UKIP members

[14]An equilibrium is type symmetric if two voters with the same type realizations behave in the same away. This means that an individual voter's choice does not depend on his player label.
[15]For each $u_i \neq u^\star$ there is an electorate size $\bar{n}(u_i)$ such that type u_i has no incentive to deviate if $n \geq \bar{n}(u_i)$.
[16]For any finite electorate size, there may be some types who do not play a best reply. In this sense, a voting equilibrium is a kind of ε-equilibrium. However, the set of types who can profitably deviate shrinks as n increases, and the payoff gain from a deviation falls in an appropriate sense. Myatt (2007) and Dewan and Myatt (2007) discussed this solution concept; a summary is repeated in Appendix A.

who wish the Conservatives to lose, then $p_H < \frac{1}{2}$ could be appropriate: in this situation it is easier for a protest by anti-EU Conservatives to cause an (unwanted) UKIP win.

The lower threshold p_L has multiple interpretations. One interpretation is that it is the fraction of Conservative voters who need to protest in order to stop an overall Conservative majority (while retaining their largest party status) and so force them to adopt anti-EU policies as a price for winning UKIP parliamentary support. A second interpretation, and the preferred one here, is that a sufficiently large protest persuades the Conservatives to change policy. If UKIP are the protest party, then such a concession might be an EU exit referendum (although this has already been promised), the modification of the UK's relationship with the EU, or the commencement of EU exit negotiations.

Here, the response to the protest is binary. Whereas the electoral success of the candidate is naturally binary, arguably the impact of protest votes may vary continuously. For example, the scale of Conservative concessions to anti-EU demands could increase as UKIP's vote share rises. The "succeed or fail" specification is simply a modeling choice. A related model can be built (one is considered in Appendix A) in which the success of the protest varies continuously with the fraction of ballots cast as protest votes.

3. Equilibrium Analysis

Optimal Voting. A voter's decision can matter in two ways: a protest vote may push the outcome above the lower threshold p_L, ensuring that the protest succeeds (good) or it may cause the outcome to cross the higher threshold p_H, causing the candidate to lose (bad). The first effect yields a gain of $U_i^{\text{win}} - U_i^{\text{fail}}$ whereas the second effect generates a loss of $U_i^{\text{win}} - U_i^{\text{lose}}$. It is strictly optimal to cast a protest vote if and only if

$$\Pr[\text{Pivotal at } L \,|\, (u^\star, u_i)] \left(U_i^{\text{win}} - U_i^{\text{fail}}\right) > \Pr[\text{Pivotal at } H \,|\, (u^\star, u_i)] \left(U_i^{\text{win}} - U_i^{\text{lose}}\right)$$
$$\Leftrightarrow \quad u_i + \log\left[\frac{\Pr[\text{Pivotal at } L \,|\, (u^\star, u_i)]}{\Pr[\text{Pivotal at } H \,|\, (u^\star, u_i)]}\right] > 0, \quad (8)$$

where "$(u^\star, u_i)$" indicates that the probabilities are evaluated conditional on the use of the cutpoint $u^\star$ by other voters and on a voter's own type u_i. A cutpoint strategy generates positive pivotal probabilities, and so re-arrangement yields the second inequality.

Inspecting this second inequality, the first term u_i can be interpreted as the sincere incentive to cast a protest vote: u_i is positive if and only if a voter cares more about a successful protest than he does about electing the mainstream candidate. The second term (the log odds of pivotal events) can be interpreted as the strategic incentive to protest: it is positive if and only if a voter is more likely to determine a successful protest than he is to determine the election of the candidate.

Pivotal Probabilities. The pivotal probabilities in inequality (8) are determined by a voter's beliefs about the votes cast by others. Given that others use a cutpoint strategy, the probability p that a given voter casts a protest ballot against the candidate is

$$p = P(\theta) \quad \text{where} \quad P(\theta) \equiv \Phi\left(\frac{\theta - u^\star}{\sigma}\right), \tag{9}$$

and where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. (Equivalently, the average voter type that induces the voting probability p is $\theta = u^\star + \sigma\Phi^{-1}(p)$.) Of course, θ is uncertain, and therefore so is p. A voter's beliefs about θ transform into beliefs about p, which I represent by the density $f(p \mid (u^\star, u_i))$. Those beliefs about θ depend on the prior $\theta \sim N(\mu, \sigma^2/\psi)$ and a voter's own type realization u_i. I write $g(\theta \mid u_i)$ for the (normal) posterior density of a voter's beliefs. Changing variables,

$$f(p \mid (u^\star, u_i)) = \left.\frac{g(\theta \mid u_i)}{P'(\theta)}\right|_{\theta = u^\star + \sigma\Phi^{-1}(p)} = \frac{\sigma g(u^\star + \sigma\Phi^{-1}(p) \mid u_i)}{\phi(\Phi^{-1}(p))}, \tag{10}$$

where the second equality is obtained by differentiating $P(\theta)$ from equation (9) and by substituting in $\theta = u^\star + \sigma\Phi^{-1}(p)$, and where $\phi(\cdot)$ is the density of the standard normal distribution. The density $g(\cdot \mid u_i)$ is available in closed form (it is a normal density) but I postpone briefly the substitution of the full density formula.

A voter's beliefs about p can be used to derive the probabilities of pivotal events. Conditional on p, the votes of others are binomial with parameters p and $n-1$. Taking expectations over p, the probability that an extra vote enables a successful protest is:

$$\Pr[\text{Pivotal at } L \mid (u^\star, u_i)] = \binom{n-1}{\lfloor p_L n \rfloor} \int_0^1 p^{\lfloor p_L n \rfloor} (1-p)^{(n-1)-\lfloor p_L n \rfloor} f(p \mid (u^\star, u_i))\, dp, \tag{11}$$

where $\lfloor p_L n \rfloor$ is the greatest integer that is weakly smaller than $p_L n$. A similar expression holds for the other pivotal probability. As n grows this pivotal probability shrinks. More subtly, the polynomial term in the integrand is sharply peaked around p_L, and so as n increases only the density around p_L matters. A variant of results from Good and Mayer (1975) and Chamberlain and Rothschild (1981) generates Lemma 1.

Lemma 1 (Pivotal Probabilities). *If protest votes are cast with probability p, where $p \sim f(\cdot)$, and if the density $f(\cdot)$ is positive and continuous around p_L and p_H, then* $\lim_{n\to\infty} n \Pr[\text{Pivotal at } L] = f(p_L)$ *and* $\lim_{n\to\infty} n \Pr[\text{Pivotal at } H] = f(p_H)$.

A corollary of this lemma is that the log odds ratio of pivotal events satisfies

$$\lim_{n\to\infty} \log\left[\frac{\Pr[\textbf{Pivotal at } L] \mid (u^\star, u_i)]}{\Pr[\textbf{Pivotal at } H \mid (u^\star, u_i)]}\right] = \log\left[\frac{f(p_L \mid (u^\star, u_i))}{f(p_H \mid (u^\star, u_i))}\right] \tag{12}$$

$$= \log\frac{\phi(\Phi^{-1}(p_H))}{\phi(\Phi^{-1}(p_L))} + \log\frac{g(u^\star + \sigma\Phi^{-1}(p_L) \mid u_i)}{g(u^\star + \sigma\Phi^{-1}(p_H) \mid u_i)} \tag{13}$$

$$= \frac{z_L^2 - z_H^2}{2} + \log\left[\frac{g(u^\star + \sigma z_L \mid u_i)}{g(u^\star + \sigma z_H \mid u_i)}\right] \tag{14}$$

$$\text{where } z_H \equiv \Phi^{-1}(p_H) \text{ and } z_L \equiv \Phi^{-1}(p_L). \tag{15}$$

Recall that $g(\theta \mid u_i)$ is the density of voter i's beliefs about the average voter type θ. The first equality is from an application of Lemma 1, the second from substitution of equation (10), and the third applies the formula for the standard normal density.

Properties of Best Replies. Using the expression for the log odds of pivotal events, if

$$u_i + \frac{z_L^2 - z_H^2}{2} + \log\left[\frac{g(u^\star + \sigma z_L \mid u_i)}{g(u^\star + \sigma z_H \mid u_i)}\right] > 0 \tag{16}$$

and if the electorate is sufficiently large then voter i has a strict incentive to cast a protest ballot. Note that if the density $g(\theta \mid u_i)$ is log concave in θ (this is true here because posterior beliefs are normal) then the left-hand side of the inequality is increasing in $u^\star$. This implies that a reduction in protest voting by others (an increase in the cutpoint $u^\star$) raises the incentive for voter i to protest: protest votes are strategic substitutes.

Whereas the incentive to protest is increasing in $u^\star$, it is not necessarily increasing in a voter's own type. Of course, the first term in the inequality is simply u_i, and so an increase in the relative desire for a successful protest directly increases the incentive to protest. However, the final term is decreasing in u_i. To see this, recall from equation (6) that the posterior beliefs of voter i satisfy

$$\theta \mid u_i \sim N\left(\frac{\psi\mu + u_i}{\psi + 1}, \frac{\sigma^2}{\psi + 1}\right). \tag{17}$$

Using the formula for the normal density, and re-arranging,

$$\log\left[\frac{g(u^\star + \sigma z_L \mid u_i)}{g(u^\star + \sigma z_H \mid u_i)}\right] = -\frac{\sigma(\mathrm{E}[\theta \mid u_i] - u^\star)(z_H - z_L)}{\mathrm{var}[\theta \mid u_i]} + \frac{\sigma^2(z_H^2 - z_L^2)}{2\,\mathrm{var}[\theta \mid u_i]} \tag{18}$$

$$= \frac{(1+\psi)(z_H - z_L)}{\sigma}\left(u^\star - \frac{\psi\mu + u_i}{1 + \psi}\right) + \frac{(1+\psi)(z_H^2 - z_L^2)}{2}. \tag{19}$$

This is decreasing in u_i. If u_i rises then voter i receives a stronger private signal (via his preference realization) about the strength of feeling for the protest issue. Given the play of cutpoint strategies by others, this leads him to believe that his vote is relatively more likely to save the mainstream candidate from defeat than to enable the success of the protest. If the size of this effect (which is driven by the strategic-substitutes logic) is sufficiently strong then an increase in a voter's preference type (in favor of the protest) can cause him to switch his vote back to the mainstream candidate.

Lemma 2 (Strategic Substitutes)**.** *If other voters use a cutpoint strategy then more protest voting by others (a fall in $u^\star$) reduces the incentive for a voter to engage in protest voting. The incentive to protest is increasing in a voter's own type if and only if $\sigma > z_H - z_L$.*

Equilibrium. A consequence of Lemma 2 is that the best reply to a cutpoint strategy can be a "flipped" strategy in which those who are less enthusiastic about the protest issue cast protest ballots. This implies that a voting equilibrium cannot exist if $\sigma < z_H - z_L$. In fact, there is no type-symmetric monotonic equilibrium in this case.[17]

If voters are sufficiently heterogeneous, however, so that $\sigma > z_H - z_L$, then a best reply to the use of a cutpoint strategy by others is itself a cutpoint strategy. The cutpoint $u^\star$ yields an equilibrium if the inequality (16) holds as an equality when $u_i = u^\star$. That is,

$$u^\star + \frac{z_L^2 - z_H^2}{2} + \log\left[\frac{g(u^\star + \sigma z_L \,|\, u_i = u^\star)}{g(u^\star + \sigma z_H \,|\, u_i = u^\star)}\right] = 0. \tag{20}$$

Using equation (19) to substitute in for the final term yields a linear equation in $u^\star$. This solves straightforwardly to yield a unique solution for the equilibrium cutpoint.

Proposition 1 (Equilibrium)**.** *If $\sigma < z_H - z_L$ then a voting equilibrium does not exist. If $\sigma > z_H - z_L$ then there is a unique voting equilibrium with cutpoint*

$$u^\star = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma}\left[\mu - \frac{\sigma(z_H + z_L)}{2}\right] \tag{21}$$

where $z_L \equiv \Phi^{-1}(p_L)$, $z_H \equiv \Phi^{-1}(p_H)$, and $\Phi(\cdot)$ is the standard normal distribution.

