Asymmetric Models of Sales

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Abstract. We generalize the captive-and-shopper model of sales to allow asymmetries in production costs and captive audiences in an oligopoly. Both kinds of asymmetry determine the firms that compete (via randomized sales) to serve the price-comparing shoppers, while other firms exploit their captive audiences. In contrast to a model with symmetric costs (but asymmetric captive audiences) there are natural situations in which more than two firms use sales by engaging in pairwise battles across different price intervals. We then study the choice of production technologies via costly process innovations. A distinctive asymmetry emerges endogenously: one firm innovates more and becomes the dominant supplier of shoppers. The pattern of innovations connects to the size of firms' captive bases and the shape of technological opportunity. We also provide a trio of extensions to consider costly acquisitions of captives and shoppers, and captives' choice of captor.

The dispersion of prices for similar products is a well-established regularity. Some researchers have observed some stability of dispersion, with different firms maintaining distinct price positions for some period of time, while others have emphasized inter-temporal price movements.[2] In any case, the existence of different prices for the same product means that buyers benefit by considering multiple offers. Of course, not all buyers may be able to shop around. Sales, which are reductions

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[2]Established studies have reported a standard deviation (relative to the mean) at brick-and-mortar stores ranging from 19% to 36% (Kaplan and Menzio, 2015) with most variation attributed to persistent differences (Nakamura and Steinsson, 2008; Wulfsberg, 2016; Kaplan et al., 2019; Moen, Wulfsberg and Aas, 2020) but with more intertemporal changes in online prices (Gorodnichenko, Sheremirov and Talavera, 2018). Several industry studies also support this summary (Sorensen, 2000; Moraga-González and Wildenbeest, 2008; Hong and Shum, 2010; Galenianos, Pacula and Persico, 2012). Nevertheless, other have emphasized movement within the price distribution (Lach, 2002) and substantial dynamic movements in specific industries (Chandra and Tappata, 2011; Pennerstorfer et al., 2020).

from established or regular prices, are a long-standing and common practice and can offer firms a resolution to the trade-off presented by such a heterogeneous population of buyers: either price high to exploit those customers who cannot access other offers, or price low to capture those who compare many prices. A central task for theorists, therefore, is to predict the pattern of sale prices and to relate that pattern to the awareness of firms' offers and other firm characteristics.

The canonical "model of sales" (Varian, 1980) parsimoniously captures the key pricing trade-off by specifying buyers who are either "shoppers" (who consider every price) or "captive" customers (who are locked in to a single firm). This succinctly represents situations in which, for example, firms experience demand from local clientele as well as from a central clearinghouse for those who see all offers. The classic model-of-sales pricing game features firms with symmetric costs and captive bases and has many Nash equilibria, all involving the play of mixed strategies. The natural focus for Varian (1980) was on the symmetric equilibrium in which all firms compete for shoppers (or "use sales") by mixing continuously over the same interval of prices stretching from the monopoly price (that fully exploits captive customers) down to a price at which a firm earns only its captive-exploiting monopoly profit even if it wins the business of all shoppers. This provides a rationale for dispersed and unpredictable price offers by all members of an oligopoly.

However, even small deviations from exact symmetry are important. If firms have differently sized captive-audiences (but are otherwise identical) then a unique Nash equilibrium offers a narrower prediction (Baye, Kovenock and De Vries, 1992): the two firms with the smallest captive audiences compete for the business of shoppers via randomized sales. Those authors memorably called this a "tango" danced by de facto duopolists. Other firms "stay off the dance floor" (we call them "wallflowers") and charge the monopoly price to their captive customers. The firm with the smallest captive audience sets a lower price more often and so is the dominant supplier of shoppers.

The literature omits a full treatment of the captive-and-shopper pricing game with asymmetric marginal costs and more than two firms. Our contribution does this, pushes further by allowing those costs to be influenced endogenously by firms' technology choices, and offers several extensions. This fills a notable and long-standing gap in the literature, and provides a springboard for future work and extensions. There are also important economic reasons to consider cost asymmetries.

Firstly, we already noted that the classic model is sensitive to asymmetries in the sizes of captive audiences. Natural questions arise. Is there sensitivity to asymmetries in costs? Do more than two firms engage in sales? Which firms do so? What is the pattern of sale-price offers? Efficiency is also relevant when costs are asymmetric: are shoppers served at the lowest cost?

