The Impact of Perceived Strength in the War of Attrition

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Abstract. In a war of attrition a player's *perceived strength* is the distribution describing beliefs about her valuation. Small asymmetries in strength have a large effect: in the unique equilibrium of a game with a deadline the war ends quickly (instantly, as the deadline becomes infinite) with a concession by the (perceived) weaker player. The ranking of strength compares hazard rates in the upper tails of the distributions of beliefs; players with uncertain valuations tend to have greater strength. The results hold when techniques other than a deadline are used to guarantee a unique equilibrium.

In a war of attrition, the first player to quit concedes a prize to her opponent. Costly fighting is worthwhile only if an opponent quits in the near future. These features are common to important phenomena, including labour negotiations, the voluntary provision of public goods, macroeconomic stabilisation, the adoption of technological standards, and political lobbying. Natural questions arise. Who will win the war? When will it end?

The context is a game in which players differ in their (privately known) valuations for the prize. A player's *perceived strength* corresponds to the distribution that represents beliefs about her valuation. For the main result the comparison of strength involves the hazard rates in the upper tails of the distributions. Rough answers to "who wins and when?" are these: *the war ends with the immediate concession of the perceived weaker player.*

Of course, the classic war of attrition has many equilibria. This can be resolved by making one of several small (and reasonable) modifications that dissuade a player from fighting forever. Here I do this directly by placing an upper limit on stopping times. The unique equilibrium converges to (and so selects) an equilibrium of a classic game as the time limit (in essence, a deadline) grows large. (Other methods for obtaining uniqueness yield the same results.)

The unique equilibrium is (of course) symmetric when players' beliefs are symmetric. However, asymmetric players have valuations that are (at least believed to be) drawn from different distributions. A player is then *perceived to be stronger* if her distribution of valuations stochastically dominates (in some sense) that of her opponent. I use several dominance rankings to predict properties of the equilibrium. I preview three results here.

[1]This paper is based upon elements of a long-neglected paper (Myatt, 2005) which was dormant for many years after (embarrassingly) I lost the files for it. Only now have I returned to write a successor to it. Looking back to the ancestor paper, I offer warm thanks to Jon Dworak, Justin P. Johnson, Max Kwiek, Paul Klemperer, Eric Rasmusen, Kevin Roberts, Juuso Välimäki, Chris Wallace, Paddy Wallace, Peyton Young, and the anonymous reviewers of Myatt (2005) for their comments and suggestions. Thank you also to recent commentators, including Alex Teytelboym, for encouraging me to resurrect the paper, develop its implications, and convert it to a note.

Firstly, if the distribution of the (perceived) stronger player's valuation hazard-rate dominates that of her weaker opponent then she uses a more aggressive stopping-rule strategy. Secondly, if her distribution first order stochastically dominates (a less stringent condition) and if valuations are strictly positive, then the weaker opponent exits at the beginning of the game with positive probability. Thirdly, suppose that a hazard-rate ranking applies only to the upper tails of the distributions. Under this condition: as the limit to players' stopping times increases, *the unique equilibrium converges to one in which the weaker player always exits at the beginning.*

The third result has bold implications. Firstly, true prize valuations do not (at least in the limiting case of a long time horizon) determine the outcome, and so the prize allocation may be inefficient. Secondly, the game ends quickly and so the classic war-of-attrition model might not, by itself, explain delay. Thirdly, any asymmetries in perception are critical to a player's likely success, and so she may pursue activities that enhance (a particular feature of) her perceived valuation of a prize rather than the value itself.

I also offer notes of caution. Firstly, the "instant exit" result applies only as the finite horizon grows large. If players are perceived similarly and the deadline is not too distant, then the symmetric equilibrium of a symmetric specification is a reasonable approximation. Secondly, the probability of instant concession rises slowly as the time limit grows: very long horizons are needed for the most extreme result to bite. Thirdly, the desire for the weaker player to fight rather than concede returns if there is the chance of an event that changes players' perceptions.

The most relevant notion of strength depends on an upper-tail hazard-rate comparison. This is readily satisfied if the distribution of the stronger player is an upward mean-shifted version of that of the weaker player. However, the comparison does not require a difference of means. Instead, the player perceived to be stronger is the one that is relatively more likely to have a very high prize valuation, even if she has a lower expected valuation. This holds (for leading cases) when there is greater uncertainty about a player: a combatant with uncertain real strength enjoys (according to the criterion) greater perceived strength. This final implication feeds into players' incentives to manipulate perceptions. A player wishes to prevent any activity which allows observers to learn about her own valuation (to maintain an air of mystery) while encouraging anything that allows learning about her opponent.

War-of-attrition games in the literature (Maynard Smith, 1974; Bishop and Cannings, 1978b,a; Bishop, Cannings, and Maynard Smith, 1978; Riley, 1979, 1980; Nalebuff and Riley, 1985) have many equilibria. Researchers have often focused on the symmetric equilibrium of a symmetric game (e.g. Kapur, 1995; Krishna and Morgan, 1997; Bulow and Klemperer, 1999). However, (reasonable) modifications result in a unique equilibrium: a finite time limit (Cannings and Whittaker, 1995; Ponsati, 1995), a chance of players who fight forever (Fudenberg and Tirole, 1986; Kornhauser, Rubinstein, and Wilson, 1989), or a hybrid all-pay auction specification (Güth and Van Damme, 1986; Amann and Leininger, 1996; Riley, 1999). In games with complete information such modifications predict the instant concession of the weaker player (Kornhauser, Rubinstein, and Wilson, 1989; Riley, 1999; Kambe, 1999; Abreu and Gul, 2000).

But what if privately-known valuations are perceived to be from different distributions? The literature establishes (conditions for) uniqueness, but rarely offers a cross-player comparison of strategies; nor does it usually characterise the equilibrium of a classic war of attribution that is selected as any uniqueness-generating modification is removed. I contribute by using the distinction of perceived strength (comparing distributions) to show that it is the key driver of who wins and when. In doing so, the closest work to this one is an insightful analysis by Martinelli and Escorza (2007) which extends the Alesina and Drazen (1991) model of stabilisation to asymmetric players, and shows positive-probability instant concession when players are committed to fight forever with small probability.

I also complement more recent studies of asymmetric all-pay auctions and contests (notably Siegel, 2009, 2010, 2014a,b) including scenarios with spillovers (Xiao, 2018; Betto and Thomas, 2024). Others (including Huangfu, Ghosh, and Liu, 2023) have examined resource constraints, which correspond to player-specific time limits. The selected equilibrium of this paper suggests a war of attrition without attrition. This is also a theme of recent work by Georgiadis, Kim, and Kwon (2022) in which conditions change according to a Brownian motion: Markov pure strategies (interpreted as "no attrition") are played in equilibrium.[2]

Preliminaries: Model and Equilibrium

The Game. Two players indexed by $i \in \{1, 2\}$ simultaneously choose stopping times $t_i \in [0, T]$ where T is a time limit (or deadline). The payoff of player i is

$$\pi_i(t_i, t_j) = u_i \times \left[\mathcal{I}[t_i > t_j] + \frac{\mathcal{I}[t_i = t_j]}{2} \right] - \min\{t_1, t_2\} \tag{1}$$

where "$\mathcal{I}[\cdot]$" is the indicator function. A player's privately known type u_i is her *prize valuation*. Player $j \neq i$ believes that u_i is drawn from the distribution $F_i(\cdot)$ with strictly positive and continuous density $f_i(\cdot)$ on the support $(\underline{u}_i, \infty)$ where $\underline{u}_i < 2T$. $F_i(\cdot)$ represents the *perceived strength* of player i from the perspective of $j \neq i$.

A strategy for player i maps her valuation to a stopping time: $t_i(u_i) : (\underline{u}_i, \infty) \mapsto [0, T]$. I seek Bayesian Nash equilibria, where (without loss) a player chooses the highest payoff-maximising stopping time whenever indifferent. Henceforth I refer to this as an "equilibrium."

Commentary. The specification of eq. (1) awards a prize to the player who fights for longest, with a "coin toss" tie break. Other tie-break rules may be used while retaining most results.

The fighting cost is proportional to the time elapsed (Maynard Smith, 1974). What matters for behaviour is a player's valuation relative to the cost of fighting, and so I normalise marginal costs to unity. A player chooses duration rather than intensity, and so there is no signalling from endogenous costly effort of the kind considered by Hörner and Sahuguet (2007, 2011).