For the remainder of this section I restrict to $\sigma > z_H - z_L$ so that an equilibrium exists.

The Willingness to Protest. A cutpoint $u^\star$ determines whether a voter casts a protest ballot. When conducting comparative-static exercises I say that a parameter change "increases the willingness to protest" if and only if it decreases the cutpoint.

Recall that $\theta \sim N(\mu, \sigma^2/\psi)$ is the average voter type. It is the average relative preference for a successful protest versus the election of the mainstream candidate, and so captures the strength of voters' enthusiasm about the protest issue.

The equilibrium cutpoint is increasing in μ and so greater expected enthusiasm lowers the willingness to protest. The effect of the precision ψ of beliefs depends upon the sign of the bracketed term of equation (21). Similarly, the effect of σ (the heterogeneity of voters' preferences) depends upon the region of the parameter space considered.

[17] If voters share the same beliefs about θ (so their beliefs are independent of their type realizations) then this is no longer a problem: a cutpoint voting equilibrium exists for all parameter values (Appendix A).

Proposition 2 (Comparative Statics)**.** *The willingness of voters to protest is*

(i) decreasing in the expected enthusiasm for the protest issue;

(ii) increasing in the precision of voters' prior beliefs if and only if $\mu < \sigma(z_L + z_H)/2$; and

(iii) increasing in the heterogeneity of voters' preferences if and only if $\mu > \psi(z_L^2 - z_H^2)/2$.

The first claim is a consequence of the logic (protest votes are strategic substitutes) underpinning Lemma 2, which yields an offset effect from increased expected enthusiasm. $u^\star$ is also supermodular in the mean and precision of beliefs ($\partial u^\star/\partial\psi\partial\mu > 0$) and so the offset effect becomes stronger as voters become more certain of how others feel.

The second claim, concerning the effect of belief precision, is more intricate. It is easiest to understand when $p_H = 1 - p_L$, so that coordination needed for the candidate to win equals that needed for a successful protest. In this case $z_L + z_H = 0$, and so the willingness to protest is increasing in the precision of beliefs if and only if $\mu < 0$. In this case $u^\star$ satisfies $\mu < u^\star < 0$. Thus an indifferent voter (with type $u_i = u^\star$) thinks that others are more likely than not to be more in favor of the candidate than he is. He concludes that they are more likely to vote for the candidate, and so the pivotal event L is relatively more likely than the pivotal event H. As the precision of beliefs increases (ψ rises) the pivotal event L becomes relatively more likely, and so protest voting increases.

The inequality in the third claim, concerning type heterogeneity, holds if and only if $u^\star < \mu$. Consider a voter who is indifferent between protesting and not. If $u^\star < \mu$ then he expects most others to be above him. An increase in σ^2 pushes more types below $u^\star$. This reduces protest voting by others, hence increasing his incentive to cast a protest ballot.

The equilibrium cutpoint also depends on p_H and p_L. (Recall that p_L is the coordination required for the protest to succeed, whereas $1 - p_H$ is the coordination,in the opposite direction, required to ensure that the candidate wins the election.) This dependence of $u^\star$ operates via $z_H - z_L$ and $z_H + z_L$. The former term is the size of the region within which voters' collectively achieve their ideal outcome. As this gap narrows (fixing $z_H + z_L$) the absolute size of $u^\star$ falls, and so $u^\star$ moves toward zero. When p_H and p_L are close the key pivotal events become equally likely, and so a voter makes his decision based upon whether the protest issue is more important (to him) than a win for the candidate. Fixing $z_H - z_L$, an increase in $z_H + z_L$ simultaneously increases the coordination needed for a successful protest (p_L is larger) and reduces the coordination needed for the candidate to win ($1 - p_H$ is smaller). This leads to greater protest voting (lower $u^\star$).

Changes in the individual parameters p_L and p_H are more involved. For example, an increase in p_L increases $z_H + z_L$ which strengthens protest voting; however, it also narrows the gap $z_H - z_L$ which shrinks the absolute size of the equilibrium cutpoint. If $u^\star > 0$ then the two effects play out in the same direction; however, if $u^\star < 0$ then they conflict.

Things are more straightforward when the precision of voters' prior beliefs is high:

$$\lim_{\psi \to 0} u^\star = \mu - \frac{\sigma(z_H + z_L)}{2} \tag{22}$$

This reveals the effect of the coordination (p_L and $1 - p_H$, respectively) needed for voters to achieve their objectives of a successful protest and the election of the candidate.

Proposition 3 (Need for Coordination)**.** *If the precision of voters' prior beliefs is sufficiently large then the willingness to protest is increasing in the coordination needed for a successful protest, but decreasing in the coordination needed for the candidate to win.*

Appendix A reports additional results concerning the effect of p_L and $1 - p_H$ (the levels of coordination required) that do not require ψ (the precision of prior beliefs) to be large.

The Impact of Protest Voting. Propositions 2 and 3 report how changes in the electoral environment (p_L and p_H) and the distribution of voters' preferences and beliefs about those preferences (μ, ψ, and σ^2) determine the willingness to protest. Fixing $u^\star$, however, these parameters also influence the amount of protest voting (via a change in the distribution of voters' preferences) and the outcome of the election (via p_L and p_H).

Of most interest are the competing effects of increased preference for the success of the protest. This is a double-edged sword for a single-issue protest party. An increase in the true average enthusiasm for the protest (that is, an increase in the true value θ) directly increases the number of protest votes. However, an increase in the perception of this (that is, an increase in μ) harms the chances of the protest's success via a lessened willingness to engage in protest voting (Proposition 2).

To examine directly the impact of the environment's parameters on protest voting, recall that $u_i \sim N(\theta, \sigma^2)$ and that $\theta \sim N(\mu, \sigma^2/\psi)$. Combining these two elements,

$$u_i \sim N\left(\mu, \frac{\sigma^2(1+\psi)}{\psi}\right). \tag{23}$$

A voter protests if u_i exceeds the equilibrium cutpoint $u^\star$. Hence

$$\Pr[\text{Protest}] = \Phi\left(\frac{\mu - u^\star}{\sigma}\sqrt{\frac{\psi}{1+\psi}}\right) = \Phi\left(\left[\mu + \frac{\psi(z_H^2 - z_L^2)}{2}\right]\frac{\sqrt{\psi/(1+\psi)}}{\psi(z_H - z_L) + \sigma}\right). \tag{24}$$

This is increasing in μ, and so the probability of a protest vote is increasing in the expected enthusiasm of voters for the protest issue. Thus, the direct effect of an increase in enthusiasm is not fully offset by the reduced willingness of individual voters to protest.

Nevertheless, the offset effect can be significant. This is easiest to see when voters have good prior knowledge about the average enthusiasm for the protest issue, so that $\psi \to \infty$. In this case, changes in μ have no effect on the probability of a protest vote.

Proposition 4 (Impact of Protest Voting)**.** *The probability of a protest vote satisfies*

$$\lim_{\psi\to\infty} \Pr[\textbf{Protest Vote}] = \Phi\left(\frac{z_H + z_L}{2}\right) \textit{ therefore } p_L < \lim_{\psi\to\infty} \Pr[\textbf{Protest Vote}] < p_H. \qquad (25)$$

If beliefs are precise (the limiting case as $\psi \to \infty$) then the amount of protest voting, the election result, and the protest outcome are independent of voters' preferences.

When beliefs about the enthusiasm for the protest issue are very precise (in essence, minimizing the aggregate uncertainty) then the amount of protest voting (and so the election outcome) is independent of how voters feel about the candidate and the protest issue.[18] The candidate always wins the election, and the protest always succeeds.

Aggregate Uncertainty. A feature of the model is the presence of aggregate uncertainty: θ (the median enthusiasm for the protest issue) is unknown. Other things equal, aggregate uncertainty is a desirable feature of a voting model.[19] Moreover, the analysis and results are not appreciably complicated by its presence. Nevertheless, it is instructive to consider, as a benchmark, a world in which θ is commonly known.

Consider first a completely stripped-down model in which $\theta = \sigma^2 = 0$: all voters are equally concerned with the election of the candidate and the success of the protest. In this setting an equilibrium can only be sustained when the pivotal probabilities are equal. This happens when the probability p of a protest vote lies at an (appropriately scaled) midpoint between the thresholds p_L and p_H. For example, if $p_H = 1 - p_L$ then the two pivotal events are equally likely if and only if $p = 1/2$.

Now suppose that $\theta \neq 0$. An equilibrium requires the log odds ratio of the pivotal events to equal a finite constant. If $p \neq 1/2$ then, in a large electorate, this log odds ratio diverges. To keep it finite (and so to maintain an equilibrium) requires $p \to 1/2$ as $n \to \infty$. Hence, any change in θ has (in a large electorate) no effect on the probability of a protest vote: there is a complete offset effect. This remains true if $\sigma^2 > 0$. More generally, for $p_H \neq 1 - p_L$, there is a complete offset effect of the kind described in Proposition 4.

One interpretation of the results so far, therefore, is that what we might expect to happen in a simpler world is robust to the introduction of slight aggregate uncertainty. For example, Proposition 4 may be interpreted as a robustness check. However, there are

[18] The logic is starkest when $p_L = 1 - p_H$, so that $z_H + z_L = 0$. For this case, the coordination required to elect the candidate is the same as the coordination required for a successful protest. As beliefs become precise the relative likelihood of one pivotal event versus the other diverges unless the probability of a protest vote satisfies $p \to \frac{1}{2}$. More generally, the limiting split between votes for the candidate and protest votes is tied down by the need to prevent the divergence of the ratio of pivotal probabilities.

[19] I have argued elsewhere (Myatt, 2007) that in large electorates any idiosyncratic uncertainty (the fact that individual voter types are uncertain even if the distribution from which they are independently drawn is known) is averaged out by the law of large numbers. This means that results that are driven by idiosyncratic uncertainty (as they most be if no other uncertainty is present) may be restrictive.

other results too. For example, Proposition 2 shows how the willingness to protest (and so protest voting itself) changes with the precision of voters' prior beliefs.

More importantly, however, aggregate uncertainty is central to the additional results that follow in Section 4 and 5. For example, in the next section I consider the endogenous behavior of the candidate who learns about the world (in particular, she learns about voters' true enthusiasm for the protest issue) from her observation of the election outcome. This would be impossible if aggregate uncertainty were absent: if θ were commonly known, then there would be nothing to learn, and so protest voting could not act as an informative signal to a political elite. Crucially this allows (as I will show) the offset effect to exceed the direct effect of increased enthusiasm for the protest.

4. Endogenous Candidate Response

So far, the reaction of the candidate to the protest has been exogenously specified: she concedes to the protesters' demands if the size of the protest vote exceeds p_L. Here I allow the candidate to react endogenously, hence deriving an equilibrium value for p_L.

The Candidate's Policy Choice. The candidate is now a policymaker. She chooses whether to adopt a policy demanded by the protesters. She wishes to do so if and only if she perceives the popular enthusiasm for it to be sufficiently high. For example, consider a policymaking politician contemplating a costly environmental initiative. If voters have environmental concerns then protest votes might be cast for a green candidate. A green vote is then an attempt to jam the signal of environmentalism amongst the electorate.

The candidate wishes to adopt the requested policy if and only if the average relative preference in favor of it (that is, θ) lies above a critical value $\theta^\dagger$. (A payoff $\theta - \theta^\dagger$ from adoption of the policy would imply this.) It might be expected that an office-seeking politician would seek to satisfy voters' desires and so always do what the protesters want. However, I have in mind a situation in which the set of voters playing the game is not necessarily the same as the wider population of supporters of the candidate. The candidate may be interested in pursuing the best policy as it applies to everyone, or may resist the urge to cave in to populist but (perhaps) undesirable demands.