Secondly, results in asymmetric settings allow the model of sales to be embedded in deeper frameworks. For example, in a recent paper by Hagiu and Wright (2024) a platform specifies benefits and per-sales fees to firms. The equilibrium analysis requires the platform to anticipate profits in a captive-and-shopper pricing subgame with asymmetric marginal costs.[3]

Thirdly, and relatedly, firms face a choice of technologies: choosing a higher fixed cost (for example, via an innovative investment) can yield a lower marginal cost (corresponding to a process innovation). An asymmetric-cost model of sales allows us to investigate such choices. One possibility is that firms (even if they begin symmetrically) may choose very different technologies, which (if true) reinforces the requirement for an asymmetric-marginal-cost model.

Bearing in mind these motivations, we preview here the results that follow in the rest of the paper.

Our first contribution (in Section 1) is to characterize a unique equilibrium of the captive-and-shopper model with asymmetric costs. If firms with smaller captive audiences also have lower costs then there is a de facto duopoly: the two most "aggressive" firms (with the smallest captive audiences and so the lowest costs) dance the familiar "tango" by using randomized sales while all other (less aggressive) "wallflower" firms charge the monopoly price to their captives.

The competition for shoppers moves beyond a de facto duopoly whenever a firm with a higher marginal cost has relatively few captive customers. As an illustrative example, consider a triopoly with two incumbent firms and a new entrant, where that new entrant has an inferior production technology (higher costs) and has yet to develop a locked-in customer base (fewer captives). The (lower cost) incumbents compete for sales across prices that lie below the marginal cost of the entrant. At elevated prices the entrant (with a hunger to attract shoppers) "steps on to the dance floor" to replace one of the incumbents in a "thrango" pattern of mixed-strategy pricing.

[3]Hagiu and Wright (2024) used results from an earlier version of our paper (for example, see their Footnotes 9 and 25 and their Online Appendix A.5). The payoff characterization of Siegel (2009) would also have served their needs, with an appropriate parameter mapping between contest and model-of-sales settings.

The equilibrium exhibits an interesting comparative-static result: a reduction in the marginal cost of the most aggressive firm pushes up the prices charged by the second-most aggressive firm.[4] This positive strategic effect gives that firm a distinct enhanced incentive to reduce its marginal cost.

This effect is central to our second contribution (in Section 2) in which we consider endogenous technology choice. We add a pre-pricing stage in which firms engage in process innovations: a firm can pay (via a higher fixed cost) to lower its marginal cost of production.

Asymmetric capabilities and a uniquely most aggressive firm emerge in equilibrium. For example, if firms are symmetric ex ante then exactly one firm chooses a distinct technology with more innovation, operates with a lower marginal cost, uses sales, and most often supplies the shoppers.

If firms have differently sized captive audiences then we can identify the firms that use sales. This depends on the strength of technological opportunity embodied in an innovation production function. A firm that expects to serve more customers innovates more, achieves a lower marginal cost, and so generates a greater per-unit surplus for each buyer supplied. If the relationship between per-unit surplus and expected output is elastic (in such a case we say that "technological opportunity is strong") then firms with the most captive customers innovate sufficiently (lower their marginal costs) to become those that compete for shoppers. Furthermore, the inverse relationship of marginal cost and captive-audience size is precisely the feature that can lead to more than two firms engaging in randomized sales. In contrast, if technological opportunity is weak then the unique equilibrium can predict (just as in a symmetric-cost specification) that the two firms with the fewest captives compete for shoppers; even though at least one of these firms has the (endogenously determined) highest marginal cost amongst all firms.

As well as considering welfare and efficiency (throughout Sections 1 and 2) we offer further contributions (in Section 3) via several extensions: we study the incentives to invest in acquiring captive customers, we evaluate how such captive buyers might wish to move between the firms, and we study a clearinghouse that charges fees for firms to reach price-comparing shoppers.

[4] Here we generalize an existing (but little known) finding under duopoly from Golding and Slutsky (2000), which also appears as a related result but in a narrower setting within Inderst (2002).