[2]It and other related papers by Lambrecht (2001), Murto (2004), and Steg (2015) fall within the framework of a recent and comprehensive paper by Décamps, Gensbittel, and Mariotti (2023).

Beyond the linear-in-time specification, other formulations in which "leader" and "follower" payoffs are general functions of time offer similar insights (Bishop and Cannings, 1978b; Hendricks, Weiss, and Wilson, 1988).[3] The "linear costs" approach allows the war of attrition to be interpreted as an ascending-price all-pay auction (Klemperer, 1999): the price t rises until a player concedes, and both players pay the exit price.[4]

The cost $C_i(t_1, t_2) = \min\{t_1, t_2\}$ incurred by a player depends only on the time of first exit: a player stops fighting immediately following a concession, and so further planned fighting is costless. Other specifications make it costly to plan a later exit. For example, if the cost is

$$C_i(t_1, t_2) = \min\{t_1, t_2\} + \beta \max\{t_i - \min\{t_1, t_2\}, 0\}, \tag{2}$$

then a player pays β per unit of additional planned fighting time.[5] This corresponds to a hybrid all-pay auction (Güth and Van Damme, 1986; Amann and Leininger, 1996; Riley, 1999) in which the winner pays a combination of the winning and losing bids. (The war of attrition corresponds to $\beta = 0$; a plain-vanilla all-pay auction to $\beta = 1$.) Later I note that eq. (2) yields a unique equilibrium even if there is no time limit ($T = \infty$).

This is a finite-horizon war of attrition (Cannings and Whittaker, 1995; Ponsati, 1995) in which the deadline T limits a player's aggression. This means that her opponent will fight to the end with positive probability (player j with $u_j > 2T$ prefers to do so rather than exiting earlier). This (an opponent might never quit) means that fighting for longer is always costly. This feature pins down a unique equilibrium. For this reason I insist that $T < \min\{\bar{u}_1, \bar{u}_2\}/2$ where $\bar{u}_i$ is the upper bound to the support of $F_i(\cdot)$; this is sufficient for many results to hold, including existence and uniqueness.[6] However, to allow for my central "large T" result (Proposition 3) and to simplify exposition I specify full support ($\bar{u}_1 = \bar{u}_2 = \infty$). The assumption on the lower-bound of support ($\underline{u}_i/2 < T$) allows for players who optimally exit before time T. (With other suitable specifications I can allow for bounded valuations.)

A player's (true) strength here is her valuation for a prize. A related formulation is where a player's type (and so strength) is determined by her cost of fighting.[7]

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[3]The exiting player enjoys a leader payoff of $L_i(t)$ whilst the follower receives $S_i(t)$. $L_i'(t) < 0$ ensures that the leader would rather quit sooner. If $S_i(t) > L_i(t)$ then a player is willing to wait for the anticipated exit of her opponent. One case is when a "fighting cost" corresponds to the delay before the award of a second prize: this might be implemented via $L_i(t) = Be^{-\delta_i t}$ and $S_i(t) = Ae^{-\delta_i t}$, where players differ via δ_i. Alternatively (Ponsati and Sákovics, 1995) it may be implemented via $L_i(t) = B_i e^{-t}$ and $F_i(t) = e^{-t}$ where players differ via B_i.

[4]This contrasts with a "first-price all-pay" auction (Baye, Kovenock, and De Vries, 1993, 1996) in which (unlike the second-price case) it is costly to raise a bid even if that bid is the highest.

[5]For Betto and Thomas (2024, pp. 183–184) eq. (2) specifies a war with costly preparation. Hendricks, Weiss, and Wilson (1988) and Pitchik (1981) referred to wars of attrition as "noisy" games in the sense that a player can "hear" her opponent's exit. (In contrast the "noisy players" of Anderson, Goeree, and Holt (1998a,b) choose quantal best replies in a logit equilibrium à la McKelvey and Palfrey (1995, 1996).) In a "silent" game a player does not observe such an exit; this is an all-pay auction. Equation (2) has this interpretation: a player fails to observe (or "hear") her opponent's exit with probability β and so continues to her planned exit time.

[6]Ponsati and Sákovics (1995) offered this summary: "There are a continuum of equilibria [of the classic war of attrition] characterized by a system of ordinary differential equations. Uniqueness may be achieved by perturbing the game, imposing that for a positive measure of types it is a dominant strategy not to concede."

[7]Another interpretation of strength is that a weaker player faces a risk of a random forced exit (Asako, 2015)

Equilibrium. An equilibrium has familiar properties: players fight over some time interval $(0, \bar{t})$ for $\bar{t} \in (0, T)$ using monotonic stopping rules. However, some players with high valuations fight until T, and others with low valuations may concede immediately. I note these properties here. (The literature-standard proofs of Lemmas 1 to 3 are reported in an appendix.)

Lemma 1 (Basic Properties of Equilibrium Stopping Rules). *For both players:*

(i) There is an upper-bound $\bar{u}^\star$ *above which a player fights until* T.

(ii) There is a lower-bound $\underline{u}_i^\star \in [\bar{u}_i, \infty)$ *at or below which a player* $i \in \{1, 2\}$ *exits at time zero. If* $\underline{u}_i^\star > \underline{u}_i$ *then she "instantly exits" with strictly positive probability at the beginning.*

(iii) There is a unique time $\bar{t} \in (0, T)$ *at or after which neither player exits.*

(iv) The stopping rule $t_i(u)$ *is strictly and continuously increasing from* 0 *to* $\bar{t}$ *for* $u \in (\underline{u}_i^\star, \bar{u}^\star)$.

Claim (iv) ensures that the inverses of the stopping rules are well defined for $t \in (0, \bar{t})$. I write $v_i(t)$ for such (differentiable) inverses. The distribution $G_i(\cdot) \equiv \Pr[t_i \leq t]$ of the stopping time for player i and its hazard rate are

$$G_i(t) = F_i(v_i(t)) \quad \text{and so} \quad \frac{g_i(t)}{1 - G_i(t)} = \frac{v_i'(t) f_i(v_i(t))}{1 - F_i(v_i(t))} \quad \text{for} \quad t \in (0, \bar{t}). \tag{3}$$

The optimal behaviour of player j for a stopping time $t \in (0, \bar{t})$ is straightforward: the marginal cost of fighting is equal to her valuation $v_j(t)$ multiplied by the hazard rate $g_i(t)/[1 - G_i(t)]$ of her opponent's exit. The first-order conditions for the two players are

$$\frac{v_1(t) v_2'(t) f_2(v_2(t))}{1 - F_2(v_2(t))} = \frac{v_2(t) v_1'(t) f_1(v_1(t))}{1 - F_1(v_1(t))} = 1. \tag{4}$$

This pair of differential equations can be solved for $v_1(t)$ and $v_2(t)$, and then inverted to obtain the equilibrium stopping rules. However, (a pair of) boundary conditions are needed.

Consider behaviour at $t = 0$ and (for discussion here) suppose that valuations are strictly positive: $\min\{\underline{u}_1, \underline{u}_2\} > 0$. If a player exits with positive probability at the beginning then her opponent always find it profitable to fight for some time: there cannot be (the positive probability of) "instant exit" by both players. This means that $v_i(0) = \underline{u}_i^\star = \underline{u}_i$ for one of $i \in \{1, 2\}$, while the other player $j \neq i$ must satisfy $v_j(0) = \underline{u}_j^\star \geq \underline{u}_i$. This leaves a free choice for $\underline{u}_j^\star$, and so the need for a second boundary condition.[8]

That condition is obtained by considering the end of active play, where $v_i(\bar{t}) = \bar{u}^\star$ for both players. A player with valuation $\bar{u}^\star$ exiting at time $\bar{t}$ is indifferent to staying until the deadline T, and so $\frac{\bar{u}^\star}{2} = T - \bar{t}$. This is sufficient to pin down unique solutions for $v_i(t)$.

[8]If valuations extend below zero then with probability $F_i(0)$ a player prefers to lose and so exits at $t = 0$. An opponent $j \neq i$ with $u_j > 0$ always fights for some period of time: $\underline{u}_1^\star = \underline{u}_2^\star = 0$. This pair of boundary conditions hints at a unique solution. Unfortunately, there is a failure of Lipschitz continuity at $t = 0$ (as explained by Fudenberg and Tirole, 1986): writing a first-order condition from eq. (4) as $v_i'(t) = [1 - F_i(v_i(t))]/[v_j(t) f_i(v_i(t))]$, the denominator of the right-hand side tends to zero as $t \to 0$. A terminal boundary condition is still required.