I consider strategies for which the candidate moves forward with the policy if and only if the observed support for it (measured via the number of protest votes) is sufficiently high. Equivalently, she chooses the threshold $p_L \in [0, p_H]$. Voters use a cutpoint strategy.

Definition. *A real-valued pair $(u^\star, p_L) \in \mathcal{R} \times [0, p_H]$ yields an equilibrium if (i) the cutpoint $u^\star$ is a voting equilibrium and (ii) for realized vote shares $p \neq p_L$ the policymaking politician chooses strictly optimally if the electorate size n is sufficiently large.*

This definition insists upon a finite value for $u^\star$, and so I am ruling out equilibria in which voters ignore their type realizations and all take the same action.[20] Furthermore, I assume that σ^2 is large enough to ensure the existence of an equilibrium.[21]

Equilibrium. The equilibrium behavior of voters can be built upon the earlier results. If $1 > p_H > p_L$ then the equilibrium from Proposition 1 applies here. If $p_H = p_L$, so that the candidate never adopts the policy, then a voter optimally votes for her if and only if $u_i > 0$; this corresponds to $u^\star = 0$, and equation (21) continues to hold.

To make inferences, the candidate uses the fact that the probability of a protest vote is $p = \Phi((\theta - u^\star)/\sigma)$. Inverting, if the probability is p then the true protest enthusiasm is $\theta = u^\star + \sigma\Phi^{-1}(p)$. In a large electorate, the proportion of those protesting is close to p. Hence (if the electorate size n is sufficiently large) the candidate adopts the relevant policy if and only if $u^\star + \sigma\Phi^{-1}(p) > \theta^\dagger$ or, equivalently, if and only if $p > p_L$ where $p_L = \Phi\left((\theta^\dagger - u^\star)/\sigma\right)$, so long as $p_L < p_H$. Otherwise, the candidate never adopts the policy ($p_L = p_H$). Notice that there is feedback: if the willingness to protest rises (a fall in $u^\star$) then (given p) the candidate perceives less enthusiasm for the protest issue.

From this, it follows that an equilibrium of the protest-and-response game satisfies

$$u^\star = \frac{(z_H - z_L)\psi}{(z_H - z_L)\psi + \sigma}\left[\mu - \frac{\sigma(z_H + z_L)}{2}\right] \quad \text{and} \quad z_L = \min\left\{\frac{\theta^\dagger - u^\star}{\sigma}, z_H\right\}, \tag{26}$$

where as usual $z_L = \Phi^{-1}(p_L)$ and $z_H = \Phi^{-1}(p_H)$.

One possibility is that the protest is ignored. This corresponds to $z_L = z_H$ so that (via the first condition) $u^\star = 0$. Using the second condition, this works if and only if $\theta^\dagger > \sigma z_H$; this says that the enthusiasm for the protest issue needed to persuade the politician is large.

The other possibility is that a protest can prompt a response, so that $z_L < z_H$. The solutions for $u^\star$ and z_L are most easily obtained when voters' beliefs are precise. Allowing $\psi \to \infty$ the equilibrium conditions become

$$u^\star = \mu - \frac{\sigma(z_H + z_L)}{2} \quad \text{and} \quad z_L = \frac{\theta^\dagger - u^\star}{\sigma}. \tag{27}$$

The second equation corresponds to the reaction of the candidate. An increased willingness to engage in the protest (a fall in $u^\star$) raises z_L and so p_L: the candidate is less willing to respond. From the first equation, this increase in z_L prompts a further fall in $u^\star$: if the candidate becomes less responsive, then voters compensate by protesting more. This

[20] Such fully coordinated equilibria always exist. Note, however, that when $u^\star$ is large but finite then the candidate responds by setting $p_L = p_H$ (see the discussion below) which in turn leads back to $u^\star = 0$. Hence a fully coordinated equilibrium in which every voter protests (that is, $u^\star = \infty$) is not robust to a slight shift away to a situation with very high (but nevertheless incomplete) levels of protest voting. There are also problems with fully coordinated equilibria in which no voter protests.

[21] This corresponds to the condition $\sigma > z_H - z_L$ used in earlier sections. It can be dropped if the common beliefs specification (as described in Appendix A) is used.

feedback suggests that any factor which prompts an increase in protest voting (such as a fall in μ, which is a weakening of the expected enthusiasm for the protest issue) can be amplified. This effect is confirmed in the characterization of the equilibrium that follows.

Proposition 5 (Equilibria with an Endogenous Response)**.** *If $\theta^\dagger > \sigma z_H$ then there is an equilibrium in which the candidate always ignores the outcome of the election and keeps her default policy, so that $p_H = p_L$, and each voter i protests if and only if $u_i > 0$.*

If $\theta^\dagger < \sigma z_H$ then there is a unique equilibrium in which $p_H > p_L$. This satisfies

$$\lim_{\psi\to\infty} u^\star = \theta^\dagger + 2\max\{\mu - \theta^\dagger, 0\} - \sigma z_H \quad \textit{and} \quad \lim_{\psi\to\infty} z_L = z_H - \frac{2\max\{\mu-\theta^\dagger,0\}}{\sigma}. \tag{28}$$

If $\mu > \theta^\dagger$, so that the expected enthusiasm for the protest exceeds the enthusiasm required for the candidate to respond, then the equilibrium cutpoint satisfies (for precise beliefs) $u^* \to 2\mu - \theta^\dagger - \sigma z_H$. Of interest here is the response of $u^\star$ to μ: once the endogenous reaction of the candidate is incorporated, there is a two-for-one effect of expected enthusiasm for the protest issue on the willingness to protest.

Expectations and the Size of the Protest. An implication of Proposition 5 is this: greater popular enthusiasm for the protest issue can be associated with fewer protest votes. To see why, consider again (for expositional simplicity) the case where belief precision (that is, ψ) is high. I have noted that an increase in μ (greater expected enthusiasm for the protest issue) has a two-for-one effect on the equilibrium cutpoint $u^\star$. When beliefs are precise the expected enthusiasm goes hand-in-hand with the actual enthusiasm; this is a one-for-one effect. Thus, the two-for-one indirect effect (the reduced willingness to protest) offsets the direct one-for-one effect (the increased enthusiasm). This implies that, overall, an increase in μ results in an expected fall in protest voting.

Proposition 6 (The Expected Size of the Protest Vote)**.** *If $\theta^\dagger < \sigma z_H$ then*

$$\lim_{\psi\to\infty} \Pr[\text{Protest Vote}] = \Phi\left(z_H - \frac{|\theta^\dagger - \mu|}{\sigma}\right) \le p_H. \tag{29}$$

(i) The expected size of the protest is quasi-concave in the expected enthusiasm for it. It achieves a maximum satisfying $\lim_{\psi\to\infty}\Pr[\text{Protest Vote}] = p_H$ when $\mu = \theta^\dagger$. If $\mu \neq \theta^\dagger$ and if prior beliefs are precise then protest voting is insufficient to cause the candidate to lose.

(ii) If $\mu > \theta^\dagger$, so that the expected enthusiasm for the protest issue exceeds the level needed to induce the candidate to respond, then

$$\lim_{\psi\to\infty} p_L = \Phi\left(z_H - \frac{2(\mu-\theta^\dagger)}{\sigma}\right) < \lim_{\psi\to\infty}\Pr[\text{Protest Vote}] \tag{30}$$

and so (if beliefs are precise) the protest succeeds.

(iii) If $\mu < \theta^\dagger$ then $\lim_{\psi\to\infty} p_L = p_H > \lim_{\psi\to\infty}\Pr[\text{Protest Vote}]$ and the protest always fails.

One message emerging from this proposition is that (at least when $\mu > \theta^\dagger$) the twin endogenous responses to an increase in voters' strength of feeling for the protest issue are enough to reduce support for that issue at the ballot box. However, a reduction in the strength of feeling (equivalently, an increase in the popularity of the candidate) does not generate a sufficient loss in support for her to lose. A second message is that a protest is likely to be greatest in size when a policymaking candidate is most susceptible to learning from the protest and changing policy. If beliefs are precise and $\mu < \theta^\dagger$, then the candidate is almost sure ex ante that the popular enthusiasm for the protest issue is insufficient to respond; similarly, if $\mu > \theta^\dagger$ then she would accept the protesters' demands in the absence of any information obtained from the election. However, if $\mu \approx \theta^\dagger$ then, prior to the election, the candidate is indifferent between her policy options. It is in exactly this circumstance that protest voting is a substantial phenomenon.

The Success of the Protest and the Candidate. Proposition 6 characterizes the ex ante probability of a protest vote, and so the expected size of the protest, when voters' (and the candidate's) prior beliefs are precise ($\psi \to \infty$). Here I consider the case of general ψ, and evaluate the probabilities of success for the candidate and for the protest.

Recall that, conditional on the true average enthusiasm θ for the protest issue, the probability of a protest vote is $p = P(\theta) = \Phi((\theta - u^\star)/\sigma)$. In a large electorate the candidate loses (with probability converging to one as n grows large) if and only if this p exceeds p_L. Noting that $\theta \sim N(\mu, \sigma^2/\psi)$, the probability of this event is

$$\Pr[\text{Candidate Loses}] = \Pr[\theta > u^\star + \sigma z_H] = \Phi\left(\sqrt{\psi}\left(\frac{\mu - u^\star}{\sigma} - z_H\right)\right). \tag{31}$$

A similar computation yields the probability that p exceeds p_L so that the protest succeeds. These probabilities depend on $u^\star$ and (in the latter case) on p_L. The proof of Proposition 5 calculates closed-form solutions for $u^\star$ and p_L. These can be used to generate predictions regarding the success and failure of the candidate and the protest.

Proposition 7 (The Probabilities of Protest Success and Candidate Failure)**.** *If $\theta^\dagger < \sigma z_H$ then the probability that protest voting causes the candidate to lose the election is quasi-concave in μ. This probability is maximized when $\mu = \theta^\dagger + (\sigma^2/\psi)$. The probability that the protest succeeds is increasing in μ. That probability is equal to $1/2$ when $\mu = \theta^\dagger$.*

As the expected enthusiasm of voters for the protest issue grows, the probability of a successful protest grows with it. If $\mu < \theta^\dagger$ then, other things equal, the candidate would not accept the protesters' demands. In this range, increased expected enthusiasm also raises the probability that the protest movement unseats the candidate. Once μ grows sufficiently large, the risk of this undesirable (in the eyes of voters) outcome recedes; this is also within the range of parameters for which the candidate would in any case choose the policy requested by the protest movement. This discussion reinforces the message that the candidate's position is most precarious, and the success of the protest is most uncertain, when the candidate is close to indifferent ex ante.

5. Equilibria with Private Signals

So far I have considered a model in which voters base their beliefs on their preference-type realizations. There is a tension here: an increase in u_i raises the incentive of a voter to protest, but it also raises the the likelihood that others will protest. Here I study an expanded model in which voters receive additional private information, and so a voting strategy depends not only on a voter's preference type but also on his signal realization. I incorporate any information contained within u_i into that signal, which allows me to separate out the preference and information elements of a voter's type.

The focus of this section is the theoretical properties of such a voting game. I have noted that protest voting resembles an upside-down version of the strategic voting studied by Myatt (2007) and Dewan and Myatt (2007). The difference is that the collective objective is anti-coordination rather than coordination. This section shows that the messages emerging from coordination and anti-coordination games mirror each other. For example, strategic-voting games have a unique and stable equilibrium when voters' private signals are precise relative to their prior beliefs. Here, however, a protest-voting game has a unique and stable equilibrium only when the opposite is true.