Related Literature. The original model of sales (Varian, 1980) specified complete symmetry of firms, while others (Narasimhan, 1988; Baye, Kovenock and De Vries, 1992; Kocas and Kiyak, 2006) studied firms with different captive-audience sizes. A central "tango" finding is that the two firms with fewest captives compete (via mixed strategies) to attract shoppers.

A full treatment of asymmetric marginal costs has (to our knowledge) not moved beyond the case of duopoly. The earliest duopoly analysis (which we know of), by Golding and Slutsky (2000), is not well-known in the literature. They offered welfare and comparative-static results. More recently, Shelegia and Wilson (2021) considered a model with asymmetric costs and also costly advertising to reach shoppers.[5] They required a condition that is not satisfied when more than two firms engage in sales, and so their paper is primarily relevant only for the de facto duopoly case.[6] We identify (as Proposition 2 in Section 1) conditions for such a de facto duopoly, in which the equilibrium characterization corresponds to that of Golding and Slutsky (2000).

Models of sales are closely related to all-pay auctions and contests: the contest effort corresponds to captive profit sacrificed by charging a lower price, and the prize is to sell to shoppers. Helpfully, the broad framework of Siegel (2009, 2010) can incorporate an asymmetric model of sales. Our Proposition 1 (concerning equilibrium profits) can be derived from Siegel (2009), but our other results (notably the derivation of equilibria) cannot.[7] Methodologically, our equilibrium-construction algorithm identifies "partner swapping" points at which the identities of actively mixing firms change. The construction is reminiscent of algorithms used elsewhere in contests and contest-like settings (Bulow and Levin, 2006; Siegel, 2010, 2014; Xiao, 2016, 2018).[8]

[5]In their full model description firms make utility offers à la Armstrong and Vickers (2001). However, their "Assumption U" restricts to unit demand when costs are asymmetric. They also specify a cost of advertising which is equivalent to a clearinghouse fee. We also study such fees (for an oligopoly) as an extension (in Section 3).

[6]We find that more than two firms do not (for generic parameters) mix over any common interval of prices in equilibrium. In their treatment of oligopoly, Shelegia and Wilson (2021) assumed that any firm offering randomized sales does so by mixing over some interval of prices that extends up to the monopoly price. This implies that when more than two firms mix, there must be an interval common to all those firms' equilibrium supports. Our work shows that condition is only met by equilibrium behavior in a duopoly, or a de facto duopoly (e.g., see our Proposition 2 point iv). Our analysis and this result also extend naturally to the case of a costly clearinghouse (see our Section 3).

[7]Siegel (2010) characterized equilibria with m prizes and $m+1$ players, and so covers models of sales ($m = 1$) only for duopoly. See Appendix C of our working paper and Shelegia and Wilson (2023) for a broader discussion.

[8]There are differences too, of course. For example, Xiao (2016) considered an all-pay auction with asymmetric costs and heterogeneous prizes. There can be intervals in which more than two players mix (see, for example, Xiao, 2016, Fig. 1, p. 184) whereas here battles are always pairwise. As a second example, Bulow and Levin (2006) considered a model in which employers bid for workers. The partner swapping can be such that no firm mixes across the whole price interval (illustrated in Bulow and Levin, 2006, Fig. 2, p. 658) whereas here the most aggressive firm does so.

Our buyers are either captives or shoppers. Armstrong and Vickers (2022) have made welcome progress with more general consideration sets, so revealing "patterns of competitive interaction" that emerge.[9] They focused on the demand side (with symmetry for $n > 3$) and assumed cost symmetry. Our simpler demand-side specification allows us to consider more general costs.

Beyond the incorporation of asymmetric marginal costs, our second contribution is our study of endogenous innovative activity in the spirit of Dasgupta and Stiglitz (1980). The key property that we exploit is that one firm faces a distinctly different innovation incentive to others. This property is present in work by Chioveanu (2008) which identified (in a model of sales with symmetric costs) a distinctly lower incentive for one (most aggressive) firm to retain captive customers. Relatedly, models with independent awareness of each firm (Ireland, 1993; McAfee, 1994) feature a stronger incentive for the firm with the greatest awareness to increase that awareness.