Lemma 2 (Uniqueness, Existence, and Properties of Inverse Stopping Rules). *There is a unique equilibrium. Each equilibrium inverse $v_i(t)$ is differentiable for $t \in (0, \bar{t})$. Define*

$$\Lambda_i(u) = \int_u^{\bar{u}^\star} \frac{1}{x} \frac{f_i(x)}{1 - F_i(x)}\, dx \quad \textit{where} \quad \bar{u}^\star = 2(T - \bar{t}). \tag{5}$$

For any $t \in (0, \bar{t})$ the pair of inverse stopping rules satisfy $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$.

The crucial step needed for uniqueness is the boundary condition at $\bar{t}$. This is generated by the existence of the deadline T. In its absence, there is only a single boundary of condition at $t = 0$, and so there is a family of equilibrium solutions each determined by the identity of the player $j \in \{1, 2\}$ and the probability $F_j(\underline{u}_j^\star)$ with which that player instantly exits at time zero. (If $T = \infty$ but there are other mild modifications to the game then a suitable boundary condition for $t \to \infty$ can still be obtained.)

Main Results: Perceived Strength and Instant Concession

Notions of Perceived Strength. A player i is perceived to be stronger than her opponent j (from now on, simply "stronger") if $F_i(\cdot)$ stochastically dominates $F_j(\cdot)$ in some sense. Of course, different notions of stochastic dominance are possible. I describe three here.[9]

Definition (Notions of Comparative Strength). *Player 1 is stronger than Player 2*

(i) in the sense of hazard-rate dominance, $F_1 \succ_{HRD} F_2$, *if for all* $u \in (\max\{\underline{u}_1, \underline{u}_2\}, \infty)$,

$$\frac{f_2(u)}{1 - F_2(u)} > \frac{f_1(u)}{1 - F_1(u)}; \tag{6}$$

(ii) in the sense of first-order dominance, $F_1 \succ_{FOSD} F_2$, *if for all* $u \in (\max\{\underline{u}_1, \underline{u}_2\}, \infty)$,

$$F_2(u) > F_1(u); \tag{7}$$

(iii) and in the sense of asymptotic hazard-rate dominance, $F_1 \succ_{AHRD} F_2$, *if*

$$\liminf_{u \to \infty} \left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] > 0. \tag{8}$$

The first criterion is the most stringent. It implies the second criterion, and also

$$\frac{\partial}{\partial u} \log \left[\frac{1 - F_1(u)}{1 - F_2(u)} \right] = \frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} > 0, \tag{9}$$

and so the odds ratio of $u_1 > u$ versus $u_2 > u$ is increasing in u.[10] Furthermore, and given $u_1 > u$ and $u_2 > u$, the conditional distribution of u_1 stochastically dominates (in the first-order sense) that of u_2. This is conditional stochastic dominance (Maskin and Riley, 2000).

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[9] Definitions (i) and (ii) use strict inequalities, but are readily modified. A standard definition of first-order dominance would ask eq. (7) to apply only weakly in general, but strictly for some values.

[10] When evaluating $\Pr[i = 1 \,|\, u_i > u]$ an increase in u is "good news" in the sense of Milgrom (1981).

The first criterion also implies the third criterion unless the difference between the hazard rates disappears in the upper tails. A player that is stronger in the asymptotic hazard-rate sense is far more likely (in a relative sense) to experience very high valuations:[11]

$$F_1 \succ_{\text{AHRD}} F_2 \quad \Rightarrow \quad \lim_{u\to\infty} \frac{1 - F_1(u)}{1 - F_2(u)} = \infty. \tag{10}$$

The second and third criteria are not nested. However, if a player is stronger in the asymptotic hazard-rate sense then she cannot be weaker in the first-order sense.

Sometimes the first two criteria do not apply, but the third one does. This is naturally so when a player's valuation is riskier in the spirit of Rothschild and Stiglitz (1970). To illustrate, suppose that players are characterised by mean μ_i and variance σ_i^2 parameters, and $F_i(u) = ((u - \mu_i)/\sigma_i)$ where $F(\cdot)$ has full support, zero mean, unit variance, and a hazard rate with a derivative that is bounded away from zero in the upper tails. (The standard normal, for example.)

If $\sigma_1 = \sigma_2$ and $\mu_1 > \mu_2$ then Player 1 is stronger than Player 2 according to all three of the notions here; her distribution is a mean-shift upward of that of her opponent. However, if $\sigma_1 > \sigma_2$ then Player 1 is stronger than Player 2 in the asymptotic hazard-rate sense, but not necessarily in the other senses. Her distribution is a spread of that of her opponent, which makes it more likely (relative to her opponent) that she has a very high valuation.

Instant Exit. I now relate notions of strength to the equilibrium stopping rules.

Suppose that $F_1 \succ_{\text{HRD}} F_2$, so that Player 1 is stronger in the hazard-rate sense.

Other things equal (when stopping rules intersect) Player 2's higher hazard rate means that she exits more quickly. To maintain an equilibrium, a wider interval of types of Player 1 must exit within that interval of time. More formally, if the stopping rules cross at some valuation u and at time $t = t_1(u) = t_2(u)$ then the conditions from eq. (4) reduce to

$$\frac{v_2'(t) f_2(u)}{1 - F_2(u)} = \frac{v_1'(t) f_1(u)}{1 - F_1(u)} \quad \Rightarrow \quad v_2'(t) < v_1'(t) \quad \Rightarrow \quad t_2'(u) > t_1'(u). \tag{11}$$

The stopping time of a weaker player rises more quickly. For stopping rules to intersect (and they can intersect only once) at the end-of-attrition time $\bar{t}$, they must begin where the weaker player's stopping time is below that of the stronger player.

Proposition 1 (Hazard-Rate Dominance ⇒ Quicker Exit by a Weaker Player). *If Player 1 is stronger than Player 2 in the sense of hazard-rate dominance, then Player 1 plays more aggressively than Player 2, and so Player 2 exits more quickly from the war of attrition.*

Formally, if $F_1 \succ_{HRD} F_2$ *then* $t_1(u) > t_2(u)$ *for all* $u \in (\max\{\underline{u}_1, 0\}, \bar{u}^\star)$.

Proof. If $F_1 \succ_{\text{HRD}} F_2$ then the integrand in the expression for $\Lambda_i(u)$ from Lemma 2 is lower for $i = 1$ than for $i = 2$, and so $v_1(t) < v_2(t)$ for $t < \bar{t}$ in order to satisfy $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$. □

[11] This is related to the unbounded likelihood-ratio property required for Mirrlees (1999) contracts.

An implication is that the prize allocation is biased toward the stronger player.

Another implication is obtained by tracing the stopping rules back to time zero: the ranking of those rules means (if the lower bounds of valuations are strictly positive) that $\underline{u}_1^\star < \underline{u}_2^\star$. The hazard-rate ranking of the players implies that $\underline{u}_2 \le \underline{u}_1$, and from this I conclude that $\underline{u}_2^\star > \underline{u}_2$: the weaker player must concede at time $t = 0$ with positive probability. In fact, this result holds under the weaker condition of first-order dominance.

Proposition 2 (First-Order Dominance ⇒ Some Instant Exit). *If Player 1 is stronger than Player 2 in the sense of first-order dominance, and if valuations are strictly positive, then the weaker Player 2 exits at time zero with positive probability.*

Formally, if $F_1 \succ_{FOSD} F_2$ *and* $\min\{\underline{u}_1, \underline{u}_2\} > 0$ *then* $\underline{u}_2^\star > \underline{u}_2$ *and so* $\Pr[t_2(\cdot) = 0] > 0$.