Information. Under a private signals specification, I expand a voter's type to a pair (u_i, s_i) where s_i is a private signal of θ. Conditional on θ, these are jointly normally distributed, and are (conditionally) independent of others' types. I have already observed that u_i is itself an informative signal of θ. I incorporate this into the signal s_i so that s_i is a sufficient statistic for (u_i, s_i) when conducting inference about θ. Formally:

$$s_i \,|\, \theta \sim N\left(\theta, \frac{\sigma^2}{\lambda}\right) \quad \text{and} \quad u_i \,|\, (\theta, s_i) \sim N\left(s_i, \frac{(\lambda-1)\sigma^2}{\lambda}\right). \tag{32}$$

This implies that $u_i \,|\, \theta \sim N(\theta, \sigma^2)$, as before. λ is the precision of a voter's private signal. Even if other information sources are weak, the availability of introspection results in the regularity assumption that $\lambda > 1$. Updating the prior belief from equation (5),

$$\theta \,|\, (u_i, s_i) \sim N\left(\frac{\psi\mu + \lambda s_i}{\psi + \lambda}, \frac{\sigma^2}{\psi + \lambda}\right). \tag{33}$$

Solution Concept. As before, a voter's strategy is the probability that he protests conditional on his type. Previously I considered cutpoint strategies, which correspond to a $u^\star$ such that voter i protests if $u_i > u^\star$ but votes for the candidate if $u_i < u^\star$. Here, however, I allow the cutpoint to depend on the signal realization of a voter: voter i protests if and only if $u_i > U(s_i)$ where $U(\cdot)$ is a differentiable function of the voter's signal.

Definition. *A signal-dependent cutpoint $U(s_i)$ yields a* voting equilibrium *if, given that all others use that cutpoint, each type $u_i \neq U(s_i)$ strictly prefers to use the cutpoint if the electorate size n is sufficiently large. An equilibrium is monotonic if $U'(s_i) < 1$, so that a simultaneous increase in s_i and u_i cannot cause a switch away from protest voting.*

Fixing prior expectations, the monotonicity requirement ensures that there is a positive relationship between the true enthusiasm for a protest issue and the probability of a protest vote: $P(\theta) = \Pr[u_i > U(s_i) \,|\, \theta]$ is increasing in θ if $U'(s_i) < 1$. Note also that this definition rules out strategy profiles in which all voters ignore their types and protest, or in which all voters ignore their types and vote for the candidate.

Optimal Voting. Just as before, if voters' decisions vary (that is, when voters' strategies are type responsive) then pivotal probabilities are positive. In this situation, a voter finds it strictly optimal to cast a protest vote if and only if u_i exceeds the log odds of the two pivotal events. That is, the criterion of equation (8) applies, but where the pivotal probabilities are conditioned on s_i and the use of the cutpoint function $U(\cdot)$ by other voters. In a large electorate, therefore, a voting equilibrium is obtained if and only if

$$U(s_i) = \lim_{n\to\infty} \log\left[\frac{\Pr[\text{Pivotal at } H \,|\, U(\cdot), s_i]}{\Pr[\text{Pivotal at } L \,|\, U(\cdot), s_i]}\right]. \tag{34}$$

Given that each voter protests if and only if $u_i > U(s_i)$, the probability of a protest vote, conditional on θ, is $p = P(\theta) \equiv \Pr[u_i > U(s_i) \,|\, \theta]$. $U(s_i)$ can be increasing in s_i and so $P(\theta)$ may increase or decrease in response to changes in θ. However, if the response of $U(s_i)$ is not too strong (for example, if $U'(s_i) < 1$) then $P(\theta)$ is monotonic. Given monotonicity, a voter's beliefs about p are given (following a change of variables) by the density

$$f(p \,|\, s_i, U(\cdot)) = \left.\frac{g(\theta \,|\, s_i)}{P'(\theta)}\right|_{\theta = P^{-1}(p)}, \tag{35}$$

where $g(\theta \,|\, s_i)$ is the density of a voter's posterior beliefs about the candidate's popularity.

If $P(\theta)$ is increasing then there are two critical values of enthusiasm for the protest, θ_L and θ_H, which generate support reaching p_L and p_H respectively. In fact,

$$\lim_{n\to\infty} \log\left[\frac{\Pr[\text{Piv. } H \,|\, U(\cdot), s_i]}{\Pr[\text{Piv. } L \,|\, U(\cdot), s_i]}\right] = \log\left[\frac{f(p_H \,|\, s_i, U(\cdot))}{f(p_L \,|\, s_i, U(\cdot))}\right] = \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \log\left[\frac{g(\theta_H \,|\, s_i)}{g(\theta_L \,|\, s_i)}\right]. \tag{36}$$

where $\theta_H = P^{-1}(p_H)$ and where $\theta_L = P^{-1}(p_L)$. Given the normality assumptions, the log likelihood ratio $\log[g(\theta_H \,|\, s_i)/g(\theta_L \,|\, s_i)]$ is linear in s_i.

Lemma 3 (Linearity)**.** *In a monotonic voting equilibrium, voter i casts a protest vote if and only if $u_i > U(s_i)$, where the cutpoint function is linear: $U(s_i) = u^\dagger + rs_i$ for $0 < r < 1$.*

Henceforth I restrict attention to monotonic linear strategies. That is, strategies where each voter casts a protest vote if and only if $u_i > u^\dagger + rs_i$ where r satisfies $0 < r < 1$.

Feedback Effects. A voter's posterior beliefs about θ are normal with mean $(\psi\mu + \lambda s_i)/(\psi + \lambda)$ (this is the precision-weighted average of the prior mean and the signal) and variance $\sigma^2/(\psi+\lambda)$. Hence, straightforwardly, and as the proof of Lemma 3 confirms,

$$\log\left[\frac{g(\theta_H \mid s_i)}{g(\theta_L \mid s_i)}\right] = \frac{\theta_H - \theta_L}{\sigma^2}\left[\lambda(s_i - \mu) + (\psi + \lambda)\left(\mu - \frac{(\theta_H + \theta_L)}{2}\right)\right]. \tag{37}$$

The coefficient on the voter's signal is $\lambda(\theta_H - \theta_L)/\sigma^2$, and so his response to that signal is stronger when his signal is more precise (so that λ is higher) and when the gap between θ_H and θ_L is large. Those critical values depend upon the behavior of others, and they depend on the response of others to their own signals.

A protest vote is cast if and only if $u_i > U(s_i)$, or equivalently if $u_i - U(s_i) > 0$. Given that $U(s_i) = u^\dagger + rs_i$, an increase in the underlying value of θ shifts $u_i - U(s_i)$ with the coefficient $(1 - r)$. Hence, if others respond relatively strongly to their own signals (if r is higher) then in aggregate their behavior responds sluggishly to changes in enthusiasm for the protest issue. Of course, a change in r also changes the conditional variance of $u_i - U(s_i)$, but nevertheless the overall effect of an increase in r is maintained. In fact

$$u_i - U(s_i) \mid \theta \sim N\left((1-r)\theta - u^\dagger, \frac{\sigma^2(\lambda - 1 + (1-r)^2)}{\lambda}\right), \tag{38}$$

and so the probability of a protest vote, conditional on θ, is

$$P(\theta) = \Phi\left(\frac{\theta - [u^\dagger/(1-r)]}{(\sigma/\lambda)S(r)}\right) \quad \text{where} \quad S(r) \equiv \sqrt{\lambda + \frac{\lambda(\lambda-1)}{(1-r)^2}}. \tag{39}$$

The response of p (the probability of a protest vote) to θ (enthusiasm for the protest) is inversely related to $S(r)$. This is increasing in r, and so an increase in r (greater sensitivity to signals) implies that the ballot-box support for the protest responds more weakly to true enthusiasm. Inverting $P(\theta)$ and evaluating at p_L and p_H yields, for $r < 1$,

$$\theta_L = \frac{u^\dagger}{1-r} + \frac{\sigma z_L S(r)}{\lambda} \quad \text{and} \quad \theta_H = \frac{u^\dagger}{1-r} + \frac{\sigma z_H S(r)}{\lambda}. \tag{40}$$

An increase in r, and so in $S(r)$, pushes apart the critical values θ_H and θ_L. A voter contemplates the relative likelihood of values for θ which are further apart, and he places more weight on the informative signals at his disposal. With these solutions in hand the coefficient on the private signal s_i in the log likelihood ratio term is

$$\frac{\lambda(\theta_H - \theta_L)}{\sigma^2} = \frac{(z_H - z_L)S(r)}{\sigma}. \tag{41}$$

As noted, $S(r)$ is increasing in r and so a heightened response of others to their signals further encourages a voter to react to his own signal. An increase in r is an increased willingness to protest when the candidate is seen as popular, and so one aspect of protest voting can involve strategic complements (cf. Lemma 2).

Lemma 4 (Best Replies with Private Signals)**.** *If voters use a linear cutpoint* $U_i(s) = u^\dagger + rs_i$*,* $0 < r < 1$*, then the unique best reply is to use the cutpoint* $\hat{U}_i(s_i) = \hat{u}^\dagger + \hat{r}s_i$ *where*

$$\hat{r} = \frac{(z_H - z_L)S(r)}{\sigma} \quad \textit{and} \quad \hat{u}^\dagger = \frac{\hat{r}}{\lambda}\left[\psi\mu - \frac{(\psi+\lambda)u^\dagger}{1-r}\right] - \frac{z_H^2 - z_L^2}{2\lambda}\left[\psi + \frac{(\psi+\lambda)(\lambda-1)}{(1-r)^2}\right], \quad (42)$$

where $S(r)$ *is increasing in* r*. An increase in the response of others to their signals raises a voter's response to his own signal; however, an unconditional increase in protest voting by others (a fall in* $u^\dagger$*) lowers a voter's own tendency to protest (*$\hat{u}^\dagger$ *rises).*

Corollary (to Lemma 4)**.** *Raw protest voting (determined by* $u^\dagger$*) is a strategic substitute; however, the response of voters to their signals (determined by* r*) is a strategic complement.*

Equilibrium. Given Lemma 4, matters are straightforward: from equation (42) a monotonic voting equilibrium corresponds to a pair $(r, u^\dagger)$ satisfying $0 < r < 1$ and

$$r = \frac{(z_H - z_L)S(r)}{\sigma} \quad \text{and} \quad u^\dagger = \frac{r}{\lambda}\left[\psi\mu - \frac{(\psi+\lambda)u^\dagger}{1-r}\right] - \frac{z_H^2 - z_L^2}{2\lambda}\left[\psi + \frac{(\psi+\lambda)(\lambda-1)}{(1-r)^2}\right]. \quad (43)$$

The second equation is linear in $u^\dagger$, and is solved straightforwardly. The first equation, however, does not necessarily have a solution. For $r \in (0,1)$ the function $S(r)$ is positive, increasing, convex, and satisfies $S(r) \to \infty$ as $r \to 1$. This means that $(z_H - z_L)S(r) > \sigma r$ for all $r \in (0,1)$ if the gap $z_H - z_L$ is sufficiently large. If not, then there is a solution to this equation; however, the properties mentioned here ensure that if there is a solution (so that a monotonic voting equilibrium exists) then there is a second solution.