One of our extensions (in Section 3) specifies a "clearinghouse" for access to shoppers. Pioneering work on clearinghouses (Baye and Morgan, 2001) has been extended to include brand advertising (Baye and Morgan, 2009); asymmetric distributions of customers (Arnold et al., 2011); differentiation and discrimination (Moraga-González and Wildenbeest, 2012); per-sale fees (Baye, Gao and Morgan, 2011; Ronayne, 2021); and auto-switching services (Garrod, Li and Wilson, 2023).[10] See Baye, Morgan and Scholten (2004*a*, 2006) for empirical tests and a review of the earlier work.

1. A Model of Sales with Fully Asymmetric Firms

Model. There are $n \geq 2$ firms who simultaneously choose their prices, where $p_i \in [0, v]$ is the price chosen by firm $i \in \{1, \ldots, n\}$ and $v > 0$ is customers' (common) maximal willingness to pay. Firm i faces a constant marginal cost $c_i \in [0, v)$ to serve any customer.[11]

A mass of $\lambda_i > 0$ customers are "captive" to firm i. A mass of $\lambda_S > 0$ customers are "shoppers" who buy from the cheapest firm, or from one of the cheapest (in the event of a tie).[12]

[9] In other work (Myatt and Ronayne, 2023) we study richer consideration sets in our theory of stable price dispersion, but make progress by joining Armstrong and Vickers (2022) in specifying symmetric marginal costs.

[10] Arnold and Zhang (2014) showed that the introduction of a fixed access fee to the Varian (1980) model leaves the symmetric equilibrium (like the one studied by Baye and Morgan, 2001) as the unique equilibrium.

[11] Equilibria are unaffected if the cost of serving captive customers is different from serving shoppers.

[12] For technical convenience we break any ties in favor of a lowest-cost firm. This allows us to apply an off-the-shelf equilibrium-existence result (Dasgupta and Maskin, 1986, Theorem 5).

Firm i earns $\lambda_i(p_i - c_i)$ from its captive customers and $\lambda_S(p_i - c_i)$ from the shoppers if it sells to them. These components sum to form a (risk neutral) firm's payoff. This specification nests that of Varian (1980) if $\lambda_i = \lambda$ and $c_i = c$ for every i, so that firms are symmetric.[13]

Equilibrium Play. Central to a characterization of equilibrium play is to identify the firms that compete for sales to shoppers. This depends on a notion of a firm's potential "aggression", which is associated with a firm's willingness to set a low price in order to win over those shoppers.

Firm i guarantees a profit of at least $\lambda_i(v - c_i)$ by setting $p_i = v$ and selling only to captive customers. The lowest price it would be willing to set in order to win the business of shoppers is $p_i^\dagger$, satisfying $\lambda_i(v - c_i) = (\lambda_i + \lambda_S)(p_i^\dagger - c_i)$, or explicitly

$$p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i}{\lambda_i + \lambda_S}. \tag{1}$$

This lowest undominated price is a measure of how aggressive (in terms of pricing) a firm is willing to be. It is higher when a firm has more captive customers (it is more costly to lose revenue from them by lowering price) or when the marginal cost of serving shoppers is higher (making it less tempting to serve those shoppers). Using this measure, firm j is strictly more aggressive than firm i and if and only if $p_j^\dagger < p_i^\dagger$ which holds if and only if

$$\underbrace{(c_i - c_j)}_{\text{cost adv. } j \text{ vs. } i} > \frac{v - (c_i + c_j)/2}{\lambda_S + (\lambda_i + \lambda_j)/2} \underbrace{(\lambda_j - \lambda_i).}_{\text{captive adv. } j \text{ vs. } i} \tag{2}$$

If firm j has an advantage over firm i on the supply side (lower costs) and a disadvantage on the demand side (fewer captives) then this holds; otherwise, this inequality can break either way.

We choose labels for the firms (without loss of generality) so that the three highest-indexed firms n, $n-1$, and $n-2$ are the most aggressive: $p_n^\dagger \leq p_{n-1}^\dagger \leq p_{n-2}^\dagger \leq \min_{i \in \{1,\ldots,n-3\}} p_i^\dagger$.

Our notion of aggression, and so the ordering of the firms, focuses on the lowest price that a firm would charge in order to be sure of capturing shoppers. A more general notion asks how confident a firm would need to be of such capture in order to charge each possible price $p \leq v$.