Proof. $F_1 \succ_{\text{FOSD}} F_2$ implies that $\underline{u}_2 \le \underline{u}_1$. Consider an equilibrium in which Player 2 does not exit at time zero: $\underline{u}_2^\star = \underline{u}_2 \le \underline{u}_1 \le \underline{u}_1^\star$. Taking eq. (5), integrating parts, and applying Lemma 2,

$$\Lambda_i(\underline{u}_i^\star) = \frac{\log[1 - F_i(\underline{u}_i^\star)]}{\underline{u}_i^\star} - \frac{\log[1 - F_i(\bar{u}^\star)]}{\bar{u}^\star} - \int_{\underline{u}_i^\star}^{\bar{u}^\star} \frac{\log[1 - F_i(u)]}{u^2}\, du$$

$$\begin{aligned} \Rightarrow \quad 0 = \Lambda_2(\underline{u}_2^\star) - \Lambda_1(\underline{u}_1^\star) = {} & \underbrace{\frac{\log[1 - F_2(\underline{u}_2^\star)]}{\underline{u}_2^\star}}_{\text{zero if } \underline{u}_2^\star = \underline{u}_2} - \underbrace{\int_{\underline{u}_2^\star}^{\underline{u}_1^\star} \frac{\log[1 - F_2(u)]}{u^2}\, du}_{\text{strictly negative}} - \underbrace{\frac{\log[1 - F_1(\underline{u}_1^\star)]}{\underline{u}_1^\star}}_{\text{strictly negative}} \\ & + \underbrace{\frac{1}{\bar{u}^\star} \log \frac{1 - F_1(\bar{u}^\star)}{1 - F_2(\bar{u}^\star)}}_{\text{strictly positive if } F_1 \succ_{\text{FOSD}} F_2} + \underbrace{\int_{\underline{u}_1^\star}^{\bar{u}^\star} \frac{1}{u^2} \log \frac{1 - F_1(u)}{1 - F_2(u)}\, du}_{\text{strictly positive if } F_1 \succ_{\text{FOSD}} F_2} > 0. \end{aligned} \tag{12}$$

The right-hand side is strictly positive. This contradicts the supposition that $\underline{u}_2^\star = \underline{u}_2$. □

I now turn to the third (and main) result, which uses only asymptotic hazard-rate dominance.

As the deadline T increases, the fraction of types who have a dominant strategy to fight to the deadline (with valuations $u_i \ge 2T$) falls. This also drives up the time $\bar{t}$ at which attrition stops. However, it also increases the length $T - \bar{t}$ of the "no more concessions" period, which increases the upper bound $\bar{u}^\star = 2(T - \bar{t})$ to the set of player types who fight through this period.

Lemma 3 (Receding Deadline). *The unique equilibrium satisfies* $\lim_{T\to\infty} \bar{u}^\star = \infty$, *and so as the deadline recedes the probability that players fight to that deadline falls to zero.*

Now suppose that Player 1 is stronger than Player 2 in the asymptotic hazard-rate sense.

If the deadline is distant (T is large) then hazard rates are bounded apart at the end-of-attrition time $\bar{t}$ when both marginal players share the same valuation $v_1(\bar{t}) = v_2(\bar{t}) = \bar{u}^\star$. Working back from that deadline valuation, the stopping rule of the weaker (in the "$\succ_{\text{AHRD}}$" sense) player is steeper than that of the stronger player. If that deadline is sufficiently distant and the difference in hazard rates is maintained, then the backward-induction process means that the stopping rule of the weaker player hits a time of zero for some relatively large valuation.

Proposition 3 (Asymptotic Hazard-Rate Dominance ⇒ Complete Instant Exit). *If Player 1 is stronger than Player 2 in the sense of asymptotic hazard-rate dominance, then in the limit with a long time horizon Player 2 always exits at time zero, and Player 1 always wins.*

Formally, if $F_1 \succ_{AHRD} F_2$ *and* $\underline{u}_1 > 0$ *then* $\lim_{T\to\infty} \underline{u}_2^\star = \infty$. *Equivalently,* $t_2(u) = 0$ *for any* $u \in (\underline{u}_2, \infty)$ *if* T *is sufficiently large. If* $\underline{u}_1 < 0$ *then* $\lim_{T\to\infty} t_2(u) = 0$ *for any* $u \in (\underline{u}_2, \infty)$.

In both of these subcases, $\lim_{T\to\infty} t_1(u) > 0$ *for any* $u \in (\max\{\underline{u}_1, 0\}, \infty)$

Proof. To shorten exposition, suppose that valuations are strictly positive ($\min\{\bar{u}_1, \bar{u}_2\} > 0$; the proof readily extends to other cases) and that $u_1^\star < u_2^\star$ (this is true for T sufficiently large). Fix $\varepsilon > 0$, a valuation $u_\varepsilon^\dagger$ above which the hazard rates differ by at least ε, and T large enough so that $\bar{u}^\star > u_\varepsilon^\dagger$. Construct a sequence of stopping times where $\underline{u}_2^\star$ remains bounded. Then:

$$\Lambda_1(\underline{u}_1^\star) = \Lambda_2(\underline{u}_2^\star) \quad \Leftrightarrow \quad \underbrace{\int_{\underline{u}_1^\star}^{\underline{u}_2^\star} \frac{1}{u}\frac{f_1(u)}{1-F_1(u)}\,du}_{\text{bounded if } \underline{u}_2^\star \text{ remains bounded}} = \int_{\underline{u}_2^\star}^{\bar{u}^\star} \frac{1}{u}\left[\frac{f_2(u)}{1-F_2(u)} - \frac{f_1(u)}{1-F_1(u)}\right] du$$

$$\geq \underbrace{\int_{\underline{u}_2^\star}^{\max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}} \frac{1}{u}\left[\frac{f_2(u)}{1-F_2(u)} - \frac{f_1(u)}{1-F_1(u)}\right] du}_{\text{bounded if } \underline{u}_2^\star \text{ remains bounded}} + \varepsilon \underbrace{\int_{\max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}}^{\bar{u}^\star} \frac{1}{u}\,du}_{\text{diverges as } \bar{u}^\star \to \infty}. \quad (13)$$

The final term $\varepsilon \log[\bar{u}^\star / \max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}]$, diverges because (Lemma 3) $\lim_{T\to\infty} \bar{u}^\star = \infty$, contradicting the supposition that $\underline{u}_2^\star$ remains bounded as $T \to \infty$. The conclusion is: $\underline{u}_2^\star \to \infty$. □

Unlike Propositions 1 and 2, Proposition 3 takes a parameter to the limit by extending the deadline T and so "selects" an equilibrium from those of the classic war of attrition.

This paper asks "who wins and when?" For the selected equilibrium the "who?" is the player that is perceived as stronger in the sense of asymptotic hazard-rate dominance. This player might well (be likely to) have a low valuation. What it takes to win is a belief (held by an opponent) that a player is relatively more likely to have a very high valuation.

Turning to the "when?" the answer is "immediately!" This is a somewhat uncomfortable prediction that wars of attrition will not be fought. However, if the deadline is present but not too large then there is only partial early concession.

Illustration. An illustration is obtained when each player $i \in \{1, 2\}$ is believed to have exponentially distributed valuations with hazard rate λ_i: the distribution is $F_i(u) = 1 - e^{-\lambda_i(u - \underline{u}_i)}$. $F_1 \succ_{\text{AHRD}} F_2$ holds if and only if $\lambda_1 < \lambda_2$.

The solution to eq. (5) is $\Lambda_i(u) = \lambda_i \log[\bar{u}^\star/u]$. From Lemma 2, the inverse-stopping rules are related by $\lambda_2 \log v_2(t) = (\lambda_2 - \lambda_1)\log \bar{u}^\star + \lambda_1 \log v_1(t)$, and so those rules move apart as T increases and $\bar{u}^\star$ grows with it. To simplify expressions I set $\underline{u}_1 = \underline{u}_2 = 0$. I also write

$\mu_i \equiv \mathrm{E}[u_i] = 1/\lambda_i$. The equilibrium stopping-rule solutions are:

$$t_1(u) = \frac{u^{1+(\mu_2/\mu_1)}(\bar{u}^\star)^{1-(\mu_2/\mu_1)}}{\mu_1+\mu_2} \quad \text{and} \quad t_2(u) = \frac{u^{1+(\mu_1/\mu_2)}(\bar{u}^\star)^{1-(\mu_1/\mu_2)}}{\mu_1+\mu_2}$$
$$\text{where} \quad \bar{u}^\star = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4T}{\mu_1+\mu_2}}\right)\frac{\mu_1+\mu_2}{2}. \quad (14)$$

The stopping time (for a particular valuation) is increasing (without bound) in the deadline for the stronger player, but decreasing (in the limit, to zero) for the weaker player. However, the weaker player reacts more strongly to her actual valuation.

Extensions and Implications

Sensitivity. The prediction of rapid concession (Proposition 3) is obtained in the limit as $T \to \infty$, based on an argument that uses the final term of eq. (13). This diverges with $\log \bar{u}^\star$, and so at a slow rate. Very long deadlines might be required for near-instant exit.