Proposition 8 (Equilibrium with Private Information)**.** *For* $\lambda > 1$*, define*

$$\bar{\sigma} \equiv (z_H - z_L)\min_{r\in[0,1]}\left[\frac{S(r)}{r}\right] \quad \textit{where} \quad S(r) \equiv \sqrt{\lambda + \frac{\lambda(\lambda-1)}{(1-r)^2}}, \quad (44)$$

where $z_L \equiv \Phi^{-1}(p_L)$ *and* $z_H \equiv \Phi^{-1}(p_H)$*. If heterogeneity is low,* $\sigma < \bar{\sigma}$*, then a monotonic equilibrium does not exist. If* $\sigma > \bar{\sigma}$*, then there are two monotonic equilibria. Each equilibrium* $r \in \{r_1, r_2\}$ *is a solution to the equation* $\sigma r = (z_H - z_L)S(r)$*. In each equilibrium, a voter protests if and only if* $u_i > u^\dagger + rs_i$ *where*

$$u^\dagger = \frac{1-r}{\lambda + r\psi}\left(r\psi\mu - \frac{z_H^2 - z_L^2}{2}\left[\psi + \frac{(\psi+\lambda)(\lambda-1)}{(1-r)^2}\right]\right). \quad (45)$$

Note that $\bar{\sigma}$ is increasing in p_H, decreasing in p_L, and increasing in the precision λ.[22]

[22] A special case is when a voter has no additional private information, so that $\lambda = 1$. To reach this case, I take the limit as $\lambda \downarrow 1$. Notice that $S(r) \to 1$ for $r < 1$ and so, taking the lowest equilibrium solution for r,

$$r \to \frac{z_H - z_L}{\sigma} \quad \text{and} \quad \frac{u^\dagger}{1-r} \to \frac{\psi(z_H - z_L)}{\psi(z_H - z_L)\psi + \sigma}\left[\mu - \frac{\sigma(z_H + z_L)}{2}\right]. \quad (46)$$

The Effect of the Precision of Beliefs. I conclude this section by investigating the effect of precise beliefs. I begin by considering the properties of equilibria as $\psi \to \infty$, so that the prior becomes sharp. An equilibrium response r to the private signal does not depend on ψ. However, the constant $u^\dagger$ does. From the proof of Proposition 9,

$$\lim_{\psi\to\infty} \frac{u^\dagger}{1-r} = \mu - \frac{(z_H + z_L)}{2}\frac{\sigma S(r)}{\lambda}. \tag{47}$$

Examining the probability of a protest vote as $\psi \to \infty$ generates the next proposition. This verifies that Proposition 4 also holds in a world with general private signals.

Proposition 9 (Protest Voting with a Precise Prior). *In either monotonic equilibrium,*

$$\lim_{\psi\to\infty} \Pr[\textbf{Protest Vote}] = \Phi\left(\frac{z_H + z_L}{2}\right) \quad \textit{and so} \quad p_L < \lim_{\psi\to\infty} \Pr[\textbf{Protest Vote}] < p_H, \tag{48}$$

and so in the limit as prior aggregate uncertainty is eliminated, the fraction of voters who protest and the election outcome are independent of voters' preferences.

A message from Section 3 is that the amount of protest voting is independent of how voters feel about the value of a protest whenever aggregate uncertainty is mild. (As noted in Section 4, things are more intricate when the reaction to the protest is endogenous.) This message also holds when voters have better private signals than introspection.

Nevertheless, improving private information is problematic. Notice that $S(r) \to \infty$ as $\lambda \to \infty$, and this implies that $\bar\sigma \to \infty$. Recall that $\bar\sigma$ is the critical value of preference heterogeneity from Proposition 8; if $\sigma < \bar\sigma$ then there is no monotonic equilibrium.

Corollary (to Proposition 8). *If private signals about the popularity of the protest issue are sufficiently precise then a monotonic voting equilibrium does not exist.*

When signals are very precise then voters react strongly to them. This means that voters refrain from protesting (expecting that others will do so) when they see that popular enthusiasm is high. This leads to a back-to-front situation in which a protest fails only when enthusiasm for it is strong; a monotonic equilibrium falls apart.

Protest vs. Strategic Voting. I have observed that payoffs in the protest-voting game studied here flips the structure of a typical strategic-voting game. To see this, it is helpful to review a related strategic-voting model (Fisher and Myatt, 2002; Myatt, 2007).

Consider an election with three competing candidates: a right-wing candidate, and two left-wing candidates. Suppose that all right-wing voters cast their ballots for the right-wing candidate, and that they represent more than one-third but less than one-half of

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A voter protests if and only if $u_i < u^\dagger + rs_i$. In the limiting case $\lambda = 1$, the signal s_i is equal to u_i and so this inequality becomes $u_i < u^\dagger + ru_i$, or equivalently $u_i < u^\dagger/(1-r)$. From equation (46), observe that $u^\dagger/(1-r) \to u^\star$ where $u^\star$ is the cutpoint from Proposition 1.

the electorate. Left-wing voters face a coordination problem: if they coordinate behind one of the left-wing candidates, then the disliked right-wing candidate will lose; however, if they split evenly between the two candidates then the right-wing candidate will win. This situation is common. For example, in the United Kingdom's 1997 General Election many Labour and Liberal Democrat supporters wished to ensure the defeat of the then-incumbent Conservative government. The Conservative party's vote share lay between one-third and one-half in 270 of the 529 English constituencies. Thus, in such constituencies, the left-of-centre anti-Conservative voters faced a coordination problem.

More formally, consider a three-candidate election (with candidates A, B, and C) in which candidate A is disliked by n left-wing voters. These n voters choose between (left-wing) candidates B and C. I write b for the number of votes cast for B, so that $n - b$ is the number cast for C. If there are $\bar{p}n$ votes for candidate A, where $1 > \bar{p} > \frac{1}{2}$, then a fraction $\bar{p}$ of left-wing voters must coordinate in order to defeat A. Specifically,

$$\text{outcome} = \begin{cases} \text{candidate } C \text{ wins} & \text{if } \frac{b}{n} < 1 - \bar{p}, \\ \text{coordination fails} \Rightarrow \text{candidate } A \text{ wins} & \text{if } 1 - \bar{p} < \frac{b}{n} < \bar{p}, \text{ and} \\ \text{candidate } B \text{ wins} & \text{if } \bar{p} < \frac{b}{n}. \end{cases} \tag{49}$$

For the n left-wing voters an appropriate payoff specification is $U_i^A > \min\left\{U_i^B, U_i^C\right\}$. For the case in Figure 2, voter i satisfies $U_i^C > U_i^B > U_i^A$. Hence, for this voter, a vote for C is a sincere vote whereas one for B is a strategic vote. Equivalently, $u_i < 0$ where

$$u_i = \log\left[\frac{U_i^B - U_i^A}{U_i^C - U_i^A}\right]. \tag{50}$$

This scenario shares many features with the protest-voting model of this paper. In particular, a voter's optimal decision compares his preference type u_i to the log odds of pivotal events. However, the payoff structure is inverted: this can be seen from a comparison of Figures 1 and 2. There is an analogy between casting a protest vote and voting for candidate B rather than C. However, if the pivotal event B (by which I mean a situation in which B is one vote away from beating the disliked candidate A) becomes more likely then this raises the incentive to vote for B; in contrast, back in the protest-voting model, if the pivotal event H is more likely then a voter shifts away from protest voting.

The flipped nature of incentives carries over to how voters respond to their types.

Consider the protest-voting scenario of this paper. If voters base their decisions solely on their preferences then protest voting exhibits the properties of a strategic substitute, in the following sense: if the range of types who protest widens (a fall in $u^\star$) then there is a weakened incentive to protest. However, in the private-signal world of this section, the reaction of voters to their private signals exhibits the properties of a strategic complement: if the reaction of others to their private signals is stronger (r is higher) then

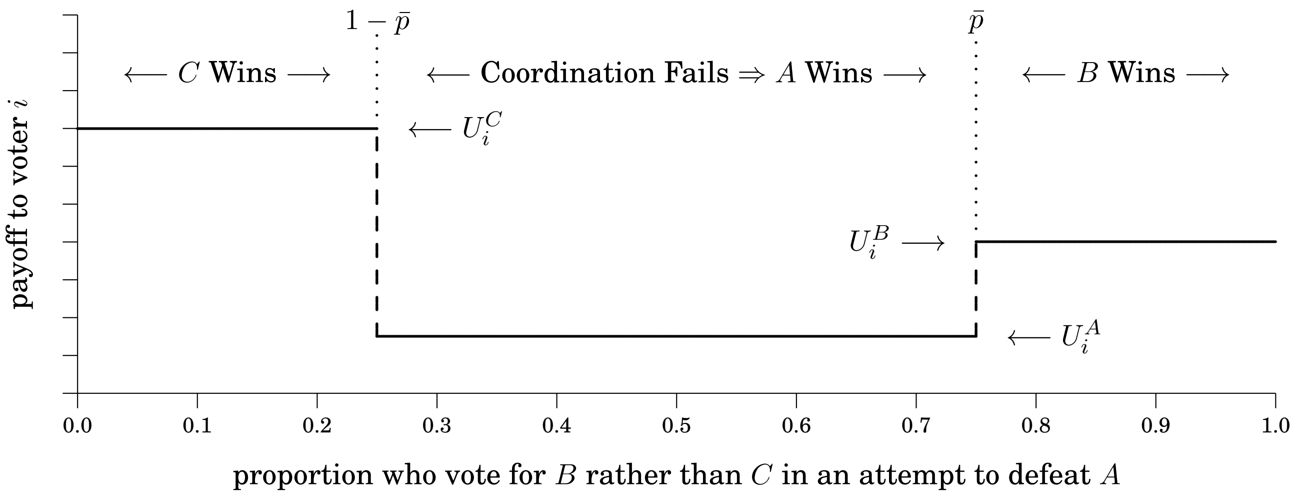

FIGURE 2. Outcomes and Voters' Payoffs in a Strategic-Voting Game (cf. Figure 1)

a voter's best reply is to react more strongly to his own private signal. An increase in r slows the reaction of protest voting to the electorate's true enthusiasm for the protest issue. This pushes apart the two critical values of θ which generate pivotal events. Given that those critical values are further apart, a voter's private signal is relatively more informative about them. This leads him to place greater emphasis on his private signal. The self-reinforcing nature of reactions to private signals can (when signals are precise) prevent the existence of nicely behaved monotonic equilibria.

Now consider (in contrast) the classic strategic-voting problem. Putting aside private signals, the properties of the flipped payoffs (compare again Figures 1 and 2) mean that strategic voting exhibits the properties of a strategic complement: if the range of types who vote strategically increases then others should do so too. This generates a force toward what are known as Duvergerian equilibria: in the context of the strategic-voting scenario described above, this means full coordination behind one of the left-wing challengers. Despite this, Myatt (2007) showed that voters' responses to their private signals exhibit the properties of a strategic substitute: if other voters respond strongly to their private signals (they back whomever their signals indicate is the stronger challenger) then the incentive of a voter to react to his own signal is weakened. What is happening here is that the pivotal events which enter into his decision making become closer and so more evenly matched, which drives him to follow his personal preference rather than act strategically. Myatt (2007) demonstrated that the self-correcting nature of reactions to private signals can enable the existence of nicely behaved monotonic equilibria, so long as those private signals are sufficiently precise relative to any public information.

A conclusion here is that the protest voting and strategic voting are closely related; the inverted nature of the findings arises because (heuristically, at least) strategic voting is a coordination game whereas protest voting is an anti-coordination game.

6. Concluding Remarks

I have considered a model of protest voting in which voters face an anti-coordination problem: ideally, they would like the protest to be large enough to succeed, but not large enough to prevent their preferred candidate from winning the election. Heuristically, at least, protest votes are strategic substitutes: if others are likely to engage in widespread protest voting then an individual voter prefers to back the favored candidate.

The comparative-static analysis shows that voters usually respond to the need for coordination: protest voting increases as the need to coordinate for a successful protest rises, but falls as more coordination is required for the candidate to win rather than lose. The most interesting results, however, concern the expected enthusiasm for the protest issue: an increase in this generates a reduced willingness to protest.

The double-edged nature of greater enthusiasm for the protest issue is sharpest when prior beliefs are precise: the offset effect is exact. However, a contribution of this paper is to show that this effect is even stronger (and sufficient to overturn the direct effect) when the candidate's reaction to a protest is endogenous. If she anticipates a lessened willingness to protest then a poor showing in the election is more strongly indicative of popular disquiet, and so she is more willing to react. This heightened reaction further reduces the willingness to protest. The net effect is that an increase in expected enthusiasm for the protest issue can lower the expected size of the protest. Protest voting is at its highest when, prior to the election, a candidate is indifferent between accepting and rejecting the demands of the protesters. More generally, there is a non-monotonic relationship between the strength of feeling for a protest issue and the electoral outcome.