[13]It also nests Baye, Kovenock and De Vries (1992) and Kocas and Kiyak (2006), which allowed for asymmetry in "captives", $\lambda_i \neq \lambda_j$ for $i \neq j$, but retained common (and so without further loss of generality, zero) marginal costs.

To explore briefly a more general notion of pricing aggression, we write $\underline{w}_i(p)$ for the "minimum win probability" (of winning the business of shoppers) for firm i contemplating charging a price p instead of simply exploiting captive customers. Explicitly, this is the probability that satisfies

$$\lambda_i(v - c_i) = (\lambda_i + \lambda_S \underline{w}_i(p))(p - c_i) \quad \text{and so} \quad \underline{w}_i(p) = \frac{\lambda_i(v-p)}{\lambda_S(p-c_i)}. \tag{3}$$

In a contest setting (as in, for example, Siegel, 2009, 2010, 2012, 2014) the numerator (this is the lost profit from sales to captive customers from a discount of size $v - p$) is the cost of effort, while the denominator (the profit from sales to shoppers made at price p) is the value of the prize. Using this notation, the lowest undominated price for firm i satisfies $\underline{w}_i(p_i^\dagger) = 1$.

An unambiguous "aggression" ranking of firms i and j is obtained if $\underline{w}_i(p) < \underline{w}_j(p)$ for all relevant p: firm i is more willing to drop to price p to win the shoppers' business. From eq. (3) this is true if i has both a supply-side advantage ($c_i < c_j$) and demand-side disadvantage ($\lambda_i < \lambda_j$). However, if one inequality is switched then $\underline{w}_i(p)$ and $\underline{w}_j(p)$ can readily cross. This possibility arises naturally when one firm has an advantage in both dimensions. In Appendix B we build a full characterization of equilibria using minimum win probabilities. Here, we simply note that such functions that cross are required for equilibria in which more than two firms actively use sales.

Baye, Kovenock and De Vries (1992, Section V) found a unique Nash equilibrium when firms with the same marginal cost ($c_i = c$ for all i) have differently sized captive audiences: ordering firms so that $\lambda_n < \lambda_{n-1} < \lambda_{n-2}$, the equilibrium involves mixing (the "tango" of their paper) by firms $n-1$ and n while firms $i \in \{1, \ldots, n-2\}$ set $p_i = v$. In other situations (including symmetry) there can be many equilibria, all of which generate the same expected profits.[14]

In our more general setting we find (just as with cost symmetry) that a firm earns its "captive-only" profit, $\lambda_i(v - c_i)$, for $i < n$. The exception is firm n. By setting its price equal to $p_n^\dagger$ it would guarantee to sell to all shoppers (recall that we order firms so that $p_n^\dagger$ is the lowest) and so would earn $p_n^\dagger(\lambda_n + \lambda_S)$ which (by construction of $p_n^\dagger$) is equal to its captive-only profit. However, it has the option to raise its price to match the lowest undominated price of the next-most aggressive

[14]Under symmetry there are infinitely many Nash equilibria (Baye, Kovenock and De Vries, 1992, Theorem 1). Two firms mix over a common interval, while others may mix over any low subinterval of that interval. Johnen and Ronayne (2021, Proposition 1) showed that multiplicity depends on the absence of customers who compare exactly two prices.

competitor by setting $p_n = p^\dagger_{n-1}$. Following this price rise of $p^\dagger_{n-1} - p^\dagger_n$ it continues to sell to all shoppers and its captives, and so earns $(p^\dagger_{n-1} - p^\dagger_n)(\lambda_n + \lambda_S)$ on top of its captive-only profit.

Proposition 1 (Nash Equilibrium and Profits). *For any parameter values, there exists a Nash equilibrium of the single-stage game in which firm i's expected profit is given by*

$$\pi_i = \underbrace{\lambda_i(v - c_i)}_{\textit{captive-only profit}} + \begin{cases} (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n) & \textit{if } i = n\textit{, and} \\ 0 & \textit{otherwise,} \end{cases} \tag{4}$$

so that only a most aggressive firm earns (weakly) more than its captive-only (expected) profit.