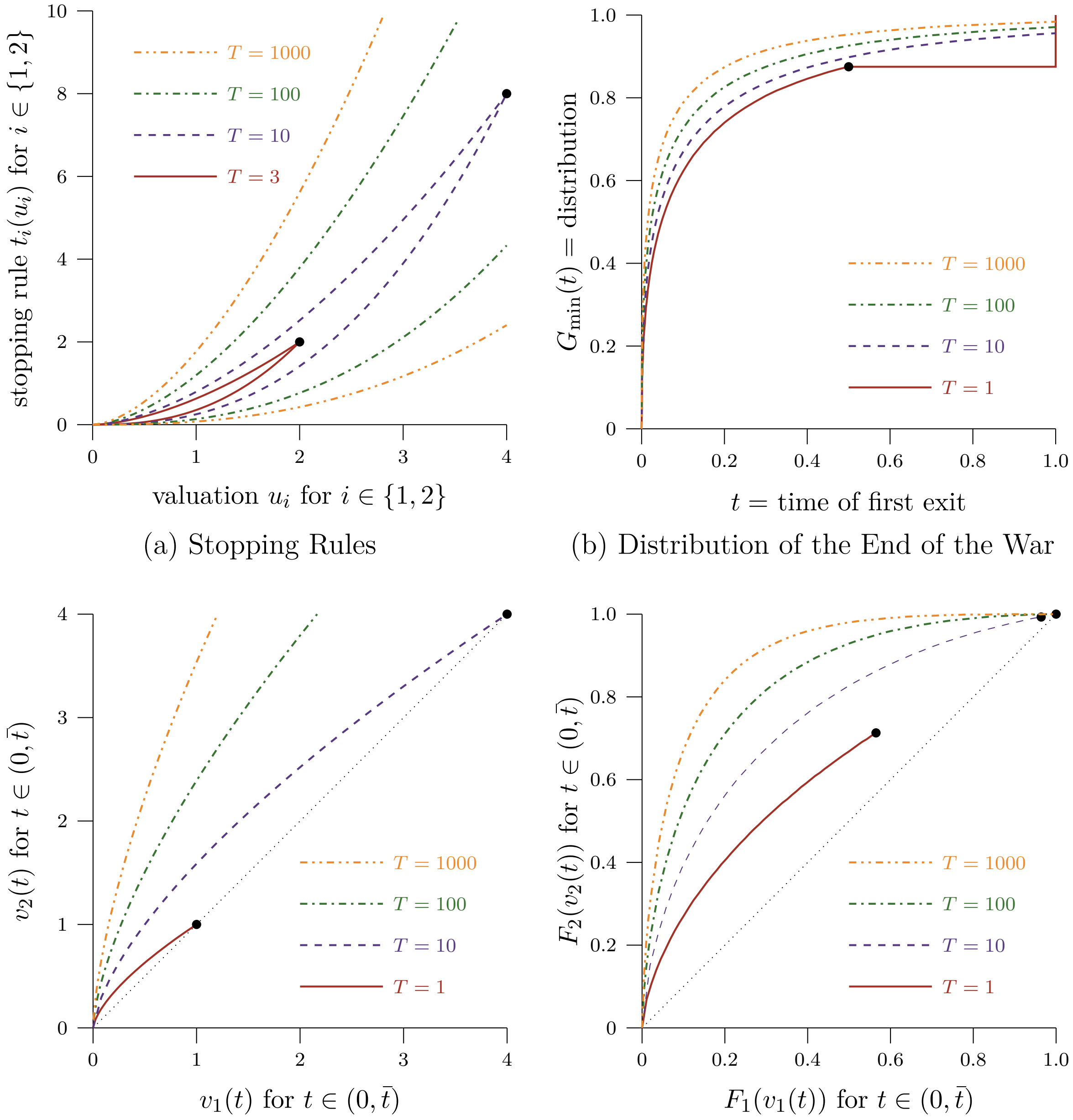
I evaluate this hypothesis using an exponential specification with hazard rates $\lambda_1 = \frac{5}{6}$ and $\lambda_2 = \frac{5}{4}$; the expectation $\mu_1 = \frac{6}{5}$ is 50% higher than $\mu_2 = \frac{4}{5}$. I set $\underline{u}_1 = \underline{u}_2 = 0$, so the valuation supports extend down to zero. This means that $\underline{u}_1^\star = \underline{u}_2^\star = 0$, and so there is no literal instant exit (Proposition 2 has no bite). Nevertheless, there can be relatively rapid exit.

In Figure 1 I illustrate equilibria. From panel (a), the stopping rules are ranked (from Proposition 1) and move apart as the T grows, while panel (b) shows how the game ends more quickly. Panels (c) and (d) compare the valuation pairs (and quantiles) at which the two players fight to the same time. This illustrates the valuation combinations which result in wins for the different players. As the time limit grows the prize shifts toward the perceived stronger player and the war ends more quickly. However, fighting may persist for some time with a chance that the perceived weaker player wins. For example, with $T = 100$, there is a 15% chance that the war lasts longer than 25% of the cross-player expected valuation. Also, if the stronger player has a median valuation then she still faces a 7% chance of losing to the weaker player.

A nuanced message is that a finite-horizon war of attrition (with a reasonably long deadline) is strongly (but not completely) biased towards an early win by a perceived-stronger player. However, if the deadline is not too distant and players are not too dissimilar then the equilibrium may be closer to a symmetric-equilibrium-style prediction.

Other Specifications. There are game modifications (other than a deadline) that yield uniqueness. Set $T = \infty$ (players can fight forever) but adopt the cost function $C(t_i, t_j)$ from eq. (2): a player pays β per unit of planned fighting time beyond her opponent's exit. Equivalently,

$$\pi_i(t_i, t_j) = u_i \times \left[\mathcal{I}[t_i > t_j] + \frac{\mathcal{I}[t_i = t_j]}{2}\right] - \min\{t_1, t_2\} - \beta \max\{t_i - \min\{t_1, t_2\}, 0\}. \quad (15)$$



(a) Stopping Rules

(b) Distribution of the End of the War

(c) Intersecting Valuations

(d) Intersecting Quantiles

These four panels illustrate the equilibria of a war-of-attrition game where players' valuations are exponentially distributed with hazards $\lambda_1 = \frac{5}{6}$ and $\lambda_2 = \frac{4}{5}$.

Any bullet points in the plots correspond to $t = \bar{t}$ or $u_1 = u_2 = \bar{u}^\star$.

Panel (a) illustrates the equilibrium stopping rules for the two players for different values of the time-horizon deadline T. The higher stopping rule is Player 1.

Panel (b) illustrates the distribution of the earliest stopping time, and so the distribution over the length of the war of attrition. This is a distribution $G_{\min}(t)$ which satisfies $1 - G_{\min}(t) = (1 - G_1(t))(1 - G_2(t))$ where $G_i(t) = F_i(v_i(t))$.

Panel (c) illustrates the valuation pairs $\{v_1(t), v_2(t)\}$ for which players both fight to time t. Any point above and to the left of the relevant line corresponds to a valuation combination leading to a win for Player 2; similarly points below and to the right generate wins for Player 1. Any combination "north east" of a bullet point corresponds to valuation pairs that are both above $\bar{u}^\star$ and so both players fight to the deadline. Panel (d) translates those valuation pairs into quantiles for the two players.

Figure 1. Equilibrium Properties for an Exponential Specification

This is equivalent to a hybrid all-pay auction.[12] It is now always costly to plan to fight for longer. This rules out equilibria in which a player plans to fight forever in the expectation that the opponent will always concede at the beginning.

The first-order condition for a player i with valuation $v_i(t)$ planning to quit at time t becomes

$$v_i(t)f_j(v_j(t))v_j'(t) = \underbrace{1 - F_j(v_j(t))}_{\text{marginal cost if loser}} + \underbrace{\beta F_j(v_j(t))}_{\text{marginal cost if winner}} \quad \Leftrightarrow \quad 1 = \frac{v_i(t)f_j(v_j(t))v_j'(t)}{1-(1-\beta)F_j(v_j(t))}. \quad (16)$$

A variant of Lemma 2 holds, including $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$, where I re-define $\Lambda_i(u)$ as

$$\Lambda_i(u) = \int_u^\infty \frac{1}{x}\frac{f_i(x)}{1-(1-\beta)F_i(x)}\,dx. \quad (17)$$

In essence, a boundary condition at time $\bar{t}$ (when players stay to the deadline T) is replaced by a condition as $t \to \infty$. However, for $\Lambda_i(u)$ to be well defined requires $\beta > 0$.[13]

A specification with the same effect is obtained when each player believes her opponent to be "crazy" and exogenously fight forever with probability $\xi > 0$.[14] Equivalently, a player may suffer exit failure with this probability. In either case, each first-order condition becomes

$$v_i(t)(1-\xi)f_j(v_j(t))v_j'(t) = 1-(1-\xi)F_j(v_j(t)). \quad (18)$$

To proceed, change the definition of $\Lambda_i(u)$ by switching "β" to "ξ" and continue as before.