Three central take-home messages are these: firstly, the presence of aggregate uncertainty means that there is relationship between the preference intensity of voters and the size of a protest, even though there would be no such relationship in the absence of such uncertainty; secondly, this relationship can be negative, so that greater intensity in favor of the protest can be associated with a smaller protest; and, thirdly, protests are largest when a winning candidate is most open to engaging in policy changes.

Throughout, these messages concern the preference intensity of voters: recall that (in my model) all relevant voters (those who form the player set) unanimously wish the protest to succeed. However, this is only a modeling choice. In fact, these key messages continue to hold when there is aggregate uncertainty over the fraction of the electorate who wish the protest to succeed. In this case, there can be a negative relationship between the number of protest-supporting voters and the number of actual protest ballots.

I have noted that the anti-EU party UKIP has enjoyed recent electoral success in the United Kingdom. The logic of protest voting explored here shows that this does not necessarily imply a strengthening of anti-EU sentiment. This finding can also apply to other recent events. Consider, for example, the recent independence referendum in Scotland. One interpretation is that Scottish voters would prefer to retain strong connections within the United Kingdom (they do not want complete independence) but also wish to obtain substantial devolved authority (via Westminster's acceptance of the protesters' demands). Hence, a vote for independence can be interpreted as a protest vote. In this case, $p_H = \frac{1}{2}$: the (undesirable) separation of Scotland happens if a majority vote for independence. In turn, the Westminster establishment chooses a critical value p_L above which it devolves further powers. The results here suggest that protest voting is maximized, and the risk of accidental separation of the United Kingdom is at its highest, when μ and $\theta^\dagger$ are close. This can be interpreted as a situation in which the Westminster government was, in any case, close to deciding to more forward with further devolution. Here, the model predicts extensive protest voting and a close race for the referendum.

Appendix A. Extensions

I describe briefly five extensions: (i) variation in and aggregate uncertainty over the fraction of voters who wish the protest to succeed; (ii) the use of a specification in which a voter's beliefs about θ do not vary with his type realization; (iii) the properties of cutpoint equilibria in finite electorates; (iv) additional comparative-static results regarding the need for coordination; and (v) a specification in which the candidate responds continuously to the size of the protest vote.

Aggregate Uncertainty over the Size of Voter Groups. The specification of this paper focuses on voters who (unanimously) wish the candidate to be elected and who wish the protest to succeed. Changes in θ influence, for the median voter, the importance of the protest's success relative to the candidate's election. In this sense, θ represents the strength of enthusiasm for the protest. However, the popularity of the protest, in the sense of the number of voters who want the protest to succeed once the candidate's position is secure, is fixed. Another reasonable situation is one in which increases in θ also increase the popularity of the protest measured by the number of voters who support it: enthusiasm and popularity should be correlated.

This correlation can be incorporated via a more general specification in which

$$\frac{U_i^{\text{win}} - U_i^{\text{fail}}}{U_i^{\text{win}} - U_i^{\text{lose}}} \equiv H(u_i) \quad \text{where} \quad u_i \,|\, \theta \sim N(\theta, \sigma^2), \tag{51}$$

and where $H(u_i)$ is increasing in a voter's type. The specification in the main text corresponds to $H(u_i) = e^{u_i}$, which satisfies $H(u_i) > 0$ for all u_i and so every voter wants the protest to succeed. If, for example, $H(u_i) = u_i$ then the probability that a voter is a supporter of the protest is $\Phi(\theta/\sigma)$ and (via uncertainty over θ) there is aggregate uncertainty over this probability.

Given that others use a cutpoint $u^\star$, the optimality criterion of equation (16) for a protest vote is maintained so long as the original term u_i is replaced with $\log H(u_i)$. Substituting in the normal specification, as I did in the main text, yields

$$\log H(u_i) + \frac{z_L^2 - z_H^2}{2} + \frac{(1+\psi)(z_H - z_L)}{\sigma}\left(u^\star - \frac{\psi\mu + u_i}{1+\psi}\right) + \frac{(1+\psi)(z_H^2 - z_L^2)}{2} > 0. \tag{52}$$

As before, an increase in the willingness to protest by others (a reduction in $u^\star$) lowers the incentive to protest. However, it is not always true that a higher type faces a stronger incentive to protest: the left-hand side of (53) is not necessarily increasing in u_i. For example, if $H(u_i) = u_i$ then the left-hand is single-peaked in u_i and the inequality fails for very high types. To ensure that best replies are monotonic in a voter's type we need to ensure that $H(u_i)$ increases sufficiently quickly for higher values of u_i. These concerns can be put aside for common-beliefs specification that is discussed in the next subsection. Once these concerns are resolved (so that any best reply involves the use of a single cutpoint) an equilibrium is can be found by solving

$$\log H(u^\star) + \frac{z_L^2 - z_H^2}{2} + \frac{\psi(u^\star - \mu)(z_H - z_L)}{\sigma} + \frac{(1+\psi)(z_H^2 - z_L^2)}{2} = 0, \tag{53}$$

and key findings of the results reported in the main text are maintained.

Common Beliefs. In the main text, voters introspectively update a common prior to form different posterior beliefs based on their type realizations. Another specification is when voters share the same beliefs independent of their types. (Recall that the description of a game of incomplete information requires only the specification of beliefs conditional on types; a common prior updated via type realizations is conventional but is not strictly necessary.) In this common beliefs case, voter i's beliefs about voters j and j' are joint normal, with moments

$$\mathrm{E}[u_j] = \mathrm{E}[u_{j'}] = \mu, \quad \mathrm{var}[u_j] = \mathrm{var}[u_{j'}] = \frac{(1+\psi)\sigma^2}{\psi}, \quad \text{and} \quad \mathrm{cov}[u_j, u_{j'}] = \frac{\sigma^2}{\psi}. \tag{54}$$

This specification is helpful because a cutpoint strategy is optimal for any σ^2 (cf. Lemma 2).

Proposition 10 (Equilibrium with Common Beliefs). *If voters believe that $\theta \mid u_i \sim N(\mu, \sigma^2/\psi)$ then there is a unique voting equilibrium. Voters use a cutpoint $u^\ddagger$ which satisfies*

$$u^\ddagger = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma}\left[\mu - \frac{\psi - 1}{\psi}\frac{\sigma(z_H + z_L)}{2}\right]. \tag{55}$$

Relative to the introspective-beliefs case, there are more protest votes, if and only if $p_L < 1 - p_H$.

The final statement compares the equilibrium cutpoint with that in Proposition 10. The difference between the equilibrium thresholds reported in equations (55) and (21) is

$$u^\ddagger - u^\star = \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma}\frac{\sigma(z_H + z_L)}{2\psi}, \tag{56}$$

and so introspection (that is, the updating of a common prior following a voter's observation of his own preference type) results in less protest voting (that is, $u^\ddagger < u^\star$) if and only if $z_H + z_L < 0$, or equivalently $p_L < 1 - p_H$. Notice that $1 - p_H$ is the fraction of voters that need to coordinate behind the candidate if she is to avoid defeat, whereas p_L is the fraction that need to coordinate behind a protest for it to succeed. Hence, $p_L < 1 - p_H$ says that coordination to avoid the candidate's defeat is harder than the coordination needed for a successful protest.

Equilibria in Finite Electorates. A voting equilibrium is a single voting strategy, defined over a sequence of voting games, which specifies a best reply for almost every type so long as the electorate is sufficiently large. Here I consider orthodox Bayesian Nash equilibria in finite electorates. For simplicity of exposition I use the common-beliefs specification described above. However, what I say here also applies to the introspective specification from the main text.

Fixing n, consider a symmetric Bayesian Nash equilibrium in which voters respond to their types. If decisions are type-responsive then the probabilities of the pivotal outcomes are positive. A voter's best reply is a cutpoint strategy. A symmetric type-responsive equilibrium in a finite electorate corresponds to a cutpoint $u_n^\ddagger$ which satisfies $u_n^\ddagger = L_n(u_n^\ddagger)$ where

$$L_n(\hat{u}) \equiv \log \left[\frac{\Pr[\text{Pivotal at } H \mid \hat{u}]}{\Pr[\text{Pivotal at } L \mid \hat{u}]} \right] \tag{57}$$

is the log odds ratio of the pivotal events given that others use a cutpoint $\hat{u}$. A Bayesian Nash equilibrium corresponds to a fixed point of $L_n(\hat{u})$. Now, $L_n(\hat{u})$ is continuous in $\hat{u}$, and

$$\lim_{n\to\infty} L_n(\hat{u}) = \frac{(1-\psi)(z_H^2 - z_L^2)}{2} - \frac{\psi(\hat{u}-\mu)(z_H - z_L)}{\sigma}. \tag{58}$$

This limit is decreasing in $\hat{u}$. It lies strictly above $\hat{u}$ if $\hat{u} < u^\ddagger$ (where $u^\ddagger$ is the equilibrium cutpoint described in Proposition 10 above) and it lies strictly below $\hat{u}$ if $\hat{u} > u^\ddagger$. Hence, for any $\varepsilon > 0$, if n is sufficiently large then $L_n(\hat{u})$ has a fixed point satisfying $u_n^\star \in (u^\ddagger - \varepsilon, u^\ddagger + \varepsilon)$. From this, a sequence of equilibrium cutpoints can be constructed which satisfy $u_n^\ddagger \to u^\star$ as $n \to \infty$. A similar argument establishes that any other sequence of cutpoints cannot converge to a finite limit; any other equilibrium sequence (if such a sequence exists) must diverge to $\pm\infty$.

I have noted that the solution concept entails a best reply by (almost) all voters in sufficiently large electorates. Of course, there may be some who do not play a best reply for a particular finite n. This would suggest that the solution concept is loose. In a second sense, however, the solution is rather stringent. Whereas a Bayesian-Nash equilibrium would involve the play of a best reply by almost all voters for a particular finite n, it would not necessarily be robust to increases in n. If a sequence of Bayesian-Nash equilibria converges, then the limit is the equilibrium used here. However, if it does not converge, then it must mean that the Bayesian-Nash equilibria depend sensitively on the precise size n of the electorate. In this setting, a desire for robustness suggests the use of the solution concept used here and in Dewan and Myatt (2007) and Myatt (2007).

Additional Comparative-Static Results. Proposition 3 in the text reports comparative-static results for changes in p_L and p_H, but restricts to a situation in which prior beliefs are precise. Without restrictions on ψ, related comparative-static predictions hold. These are reported here.

Proposition 11 (Additional Properties of the Need for Coordination)**.** *(i) If the expected enthusiasm for the protest is sufficiently high, $\mu > \sigma z_H$, then protest voting is increasing in the coordination p_L needed for the protest to succeed. However, if that expected enthusiasm is low, $\mu < \sigma z_H$, then protest voting is decreasing in p_L if p_L is sufficiently close to p_H.*

(ii) If $\mu < \sigma z_L$, so that the expected enthusiasm for the protest is low, then protest voting is decreasing in the coordination $1 - p_H$ needed for the candidate to avoid defeat. However, if $\mu > \sigma z_L$, then protest voting is increasing in $1 - p_H$ if p_L and p_H are sufficiently close.

If p_H and p_L are not too close then their effects are straightforward. Increasing $1 - p_H$ makes the candidate's position more precarious; protest is more costly. An increase in p_L makes a successful protest more difficult, increasing the pressure to respond. A voter cares about the relative likelihood of the pivotal events. The odds move in the natural direction as the need-for-coordination parameters change. However, if p_H and p_L are close the effect via the gap $z_H - z_L$ can dominate.

Continuous Response to the Protest. In the main text, the reaction to the protest is binary: the protest either succeeds or it does not. As noted in the text, other specifications are possible. For example, consider a situation in which the mainstream candidate varies her position on the protest dimension in response to the number of protest ballots. A suitable specification is

$$\text{voter payoff} = \begin{cases} 1 - u_i \left(p_H - \frac{b}{n}\right) & \text{if } \frac{b}{n} < p_H, \\ 0 & \text{if } p_H < \frac{b}{n}, \end{cases} \tag{59}$$

where u_i is the preference type of voter i and where b is the number of ballots cast in favor of the protest. Hence, a voter's ideal situation is when the candidate wins with no margin of victory. Given that she wins, he would like her victory to be as small as possible: he loses u_i per unit of vote share as the size of the candidate's win increases.