For generic parameter values satisfying $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$, eq. (4) *holds in any Nash equilibrium.*

Our main proofs are contained within Appendix A; some details are relegated to Appendix B.

We noted (in our introduction) that a model of sales is a contest: the effort cost is the captive profit lost from a lower price, while the shoppers are the prize. A statement equivalent to Proposition 1 using a contest setting can be derived from Theorem 1 and Corollary 2 of Siegel (2009).[15]

Proposition 1 says that the profits given by eq. (4) arise in the non-generic case of $p^\dagger_n = p^\dagger_{n-1}$ or $p^\dagger_{n-1} = p^\dagger_{n-2}$, but leaves scope for other profit levels too. However, we deem any other equilibria to be "pathological" in the sense that a small perturbation to parameters away from such a case would cause a discontinuous change to equilibrium profits—back to those given by eq. (4).

Definition. *An equilibrium is* pathological *if it gives payoffs that differ from those of eq.* (4).

For the remainder of the main paper, we do not consider pathological equilibria further.[16]

Note that (at most) one firm strictly benefits from its access to shoppers beyond its captive-only profit. With symmetric captive bases ($\lambda_i = \lambda$ for all i), profits are the same as if firms offered a discriminatory price to shoppers: firm n earns $\lambda(v - c_n) + \lambda_S(c_{n-1} - c_n)$, which is what it would earn if shoppers are served by it (the lowest-cost firm) at a price equal to the second-lowest cost.[17]

[15] Appendix C of our working paper provides an explicit mapping between Siegel's contest setting and models of sales.

[16] We give an example of one in Appendix B and show that in any, profits differ from eq. (4) for exactly one firm.

[17] As usual, a Bertrand construction requires a careful treatment of tie-break rules; for example by breaking a tie in favor of a lowest-cost firm. Our reference to discriminatory pricing refers to unit-demand customers. A more general analysis of captive-vs-shoppers discrimination was reported by Armstrong and Vickers (2019).

More generally, by pricing below $p^\dagger_{n-1}$, the most aggressive firm n is sure to sell to all shoppers. Any higher price invites an "undercut" from the next-most aggressive firm, $n-1$, and for prices ranging upward from $p^\dagger_{n-1}$ we see the (familiar) mixing from (at least) two firms. In the classic setting (with asymmetric captive shares) only two firms are involved in such randomized sales.[18]

The "two to tango" result does not extend fully. Such a tango is danced by the two most aggressive firms over some interval ranging upward from $p^\dagger_{n-1}$. However, there is a possibility that (for higher prices) other firms step onto the dance floor. For strictly asymmetric firms, there are situations (we describe these later) in which more than two tango. We also find conditions so that only two tango, e.g., if the most aggressive firms have the two smallest captive audiences.

Appendix B describes the (unique, generically) construction of an equilibrium for all parameters.[19] Here, however, to ease exposition we focus on (generic) cases satisfying $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$. We use F_i to refer to the cdf of firm i's prices, which may be in mixed strategies in equilibrium.

Proposition 2 (Nash Prices I: When Two Tango). *Suppose that* $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$.

(i) Firms $i < n$ *place an atom at* $p_i = v$*, while* n *mixes continuously over all* $p \in [p^\dagger_{n-1}, v]$.

(ii) There is $p^\ddagger \in (p^\dagger_{n-2}, v]$ *such that for* $p \in [p^\dagger_{n-1}, p^\ddagger)$, $F_i(p) = 0$ *for* $i \le n-2$*, while*

$$F_n(p) = \frac{(p - p^\dagger_{n-1})(\lambda_{n-1} + \lambda_S)}{\lambda_S(p - c_{n-1})} \quad \textit{and} \quad F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda_n + \lambda_S)}{\lambda_S(p - c_n)}. \tag{5}$$

(iii) If $c_n \le c_{n-1}$ *then* $F_{n-1}(p)$ *first order stochastically dominates* $F_n(p)$.

(iv) If $\lambda_{n-1} \le \min_{i \in \{1,\dots,n-2\}}\{\lambda_i\}$*, then* $p^\ddagger = v$ *and so all firms* $i \in \{1, \dots, n-2\}$ *choose* $p_i = v$ *and serve only captives, while firms* n *and* $n-1$ *mix via eq.* (5) *over prices* $p \in [p^\dagger_{n-1}, v)$.