Although the details are omitted, the fundamental findings are maintained with these specifications.[15] In particular, as $\max\{\beta, \varepsilon\} \to 0$ the perceived weaker player concedes immediately.

Proposition (Other Specifications). *Consider a hybrid all-pay auction with parameter β. Equivalently, either (i) players pay a cost β for planned fighting beyond the exit of an opponent; or (ii) a player fails to see the exit of an opponent with probability β. Suppose also that an opponent is believed to be crazy (and so exogenously fight forever) with probability ξ.*

There is a unique equilibrium if $\max\{\beta, \xi\} > 0$. Proposition 1 holds if $\max\{\beta, \xi\}$ is sufficiently small. Proposition 2 holds. Proposition 3 holds if "$T \to \infty$" is replaced by "$\max\{\beta, \varepsilon\} \to 0$".

[12] Related features are present in auction-theoretic treatments of wars of attrition. Bulow and Klemperer (1999), for example, considered $N + K$ symmetric bidders competing for N prizes. When $K > 1$ such a game has no symmetric equilibrium (Haigh and Cannings, 1989). To circumvent this, Bulow and Klemperer (1999) perturbed the game: a conceding player continues to pay a (perhaps small) fraction of her fighting costs until the game ends. The perturbed game has a symmetric equilibrium involving rapid exit of $K - 1$ players until the $N + 1$ highest-valued players remain. In contrast, I suggest the instant exit of K players. Bulow and Klemperer (1999, p. 178, note 15) acknowledged this possibility, noting that "[a]symmetric perfect-Bayesian equilibria include those in which K (pre-identified) firms quit in zero time ... [e]quilibria of this kind seem particularly natural if (in contrast to our model) there any asymmetries between players."

[13] If $\beta = 0$ then $\Lambda_i(u)$ can fail to be well defined, and there are multiple equilibria.

[14] This corresponds to the approach of Kornhauser, Rubinstein, and Wilson (1989) and (more recently) Martinelli and Escorza (2007) and Kambe (2019) who, following Kreps and Wilson (1982), Milgrom and Roberts (1982), and Kreps, Milgrom, Roberts, and Wilson (1982), introduced a kind of "irrationality" into the model.

[15] These specifications can be further generalised by specifying player-specific parameters β_i and ξ_i for $i \in \{1, 2\}$. The relative rates at which these parameters vanish to zero then matters for results analogous to Proposition 3.

This specification can also allow for upper bounds to valuations, so that $u \in (\underline{u}_i, \bar{u}_i)$. In such a case, the hazard-rate ranking extends so that $F_1 \succ_{\text{HRD}} F_2$ if $\bar{u}_1 > \bar{u}_2$.

Direct Learning. An uncomfortable conclusion (in the $T = \infty$ limit) is that there will be little if any fighting in a war of attrition: the (perceived) weaker player concedes immediately.

What else might explain fighting? One possibility is that perceptions can change, and so players may stick around in anticipation of this. Specifically, over time players may directly learn about each other (a player already updates her beliefs as her opponent stays in the game) and such direct learning may change the strength ranking.

To explore this idea, suppose that there is an arrival rate ρ of a public revelation of players' valuations. If this happens, then players go on to play a complete-information game in which (when the deadline is distant) the lower-valuation player exits immediately.

Consider two players at time t and suppose that $v_1(t) < v_2(t)$. The potential arrival of a day-of-revelation has no effect on the incentives of Player 1 with valuation $v_1(t)$: she knows that if valuations are revealed then she will be unmasked as the lower valuation player (given that $u_2 \geq v_2(t) > v_1(t)$.) However, Player 2 (with valuation $v_2(t)$) knows that such an arrival might prompt a win: if $u_1 \in [v_1(t), v_2(t))$ then the (perceived stronger, but perhaps actually weaker) Player 1 exits immediately. The first-order conditions for the players become

$$1 = \frac{v_1(t)v_2'(t)f_2(v_2(t))}{1 - F_2(v_2(t))} = \frac{v_2(t)v_1'(t)f_1(v_1(t))}{1 - F_1(v_1(t))} + \underbrace{\frac{\rho v_2(t)(F_1(v_2(t)) - F_1(v_1(t)))}{1 - F_1(v_1(t))}}_{\text{extra incentive for weaker player}}. \tag{19}$$

Working back from the end-of-attrition time $\bar{t}$ the incorporation of the second term brings the inverse stopping rules closer together, and pushes against "instant exit" results.[16]

Pre-Play Tactics. The importance of perceived strength implies that players may wish to influence such perceptions.[17] Suppose, for example beliefs are symmetric and that players have yet to discover their true valuations. Both players have distinct preferences over what might happen before they observe their valuations and the war-of-attrition game begins.

Consider, for example, a pre-play public signal of a player's valuation. The posterior over that valuation will be more concentrated, and so (in the asymptotic hazard rate sense) such a player will be perceived as weaker. A player wishes there to be a public signal of her opponent's valuation, but not of her own. This suggests a pre-play tactic via which a player strives to maintain an air of mystery about herself, while encouraging learning about her opponent.

[16]This idea resonates with a recent paper by Gieczewski (2023) who considered a model in the spirit of Smith (1998) and (more distantly) other work in international relations (Slantchev, 2003a,b; Powell, 2004), as well as more recent theory (Ortner, 2013, 2017). Players' costs evolve over time, and so a motive to stay in the game is that conditions may (exogenously) move in a player's favour.

[17]Relatedly, players may seek to change player-specific deadlines in a pre-play stage (Foster, 2018).

Applications. A war of attrition is a stylised representation of many applied scenarios.

Workers and employers may prolong a strike in order to obtain a preferred resolution (Kennan and Wilson, 1989). Potential suppliers may delay public-good provision in an effort to free ride (Bliss and Nalebuff, 1984; Bilodeau and Slivinski, 1996). Oligopolists may incur losses in anticipation of profitability following a competitor's exit (Fudenberg and Tirole, 1986; Ghemawat and Nalebuff, 1985, 1990). Macroeconomic stabilization may be delayed in order to push the burden of an agreement towards others (Alesina and Drazen, 1991; Casella and Eichengreen, 1996). The sponsor of a technological standard may continue its promotion in the hope that a competitor will adopt it (Farrell and Saloner, 1988; David and Monroe, 1994; Farrell and Simcoe, 2012). Political actors may expend lobbying costs to gain political influence (Hillman and Samet, 1987; Hillman and Riley, 1989) while (a conclave of) voters or candidates themselves may need to concede to achieve consensus (Indridason, 2008; Kwiek, 2014; Kwiek, Marreiros, and Vlassopoulos, 2016, 2019). In a bargaining game with fixed proposals, agreement requires the acquiescence of one participant (Osborne, 1985; Ordover and Rubinstein, 1986; Chatterjee and Samuelson, 1987; Abreu and Gul, 2000; Kambe, 1999). Of course, such a war-of-attrition-style negotiation can also take place rather literally in a conflict environment (for example Fearon, 2004; Powell, 2004, 2017; Leventoğlu and Slantchev, 2007; Krainin, 2014).

In some applications, it seems reasonable to specify some kind of perceived asymmetry. In the case of a labor dispute, for example, workers and employers may hold different perceptions. Similarly, competing technologies in a standards war are likely heterogeneous, and so perceptions of valuations may differ. This point was made by Martinelli and Escorza (2007) in the context of the Alesina and Drazen (1991) macroeconomic stabilisation model. They noted (p. 1224) that ex ante asymmetry is a strong assumption and concluded (in a message common to that delivered here) that "a political group more exposed to inflation costs will be likely to cave in immediately, leading to immediate reform."[18]

The results of this paper suggest how asymmetries can be important in predicting a relatively early winner. Any prize allocation might also be inefficient, albeit with lessened rent dissipation. However, this leaves open the scope (as noted above) for players to influence perceptions. A hypothesis, then, is that a combatant in a war of attrition may be waiting for (public) perceptions to change rather than (solely) waiting for an opponent to concede.