Given this specification, it is strictly optimal for the voter to protest if

$$\frac{u_i \Pr\left[\text{Candidate Wins}\right]}{n} > \Pr[\text{Pivotal at } H]. \tag{60}$$

This naturally generates the use of a cutpoint strategy: a voter protests if $u_i > u^\star$ for some $u^\star$.

Using the common-beliefs specification (for simplicity of exposition), writing $f(p \mid u^\star)$ for a voter's beliefs about the probability that others' protest given that they use a cutpoint $u^\star$, writing $g(\theta)$ for the density of a voter's beliefs about θ, and using Lemma 1 and equation (10),

$$\lim_{n\to\infty} n \Pr[\text{Pivotal at } H \mid u^\star] = f(p_H \mid u^\star) = \frac{\sigma g(u^\star + \sigma z_H)}{\phi(z_H)}, \tag{61}$$

and similarly

$$\lim_{n\to\infty} n \Pr[\text{Candidate Wins} \mid u^\star] = \Pr\left[\Phi\left(\frac{\theta - u^\star}{\sigma}\right) < p_H\right] = G(u^\star + \sigma z_H). \tag{62}$$

Hence, in a large electorate, a voter finds it optimal to protest if

$$u_i > \frac{\sigma}{\phi(z_H)} \times \frac{g(u^\star + \sigma z_H)}{G(u^\star + \sigma z_H)}. \tag{63}$$

Under regularity conditions (this holds here when beliefs about θ are normal) the right-hand side is decreasing in $u^\star$. Hence, an increase in protest voting by others (lowing $u^\star$) reduces the incentive of a voter to protest. That is, protest votes are strategic substitutes. Furthermore, there is a unique equilibrium cutpoint which satisfies

$$u^\star = \frac{\sigma}{\phi(z_H)} \times \frac{g(u^\star + \sigma z_H)}{G(u^\star + \sigma z_H)}. \tag{64}$$

Comparative-static results are readily obtained. For example, an increase in the expected enthusiasm for the protest issue reduces the willingness of voters to engage in protest voting.

Appendix B. Omitted Proofs

Proof of Lemma 1. Good and Mayer (1975) and Chamberlain and Rothschild (1981) proved this for $p_L = \frac{1}{2}$. The technique of Chamberlain and Rothschild (1981) extends to other values of p_L and p_H. The result here is a special case of Lemma 1 of Myatt (2012); he generalized the Good-Mayer-Chamberlain-Rothschild finding to more candidates and more pivotal events. □

Proof of Lemma 2. Substituting the expression from (19), the criterion becomes

$$u_i + \frac{(1+\psi)(z_H - z_L)}{\sigma}\left(u^\star - \frac{\psi\mu + u_i}{1+\psi}\right) + \frac{(\psi - 1)(z_H^2 - z_L^2)}{2} > 0. \tag{65}$$

The left-hand side of this inequality is increasing in $u^\star$, yielding the first claim. It increasing in u_i if and only if $\sigma > z_H - z_L$, so yielding the second claim. □

Proof of Proposition 1. A voting equilibrium involves cutpoint strategies. These require a voter's incentive to protest to increase with his type. Using Lemma 2, this can be true if and only if $\sigma > z_H - z_L$. To find the equilibrium cutpoint, substitute (19) into the equilibrium condition (20):

$$u^\star + \frac{z_L^2 - z_H^2}{2} + \frac{(1+\psi)(z_H - z_L)}{\sigma}\left(u^\star - \frac{\psi\mu + u^\star}{1+\psi}\right) + \frac{(1+\psi)(z_H^2 - z_L^2)}{2} = 0. \tag{66}$$

This can be re-arranged straightforwardly to yield the reported solution. □

Proof of Proposition 2. Claims (i) and (ii) follow from an inspection of $u^\star$. To verify claim (iii):

$$\begin{aligned} \frac{\partial u^\star}{\partial \sigma} &= -\frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma}\left[\frac{(z_H + z_L)}{2} + \frac{1}{\psi(z_H - z_L) + \sigma}\left[\mu - \frac{\sigma(z_H + z_L)}{2}\right]\right] < 0 \\ &\Leftrightarrow \quad \frac{(z_H + z_L)[\psi(z_H - z_L) + \sigma]}{2} + \mu - \frac{\sigma(z_H + z_L)}{2} > 0 \quad \Leftrightarrow \quad \mu > \frac{\psi(z_L^2 - z_H^2)}{2}. \end{aligned} \tag{67}$$

Also note that

$$\mu - u^\star = \mu - \frac{\psi(z_H - z_L)}{\psi(z_H - z_L) + \sigma}\left[\mu - \frac{\sigma(z_H + z_L)}{2}\right] = \frac{\sigma}{\psi(z_H - z_L) + \sigma}\left[\mu - \frac{\psi(z_L^2 - z_H^2)}{2}\right], \tag{68}$$

which confirms what is said as part of the discussion (following the proposition) of claim (iii). □

Proof of Proposition 3. The claims follow from an inspection of equation (22) □

Proof of Proposition 4. Examine the limit of the expression in equation (24) as $\psi \to \infty$. □

Proof of Proposition 5. Ignoring the election is equivalent to choosing $p_L = p_H$. For voters, the pivotal events L and H coincide. So, a pivotal vote simultaneously enables the win and the protest's failure. Hence, a voter optimally protests if and only if $u_i > 0$. Given that $u^\star = 0$, and $\theta^\dagger > \sigma z_H$ holds, the second condition confirms that $p_L = p_H$ is optimal.

For the $p_H > p_L$ case, the relevant equilibrium conditions may be written as

$$u^\star = \frac{(z_H - z_L)\psi}{(z_H - z_L)\psi + \sigma}\left[\mu - \sigma z_H + \frac{\sigma(z_H - z_L)}{2}\right] \quad \text{and} \quad z_H - z_L = \frac{\sigma z_H + u^\star - \theta^\dagger}{\sigma}. \tag{69}$$

These conditions combine to eliminate $u^\star$ and give a quadratic in $z_H - z_L$:

$$\frac{\psi\sigma(z_H - z_L)^2}{2} + (\sigma^2 + \psi(\theta^\dagger - \mu))(z_H - z_L) + \sigma(\theta^\dagger - \sigma z_H) = 0. \tag{70}$$

A sufficient condition for this to have real roots is if the constant term is negative, which is so if and only if $\sigma z_H > \theta^\dagger$. Note that in this case there cannot be an equilibrium in which $p_L = p_H$. Moreover, in this case, there is a unique positive root. This is

$$z_H - z_L = \frac{\psi(\mu - \theta^\dagger) - \sigma^2 + \sqrt{[\sigma^2 + \psi(\theta^\dagger - \mu)]^2 + 2\psi\sigma^2(\sigma z_H - \theta^\dagger)}}{\psi\sigma}, \tag{71}$$

and so

$$u^\star = \theta^\dagger - \sigma z_H + \frac{\psi(\mu - \theta^\dagger) - \sigma^2 + \sqrt{[\sigma^2 + \psi(\theta^\dagger - \mu)]^2 + 2\psi\sigma^2(\sigma z_H - \theta^\dagger)}}{\psi}. \tag{72}$$

Allowing $\psi \to \infty$ generates the claims of the proposition. If $\theta^\dagger > \sigma z_H$ then the quadratic may still have solutions. In fact, for $\theta^\dagger \neq \mu$ there are real solutions if ψ is sufficiently large. Given that solutions exist, the limits of the roots as $\psi \to \infty$ are

$$\lim_{\psi\to\infty}(z_H - z_L) = \frac{\mu - \theta^\dagger \pm |\mu - \theta^\dagger|}{\sigma}. \tag{73}$$

This yields the claims reported in equation (28). □

Proof of Proposition 6. The probability of a protest vote satisfies

$$\begin{aligned}\lim_{\psi\to\infty}\Pr[\text{Protest Vote}] &= \lim_{\psi\to\infty}\mathrm{E}\left[\Phi\left(\frac{\theta - u^\star}{\sigma}\right)\right] = \lim_{\psi\to\infty}\Phi\left(\frac{\mu - u^\star}{\sigma}\right) = \Phi\left(\frac{\mu - \lim_{\psi\to\infty}u^\star}{\sigma}\right) \\ &= \Phi\left(\frac{\mu - \theta^\dagger + 2\max\{\mu - \theta^\dagger, 0\} + \sigma z_H}{\sigma}\right) = \Phi\left(z_H - \frac{|\theta^\dagger - \mu|}{\sigma}\right).\end{aligned} \tag{74}$$

From this, the remaining claims follow straightforwardly. □

Proof of Proposition 7. Differentiate the expression reported in equation (31) to obtain

$$\frac{\partial \Pr[\text{Candidate Loses}]}{\partial\mu} = \frac{\sqrt{\psi}}{\sigma}\phi\left(\sqrt{\psi}\left(\frac{\mu - u^\star}{\sigma} - z_H\right)\right)\left(1 - \frac{\partial u^\star}{\partial\mu}\right). \tag{75}$$

Differentiate the explicit expression for $u^\star$ reported in equation (72) to obtain

$$\frac{\partial u^\star}{\partial\mu} = 1 - \frac{[\sigma^2 + \psi(\theta^\dagger - \mu)]}{\sqrt{[\sigma^2 + \psi(\theta^\dagger - \mu)]^2 + 2\psi\sigma^2(\sigma z_H - \theta^\dagger)}} < 1 \quad \Leftrightarrow \quad \mu < \theta^\dagger + \frac{\sigma^2}{\psi}. \tag{76}$$

Turning to the probability that the protest succeeds, this is

$$\Pr[\textbf{Protest Succeeds}] = \Pr[\theta > u^\star + \sigma z_L] = \Phi\left(\sqrt{\psi}\left(\frac{\mu - u^\star}{\sigma} - z_L\right)\right). \tag{77}$$

Recall that the equilibrium condition for the candidate is $z_L = (\theta^\dagger - u^\star)/\sigma$. Hence:

$$\Pr[\textbf{Protest Succeeds}] = \Phi\left(\sqrt{\psi}\left(\frac{\mu - \theta^\dagger}{\sigma}\right)\right). \tag{78}$$

By inspection, this is increasing in μ and equal to $1/2$ when $\mu = \theta^\dagger$. □

Proof of Lemma 3. Given the normality assumptions, the density $g(\theta \mid s_i)$ takes the form

$$g(\theta \mid s_i) = \frac{1}{\sqrt{2\pi \operatorname{var}[\theta \mid s_i]}} \exp\left(-\frac{(\theta - \mathrm{E}[\theta \mid s_i])^2}{2\operatorname{var}[\theta \mid s_i]}\right). \tag{79}$$

A few straightforward algebraic steps yield:

$$\log\left[\frac{g(\theta_H \mid s_i)}{g(\theta_L \mid s_i)}\right] = \frac{(\theta_L - \mathrm{E}[\theta \mid s_i])^2 - (\theta_H - \mathrm{E}[\theta \mid s_i])^2}{2\operatorname{var}[\theta \mid s_i]} \tag{80}$$

$$= \frac{\theta_L^2 - \theta_H^2}{2\operatorname{var}[\theta \mid s_i]} + \frac{(\theta_H - \theta_L)\,\mathrm{E}[\theta \mid s_i]}{\operatorname{var}[\theta \mid s_i]} \tag{81}$$

$$= \frac{\theta_H - \theta_L}{\operatorname{var}[\theta \mid s_i]}\left(\mathrm{E}[\theta \mid s_i] - \frac{\theta_H + \theta_L}{2}\right). \tag{82}$$