Equation (5) characterizes a "tango" by firms $n-1$ and n below $p^\dagger_{n-2}$. Firm $n-1$ can earn its captive-only profit of $\lambda_{n-1}(v - c_{n-1})$ by charging v. It is indifferent to charging $p < v$ if

$$\underbrace{(v - p)\lambda_{n-1}}_{\text{loss on captives}} = \underbrace{(p - c_{n-1})\lambda_S(1 - F_n(p))}_{\text{gain from shopper sales}}, \tag{6}$$

[18]If $c_n < \cdots < c_1$, then for sufficiently symmetric captive audience sizes, firms n and $n-1$ are the most aggressive and the only two that mix (we approach the "two to tango" of Baye, Kovenock and De Vries, 1992, Section V).

[19]Siegel (2010) described an algorithm to construct an equilibrium in a related class of contest games. His approach applies to a duopoly model of sales, but not for the broader oligopoly environment studied here.

which solves for $F_n(p)$. The desire to compete for shoppers is lessened if a firm has more captives, and sales to shoppers are less valuable if its marginal cost is higher. Any lower-indexed (and less aggressive) firm $i \in \{1, \ldots, n-2\}$ that has both more captives ($\lambda_i > \lambda_{n-1}$) and higher costs ($c_i > c_{n-1}$) has a strictly weaker incentive to set a price $p < v$. Such a firm does not wish to "step onto the dance floor" and so (if this is true, and in fact under weaker conditions) we can construct a unique equilibrium in which firms $n-1$ and n "tango" with the distributions of eq. (5) all of the way up to v. In this case the oligopoly reduces to a de facto duopoly in which the analyses of Golding and Slutsky (2000) or Shelegia and Wilson (2021) apply and where a characterization based on contests (Siegel, 2010) can also be used. Other (lower indexed) firms are simply wallflowers.[20]

However, a less aggressive firm $i \leq n-2$ might have both higher costs but fewer captives. This combination can result in a higher $p_i^\dagger$ but a greater temptation to charge some intermediate price. To demonstrate this explicitly, we (attempt to) construct a "two to tango" scenario in which firms n and $n-1$ mix over $[p_{n-1}^\dagger, v)$ according to the distributions reported in eq. (5), while other firms charge v. For this to be an equilibrium, we must be sure that for all $p \in [p_{n-1}^\dagger, v)$ and $i \leq n-2$

$$(v-p)\lambda_i \geq (p-c_i)\lambda_S(1-F_n(p))(1-F_{n-1}(p)). \tag{7}$$

Suppose, however, $c_i \in (p_{n-1}^\dagger, v)$. This guarantees that $p_i^\dagger > p_{n-1}^\dagger$, and so firm i is not one of the two most aggressive firms. We can now choose λ_i sufficiently small such that eq. (7) fails. This means that there is a price at which firm i wishes to join the dance floor.

This argument adds a third firm to disrupt the tango danced by two existing firms. By choosing this firm's marginal cost to be high, we guarantee that the two existing firms compete in a lower price range. However, if this third firm has a sufficiently small captive base then it wishes to join the action at some point, to sell to shoppers at least sometimes. This implies that any equilibrium must involve mixing from that third firm. This logic extends such that if $k \geq 2$ firms play mixed strategies in equilibrium there are parameters such that a $k+1^{\text{th}}$ firm wants to join the dance.

[20] In the study of contests, "participation" in the contest is akin to the "dancing" we and others refer to in the study of models of sales. Siegel (2012) identifies when any number of players in a contest can participate in equilibrium. A difference to models of sales is that there the valuation of the prize is independent of the player's choice variable.

Proposition 3 (Nash Prices II: When Three or More Tango). *For any $k \in \{3, \ldots, n\}$ there is an open set of parameters such that k firms play mixed strategies in any equilibrium.*

If three or more firms mix, then for generic parameter values the set of the supports (excluding any atoms at v) of each firm $i < n$ that mixes is a partition of the support of firm n, $[p^{\dagger}_{n-1}, v]$.