[18]Their main result (Theorem 2, p. 1231) corresponds to Proposition 2 in an environment with commitment types and where player types correspond to fighting costs rather than prize valuations. The result is equivalent to Proposition 1 of Myatt (2005), which also applies to a model with commitment types and with bounded valuations (or bounded fighting costs). Their specification of common support for player types prevented them from obtaining the "instant exit" result of Proposition 3 or the corresponding result in Myatt (2005).

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Supplemental (and Optional) Appendix for On-Line Publication

for **"The Impact of Perceived Strength in the War of Attrition"** by **David P. Myatt**

Omitted Proofs of Lemmas 1 to 3

(Variants of) Lemmas 1 to 3 are standard in the literature, and so their formal proofs are omitted from the paper. For completeness, proofs are collected here.

Lemma 1 (concerning basic equilibrium properties) is assembled from the following claims.

Claim 1. *Sufficiently high valuation types fight until the deadline: $t_i(u_i) = T$ if $u_i > 2T$.*

Proof. A player $u_i > 2T$ will exit at time $t < T$ only if guaranteed to win: $\Pr[t_j(u_j) < t] = 1$ for $j \neq i$. This means that for $j \neq i$ all types $u_j > 2T$ stop before t, which (repeating the argument) means that $\Pr[t_i(u_i) < t] = 1$. This is a contradiction. □

Claim 2. *It is always costly to fight for longer.*

Proof. This holds because at least some opposing types fight until the deadline. □

Claim 3. *A player's stopping time is at least weakly increasing in her valuation.*

Proof. Waiting involves a strict increase in expected costs (Claim 2) and so must be weakly outweighed by a higher expected benefit. The benefit is strictly higher with a higher valuation. Hence if a lower valuation type fights for longer, then a higher valuation type will do so. □

Given that some types fight until the deadline (Claim 1) and that stopping rules are monotonic (Claim 3) I write $\bar{u}_i^\star = \inf\{u : t_i(u) = T\}$ for the threshold at which i switches to fight until T.

Claim 4. *Not all players fight forever: $\bar{u}_i^\star > \underline{u}_i$ for each $i \in \{1, 2\}$.*

Proof. If $\bar{u}_i^\star = \underline{u}_i$ then every type of player i will fight until T. Player j will either fight to T or not fight at all: $t_j(u) \in \{0, T\}$. In particular, $\bar{u}_j^\star = 2T$, so that $t_j(u) = T$ if $u_j > 2T$ but $t_j(u) = 0$ if $u_j < 2T$. (By assumption $\underline{u}_j < 2T$ and so there is positive probability of both $t_j = 0$ and $t_j = T$.) In best reply types $u_i \in (\underline{u}_i, 2T)$ will not fight until T; a contradiction. □

Claim 5. *Any discontinuity in a stopping rule must be a jump up to the deadline T.*

Proof. Suppose that there is a jump at $u^\dagger \in (\underline{u}_i, \infty)$ from $t_L = \lim_{u \uparrow u^\dagger} t_i(u)$ to $t_H = \lim_{u \downarrow u^\dagger} t_i(u)$.

Player $j \neq i$ will not quit in (t_L, t_H): a small reduction in her stopping time will strictly reduce her expected fighting cost (from Claim 2) without reducing her chance of winning.

Suppose that $t_H < T$. For any $\varepsilon \in (0, T - t_H)$ there is some $u \geq u^\dagger$ close to $u^\dagger$ satisfying $t_H \leq t_i(u) \leq t_H + \varepsilon < T$. Player i with valuation u incurs a cost to fight from t_L to $t_i(u)$ and so must expect a positive probability of exit by $j \neq i$. Taking $\varepsilon \to 0$, j must exit at t_H with strictly positive probability. This means that i will never exit at time t_H, which in turn means that j could reduce her stopping time from t_H (strictly saving costs) without reducing her chance of winning. This is a contradiction. □

Claim 6. *A player will not exit with positive probability at time $t \in (0, T)$.*

Proof. If i exits at $t \in (0, T)$ with positive probability, then $j \neq i$ will never exit at or shortly before: she would wait until after t to "capture the atom" at time t. This means that i stopping at t could instead stop earlier, save costs, and yet not reduce her chance of winning. □

Claim 7. *Players begin exiting at $t = 0$: stopping rules satisfy $\lim_{u \downarrow \underline{u}_i} t_i(u) = 0$.*

Proof. $\underline{t}_i = \lim_{u \downarrow \underline{u}_i} t_i(u)$ is well defined given that $t_i(u)$ is at least weakly increasing (Claim 3). If $\underline{t}_i > 0$ then $j \neq i$ will never exit in $(0, \underline{t}_i)$. Since i is always prepared to incur a non-negligible fighting cost to get to $\underline{t}_i$, the logic employed in the proof of Claim 6 implies that j exits at $\underline{t}_i$ with positive probability. This contradicts Claim 6. Hence $\underline{t}_i = 0$. □

I defined an upper threshold $\bar{u}_i^\star$ above which a player fights until the deadline; similarly I define a lower threshold below which a player does not fight: $\underline{u}_i^\star = \inf\{u : t_i(u) > 0\}$. If $\underline{u}_i^\star = \underline{u}_i$ then a player always fights (and so the stopping rule $t_i(u)$ is strictly increasing at the lower bound of a player's valuations), but if $\underline{u}_i^\star > \underline{u}_i$ then there is positive probability of instant exit at the beginning. Naturally, $\underline{u}_i^\star \geq 0$ and so there is always (at least some) instant exit if $\underline{u}_i < 0$.

Claim 8. *There is at least some attrition: $\underline{u}_i^\star < \bar{u}_i^\star$ for each $i \in \{1, 2\}$.*

Proof. If $\underline{u}_i^\star = \bar{u}_i^\star$ then player i chooses $t \in \{0, T\}$, placing an atom at both times. ($t = 0$ is played with positive probability, because playing only T would contradict Claim 7.) If u_j is high then her best reply is $t_j = T$. However, if u_j is low then she prefers to exit just after $t = 0$. No best reply exists; but even f it did ("$t_j = \varepsilon$") then j would place an atom there; but then i (when choosing $t_i = 0$ and when $u_i > 0$) would no longer be choosing a best reply. □

Claim 9. *Attrition stops (for both players) before the deadline: $\lim_{u \uparrow \bar{u}_i^\star} t_i(u) = \bar{t} \in (0, T)$.*

Proof. Claim 8 implies that $\bar{u}_i^\star > 0$ (types in $(\underline{u}_i^\star, \bar{u}_i^\star)$ fight for some positive time, and so have positive valuations). Write $\bar{t}_i = \lim_{u \uparrow \bar{u}_i^\star} t_i(u)$. If $\bar{t}_i = T$ then a type u_i close to $\bar{u}_i^\star$ chooses to exit at a time t close to T. However, we can find such a type and time such that $u_i > 2(T - t)$. Such a type would prefer instead to stay until the deadline. Hence $\bar{t}_i < T$. Now suppose that $\bar{t}_i < \bar{t}_j$. If this were so then types of player j for whom $t_j(u) \in (\bar{t}_i, \bar{t}_j)$ could strictly save costs (with no effect on their chance of winning) by reducing their exit times. Hence $\bar{t}_i = \bar{t}_j = \bar{t}$. □

Claim 10. *The threshold valuation at which players fight to T is $\bar{u}_1^\star = \bar{u}_2^\star = \bar{u}^\star = 2(T - \bar{t})$.*

Proof. Indifference to paying a cost $T - \bar{t}$ for the equal chance of receiving the prize $\bar{u}_i^\star$. □

Proof of Lemma 1. The statements (i) to (iv) follow from the claims proven above. □

Proof of Lemma 2. For $t \in (0, \bar{t})$ the distribution of i's stopping time is $G_i(t) = F_i(v_i(t))$. $v_i(t)$ is strictly and continuously increasing, and so differentiable almost everywhere. Consider a time t at which $v_i(t)$ is differentiable, and player j with valuation $v_j(t)$. Her expected payoff from stopping time t_j and its derivative evaluated at $t_j = t$ is

$$\pi_j(t_j \,|\, u_j = v_j(t)) = v_j(t)G_i(t_j) - (1 - G_i(t_j))t_j - \int_0^{t_j} x\, dG_i(x)$$

$$\Rightarrow \quad \left.\frac{\partial \pi_j(t_j \,|\, u_j = v_j(t))}{\partial t_j}\right|_{t_j = t} = v_j(t)g_i(t) - (1 - G_i(t))$$

$$= v_j(t)f_i(v_i(t))v_i'(t) - (1 - F_i(v_i(t))). \tag{20}$$

Setting this to zero yields the first-order conditions which apply for almost all t:

$$v_i'(t) = \frac{1 - F_i(v_i(t))}{v_j(t)f_i(v_i(t))} \quad \text{for} \quad i, j \in \{1, 2\} \quad \text{and} \quad j \neq i. \tag{21}$$

I argue that the right-hand side is Lipschitz continuous in t. This implies that $v_i(t)$ is the integral of a Lipschitz continuous function, and so is differentiable everywhere. (This supports the claim that $v_i(t)$ is differentiable for $t \in (0, \bar{t})$.) To do this, I show that $v_i(t)$ is Lipschitz continuous, and then it follows that the right-hand side of eq. (21) is also Lipschitz continuous.