The posterior precision is the sum of the prior precision and signal precision, and the posterior mean is the precision-weighted average of the prior mean and the signal. That is,

$$\mathrm{E}[\theta \mid s_i] = \frac{\psi\mu + \lambda s_i}{\psi + \lambda} \quad \text{and} \quad \operatorname{var}[\theta \mid s_i] = \frac{\sigma^2}{\psi + \lambda}. \tag{83}$$

Substitution yields equation (37). Combining equation (34), (36), and (37) yields

$$U(s_i) = \lim_{n\to\infty} \log\left[\frac{\Pr[\textbf{Pivotal at } H \mid U(\cdot), s_i]}{\Pr[\textbf{Pivotal at } L \mid U(\cdot), s_i]}\right] \tag{84}$$

$$= \log\left[\frac{f(p_H \mid s_i, U(\cdot))}{f(p_L \mid s_i, U(\cdot))}\right] \tag{85}$$

$$= \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \log\left[\frac{g(\theta_H \mid s_i)}{g(\theta_L \mid s_i)}\right] \tag{86}$$

$$= \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \frac{\theta_H - \theta_L}{\sigma^2}\left[\lambda(s_i - \mu) + (\psi + \lambda)\left(\mu - \frac{(\theta_H + \theta_L)}{2}\right)\right], \tag{87}$$

where the second equality follows from an application of Lemma 1. Defining

$$u^\dagger = \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \frac{\theta_H - \theta_L}{\sigma^2}\left[\psi\mu - \frac{(\psi + \lambda)(\theta_H + \theta_L)}{2}\right] \quad \text{and} \quad r = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2}. \tag{88}$$

it is clear that this takes the linear form $U(s_i) = u^\dagger + r s_i$, as required. □

Proof of Lemma 4. I begin by verifying the various calculations in the text leading up to the statement of the lemma. Notice that $u_i - U(s_i) = (1 - r)s_i - u^\dagger + (u_i - s_i)$. Recall that

$$s_i \mid \theta \sim N\left(\theta, \frac{\sigma^2}{\lambda}\right) \quad \text{and} \quad (u_i - s_i) \mid (\theta, s_i) \sim N\left(0, \frac{(\lambda - 1)\sigma^2}{\lambda}\right). \tag{89}$$

Conditional on θ, the mean and variance are

$$\mathrm{E}[u_i - U(s_i)\,|\,\theta] = (1-r)\theta - u^\dagger \quad \text{and} \quad \mathrm{var}[u_i - U(s_i)\,|\,\theta] = \frac{(1-r)^2\sigma^2 + (\lambda-1)\sigma^2}{\lambda}, \tag{90}$$

which can be used to give equation (38). The probability of a protest vote, conditional on θ, is

$$P(\theta) = \Phi\left(\frac{(1-r)\theta - u^\dagger}{\sigma\sqrt{(\lambda - 1 + (1-r)^2)/\lambda}}\right), \tag{91}$$

and from this the expression in equation (39) is obtained. Inverting this,

$$\theta = \frac{u^\dagger}{1-r} + \frac{\sigma S(r)\Phi^{-1}(p)}{\lambda}, \tag{92}$$

which yields the solutions for θ_L and θ_H given in equation (40). Furthermore,

$$\theta_H - \theta_L = \frac{\sigma S(r)(z_H - z_L)}{\lambda} \quad \text{and} \quad \frac{\theta_H + \theta_L}{2} = \frac{u^\dagger}{1-r} + \frac{\sigma S(r)(z_H + z_L)}{2\lambda}. \tag{93}$$

Also, differentiating $P(\theta)$ and applying equation (92),

$$P'(\theta) = \frac{\lambda}{\sigma S(r)}\phi\left(\frac{\theta - [u^\dagger/(1-r)]}{(\sigma/\lambda)S(r)}\right) = \frac{\lambda}{\sigma S(r)}\phi\left(\Phi^{-1}(p)\right), \tag{94}$$

where $\phi(\cdot)$ is the density of the standard normal distribution. Using this,

$$\log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] = \frac{z_H^2 - z_L^2}{2}. \tag{95}$$

Now recall from equation (88) in the proof of Lemma 3 that

$$u^\dagger = \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \frac{\theta_H - \theta_L}{\sigma^2}\left[\psi\mu - \frac{(\psi+\lambda)(\theta_H+\theta_L)}{2}\right] \quad \text{and} \quad r = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2}. \tag{96}$$

If all other voters use a threshold $U(s_i) = u^\dagger + rs_i$, then clearly (as claimed) a best reply is to use a threshold $\hat{U}(s_i) = \hat{u}^\dagger + \hat{r}s_i$. Using the expression above and substituting in for $\theta_H - \theta_L$,

$$\hat{r} = \frac{\lambda(\theta_H - \theta_L)}{\sigma^2} = \frac{S(r)(z_H - z_L)}{\sigma}. \tag{97}$$

Similarly, for the intercept of the threshold function,

$$\hat{u}^\dagger = \log\left[\frac{P'(\theta_L)}{P'(\theta_H)}\right] + \frac{\theta_H - \theta_L}{\sigma^2}\left[\psi\mu - \frac{(\psi+\lambda)(\theta_H+\theta_L)}{2}\right] \tag{98}$$

$$= \frac{z_H^2 - z_L^2}{2} + \frac{S(r)(z_H - z_L)}{\lambda\sigma}\left[\psi\mu - (\psi+\lambda)\left(\frac{u^\dagger}{1-r} + \frac{\sigma S(r)(z_H+z_L)}{2\lambda}\right)\right] \tag{99}$$

$$= (z_H - z_L)\left(\frac{z_H + z_L}{2} + \frac{S(r)}{\lambda\sigma}\left[\psi\mu - (\psi+\lambda)\left(\frac{u^\dagger}{1-r} + \frac{\sigma S(r)(z_H+z_L)}{2\lambda}\right)\right]\right) \tag{100}$$

$$= (z_H - z_L)\left(\frac{z_H + z_L}{2}\left[1 - \frac{[S(r)]^2(\psi+\lambda)}{\lambda^2}\right] + \frac{S(r)}{\lambda\sigma}\left[\psi\mu - \frac{(\psi+\lambda)u^\dagger}{1-r}\right]\right) \tag{101}$$

$$= (z_H - z_L)\left(\frac{S(r)}{\lambda\sigma}\left[\psi\mu - \frac{(\psi+\lambda)u^\dagger}{1-r}\right] - \frac{z_H + z_L}{2\lambda}\left[\psi + \frac{(\psi+\lambda)(\lambda-1)}{(1-r)^2}\right]\right), \tag{102}$$

where the extra step of spotting $\hat{r}$ yields the expression in the statement of the lemma. $\square$

Proof of Proposition 8. From Lemma 3, a monotonic equilibrium involves a linear cutpoint function $U(s_i) = u^\dagger + rs_i$. Using Lemma 4, $u^\dagger$ and r satisfy equation (43). Fixing r, the second equation is linear in $u^\dagger$ and yields equation (45). However, r needs to satisfy $0 < r < 1$ and

$$\sigma = \frac{(z_H - z_L)S(r)}{r}. \tag{103}$$

There can be such a solution if and only if $(z_H - z_L)S(r)/r$ falls below σ within the range $(0, 1)$; this is so if and only if $\sigma > \bar{\sigma}$. A solution for r is a fixed point of $(z_H - z_L)S(r)/\sigma$. $S(r)$ is increasing in r and $S(0) > 0$, and $S(r) \to \infty$ as $r \to 1$. If there is a solution then $(z_H - z_L)S(r)/\sigma$ must cross r from above to below; $S(r) \to \infty$ as $r \to 1$ and so $(z_H - z_L)S(r)/\sigma$ must subsequently cross r from below to above. I conclude that if there is a solution, then there must be a second solution. (The exception is when $(z_H - z_L)S(r)/\sigma$ is tangent to r at the unique fixed point, which happens when $\sigma = \bar{\sigma}$.) There can, however, be no more than two fixed points, owing to the convexity of $S(r)$. □

Proof of Proposition 9. Substituting in for $S(r)$,

$$\frac{u^\dagger}{1-r} = \frac{1}{\lambda + r\psi}\left(r\psi\mu - \frac{z_H^2 - z_L^2}{2}\left[\frac{(\psi+\lambda)[S(r)]^2}{\lambda} - \lambda\right]\right) \tag{104}$$

$$= \frac{1}{\lambda + r\psi}\left(r\psi\mu - \frac{(z_H + z_L)(z_H - z_L)}{2}\left[\frac{\sigma r(\psi+\lambda)S(r)}{\lambda(z_H - z_L)} - \lambda\right]\right), \tag{105}$$

where the second equality follows from substitution of $S(r) = \sigma r/(z_H - z_L)$. Taking the limit as $\psi \to \infty$ yields equation (47). Evaluated at $\theta = \mu$, the probability of a vote for the candidate is

$$\Pr[u_i > u^\dagger + rs_i \,|\, \theta = \mu] = P(\mu) = \Phi\left(\frac{\mu - [u^\dagger/(1-r)]}{(\sigma/\lambda)S(r)}\right) \to \Phi\left(\frac{z_H + z_L}{2}\right). \tag{106}$$

If $\psi \to \infty$ then $\mathrm{E}[P(\theta) \,|\, \mu] \to P(\mu)$, yield the proposition's claims. □

Proof of Proposition 10. The proof follows the same procedure as the proof of Proposition 1 □

Proof of Proposition 11. Differentiation of $u^\star$ with respect to z_L yields

$$\frac{\partial u^\star}{\partial z_L} = -\frac{\sigma}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)}\left[u^\star + \frac{\psi(z_H - z_L)^2}{2}\right] \tag{107}$$

$$\text{and} \quad \left.\frac{\partial^2 u^\star}{\partial z_L^2}\right|_{\partial u^\star/\partial z_L = 0} = \frac{\sigma\psi(z_H - z_L)}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} > 0, \tag{108}$$

and so $u^\star$ is a quasi-convex function of z_L. Hence, it is either everywhere decreasing in z_L, everywhere increasing in z_L, or first decreasing and then increasing. To investigate further, I evaluate the first derivative at upper bound of the range for z_L:

$$\left.\frac{\partial u^\star}{\partial z_L}\right|_{z_L \to z_H} = -\frac{\psi}{\sigma}\left[\mu - \sigma z_H\right] < 0 \quad \Leftrightarrow \quad \mu > \sigma z_H. \tag{109}$$

So, if $\mu > \sigma z_H$ then $u^\star$ must be decreasing in z_L across its entire range. However, if $\mu < \sigma z_H$ then $u^\star$ is increasing in z_L if z_L is close to z_H, or equivalently if $p_H - p_L$ is sufficiently small.

Similarly, differentiation with respect to z_H yields:

$$\frac{\partial u^\star}{\partial z_H} = \frac{\sigma}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} \left[u^\star - \frac{\psi(z_H - z_L)^2}{2} \right] \tag{110}$$

$$\text{and} \quad \left.\frac{\partial^2 u^\star}{\partial z_H^2}\right|_{\partial u^\star / \partial z_H = 0} = -\frac{\sigma\psi(z_H - z_L)}{\psi(z_H - z_L)^2 + \sigma(z_H - z_L)} < 0, \tag{111}$$

and so $u^\star$ is a quasi-concave function of z_H. Hence, it is either everywhere decreasing in z_L, everywhere increasing in z_H, or first increasing and then decreasing. Now:

$$\left.\frac{\partial u^\star}{\partial z_H}\right|_{z_H \downarrow z_L} = \frac{\psi}{\sigma}\left[\mu - \sigma z_L\right] < 0 \quad \Leftrightarrow \quad \mu < \sigma z_L. \tag{112}$$

So, if $\mu < \sigma z_L$ then $u^\star$ must be decreasing in z_H across its entire range; equivalently, this means that $u^\star$ is increasing in $1 - p_H$ across its range. However, if not then $u^\star$ is increasing in z_H (and so decreasing in the need for coordination $1 - p_H$) if $p_H - p_L$ is sufficiently small. □

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