Take a closed interval $C \subseteq (0, \bar{t})$. $v_j(t)$ is continuous and strictly positive on the compact set C, and so $\min_{t \in C}\{v_j(t)\}$ exists and is strictly positive. Set $K = 1/\min_{t \in C}\{v_j(t)\}$ as a Lipschitz constant on C. Suppose j with valuation $v_j(t_L)$ is considering delaying exit until $t_H > t_L$ where $\{t_L, t_H\} \subseteq C$. Given that t_L is an optimal stopping time,

$$G_i(t_H) - G_i(t_L) \leq \frac{t_H - t_L}{v_j(t_L)} \leq \frac{t_H - t_L}{\min_{t \in C}\{v_j(t)\}} \quad \Rightarrow \quad |G_i(t_H) - G_i(t_L)| \leq K|t_H - t_L|. \tag{22}$$

This means that $G_i(t)$ is Lipschitz continuous in t. Next note that $\min_{t \in C}\{f_i(v_i(t))\}$ exists and is strictly positive. Set $\hat{K} = K/\min_{t \in C}\{f_i(v_i(t))\}$ as a mew Lipschitz constant. Then:

$$v_i(t_H) - v_i(t_L) = F_i^{-1}(G_i(t_H)) - F_i^{-1}(G_i(t_L)) \leq \frac{G_i(t_H) - G_i(t_L)}{\min_{t \in C}\{f_i(v_i(t))\}}$$

$$\leq \frac{t_H - t_L}{\min_{t \in C}\{f_i(v_i(t))\} \times \min_{t \in C}\{v_j(t)\}} \quad \Rightarrow \quad |v_i(t_H) - v_i(t_L)| \leq \hat{K}|t_H - t_L|. \tag{23}$$

This means that $v_i(t)$ is Lipschitz continuous in t.

With basic properties established, I now show the existence and uniqueness of the equilibrium. I write $\hat{\Lambda}_i(t) \equiv \Lambda(v_i(t))$ for $t \in (0, \bar{t})$. Using Claim 10, $\hat{\Lambda}_i(\bar{t}) \equiv \lim_{t \uparrow \bar{t}} \hat{\Lambda}_i(t) = 0$. Now:

$$\hat{\Lambda}_1'(t) = -\frac{v_1'(t)f_1(v_1(t))}{v_1(t)[1 - F_1(v_1(t))]} = -\frac{v_2'(t)f_2(v_2(t))}{v_2(t)[1 - F_2(v_2(t))]} = \hat{\Lambda}_2'(t), \tag{24}$$

where the central equality is from the players' first-order conditions. I conclude that $\hat{\Lambda}_1(t) = \hat{\Lambda}_2(t)$ or equivalently $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$ as stated in the lemma.

I show uniqueness, and I begin with the case where $\max\{\underline{u}_1, \underline{u}_2\} < 0$. For this case, the lower bounds of the stopping rules of the two players are $\underline{u}_1^\star = \underline{u}_2^\star = 0$.

Fixing $\bar{u}^\star$, $\Lambda_i(u)$ is strictly and continuously decreasing, with $\Lambda_i(\bar{u}^\star) = 0$ and $\lim_{u\downarrow 0} \Lambda_i(u) = \infty$. Its inverse is well defined. I define $\gamma_i(u; \bar{u}^\star) = \Lambda_i^{-1}(\Lambda_j(u))$, meaning that an equilibrium satisfied $v_i(t) = \gamma_i(v_j(t); \underline{u}^\star)$. From the first order condition for player j,

$$t_j'(u) = \frac{\gamma_i(u; \bar{u}^\star) f_j(u)}{1 - F_j(u)} \quad \Rightarrow \quad \bar{t} = \bar{\tau}(\bar{u}^\star) \quad \text{where} \quad \bar{\tau}(\bar{u}^\star) = \int_0^{\underline{u}^\star} \frac{\gamma_i(u; \bar{u}^\star) f_j(u)}{1 - F_j(u)}\, du. \tag{25}$$

Now recall that $\bar{u}^\star = 2(T - \bar{t})$, or equivalently $\bar{t} = T - \bar{u}^\star/2$. This means that $\bar{u}^\star$ satisfies

$$T - \frac{\bar{u}^\star}{2} = \bar{\tau}(u^\star). \tag{26}$$

The left-hand side is strictly decreasing from T down to zero as $\bar{u}^\star$ increases from zero up to $2T$. The right-hand side begins at $\tau(0) = 0$, is continuous, and is strictly positive for $\bar{u}^\star > 0$. Hence there is at least one solution $\bar{u}^\star$. This pins down an equilibrium. To show that this equilibrium is unique, I show that $\bar{\tau}(\bar{u}^\star)$ is increasing. To do this, I note that

$$\bar{\tau}'(\bar{u}^\star) = \frac{\bar{u}^\star f_j(\bar{u}^\star)}{1 - F_j(\bar{u}^\star)} + \int_0^{\bar{u}^\star} \frac{\partial \gamma_i(u; \bar{u}^\star)}{\partial \bar{u}^\star} \frac{f_j(u)}{1 - F_j(u)}\, du. \tag{27}$$

The first term is positive. To evaluate the second term, note that

$$\frac{\partial \gamma_i(u; \bar{u}^\star)}{\partial \bar{u}^\star} = \frac{\gamma_i(u; \bar{u}^\star)}{\bar{u}^\star} \frac{1 - F_i(\gamma_i(u; \bar{u}^\star))}{f_i(\gamma_i(u; \bar{u}^\star))} \left[\frac{f_i(\bar{u}^\star)}{1 - F_i(\bar{u}^\star)} - \frac{f_j(\bar{u}^\star)}{1 - F_j(\bar{u}^\star)} \right], \tag{28}$$

and so the sign of that second term is determined by the ranking of the hazard rates of i and j at $\bar{u}^\star$. I am free to determine this ranking based on the labels of i and j. (That is, the function $\bar{\tau}(\bar{u}^\star)$ is equivalently defined switching the roles of players i and j.) Hence I can ensure that the second term is positive, and this means that $\bar{\tau}(\bar{u}^\star)$ is increasing in $\bar{u}^\star$.

For $\min\{\underline{u}_1, \underline{u}_2\} > 0$ and other cases, similar proofs may be used. □

Proof of Lemma 3. Using first-order condition of player j from eq. (4),

$$\begin{aligned} t_i'(v_i(t)) = \frac{1}{v_i'(t)} &= \frac{v_j(t) f_i(v_i(t))}{1 - F_i(v_i(t))} \le \frac{\bar{u}^\star f_i(v_i(t))}{1 - F_i(v_i(t))} \\ \Rightarrow \quad \bar{t} = t_i(\bar{u}^\star) &\le \int_{\underline{u}_i^\star}^{\bar{u}^\star} \frac{\bar{u}^\star f_i(u)}{1 - F_i(u)}\, du = \bar{u}^\star \log \left[\frac{1 - F(\underline{u}_i^\star)}{1 - F(\bar{u}^\star)} \right] \le -\bar{u}^\star \log[1 - F(\bar{u}^\star)]. \end{aligned} \tag{29}$$

If the claim that $\lim_{T\to\infty} \underline{u}^\star = \infty$ is false then a sequence of deadlines can be constructed such that $\bar{u}^\star$ remains bounded. The inequality above would imply that $\bar{t}$ remains bounded. But if that is true the $\bar{u}^\star = 2(T - \bar{t}) \to \infty$; this is a contradiction. □