If $p^{\dagger}_{n} < p^{\dagger}_{n-1} < p^{\dagger}_{n-2}$, then at least three firms mix in any equilibrium if and only if eq. (7) *holds with equality for some $i \leq n-2$ at some $p \in (p^{\dagger}_{n-2}, v)$, where $F_n(p), F_{n-1}(p)$ are given by eq.* (5).

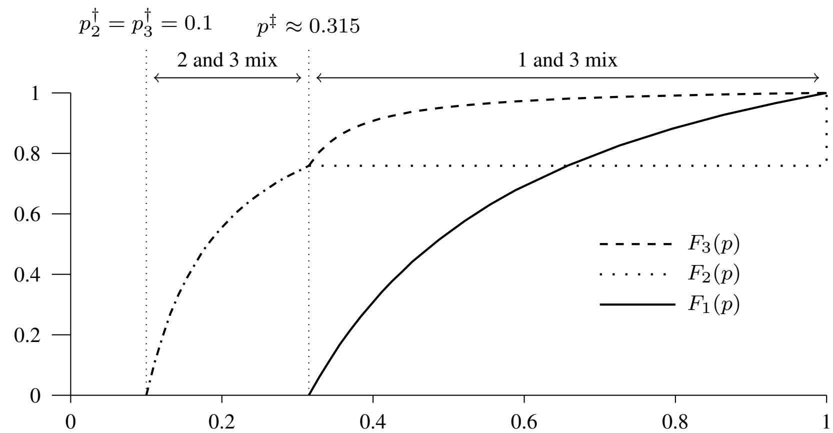
A firm that joins a dance "late" (rather than at the "start", i.e., at $p^{\dagger}_{n-1}$) may be a new entrant with higher costs and a smaller captive base. Indeed, if its captive base is small then it has an incentive to use a production technology with a relatively high marginal cost if that results in a lower fixed cost. (This corresponds to technology choice studied in Section 2.) This suggests that the presence of such a firm (and so randomized sales by more than two firms) is not only a theoretical curiosity.

Randomized Sales from Multiple Firms. In Appendix B we offer an equilibrium characterization that is unique except for knife-edge cases (such as exact symmetry). If k firms play mixed strategies then in equilibrium the interval $[p^{\dagger}_{n-1}, v)$ is partitioned into $k-1$ subintervals. The most aggressive firm n mixes over the entire range of prices. However, each of the other $k-1$ firms mix only over a single subinterval. Moving upward through prices, the "dance partner" of firm n switches. More than two firms mix; but only two firms mix within any particular interval of prices.

Propositions 2 and 3 confirm existence and properties of equilibria with $k > 2$. Here we build an illustrative "thrango" example with $k = n = 3$. For simplicity and to simplify exposition, two firms are symmetric, but this is easily modified so that the three firms are (generically) different.

Specifically, assume that firms 2 and 3 have the same characteristics, comprising low costs and larger captive audiences: $c_2 = c_3 = 0$ and $\lambda_2 = \lambda_3 = \lambda_H$. Firm 1 has higher costs, but a smaller captive audience: $c_1 = c > 0$ and $\lambda_1 = \lambda_L$ where $\lambda_L < \lambda_H$. By setting $c > (\lambda_H - \lambda_L)v/(\lambda_H + \lambda_S)$ we guarantee that firm 1 is less aggressive than the others. An equilibrium involves the two jointly most aggressive firms engaged in a "tango" over the lower interval of prices, with

$$1 - F_3(p) = 1 - F_2(p) = \frac{\lambda_H(v - p)}{\lambda_S p}. \tag{8}$$



This illustrates the mixed-strategy distribution functions for a triopoly in which all three firms use randomized sales. The specification is from the text, where $c_2 = c_3 = 0$, $\lambda_2 = \lambda_3 = \lambda_H = 0.1$, $\lambda_S = 0.9$, $v = 1$ and so $p_2^\dagger = p_3^\dagger = 0.1$. For the less aggressive firm, $\lambda_1 = \lambda_L = 0.005$ and $c_1 = c = 0.25$. There are two distinct "dance floor" segments, with firm 3 "partner swapping" from firm 2 to firm 1 at $p^\ddagger \approx 0.315$.

FIGURE 1. Mixing CDFs in a Three-Firm "Thrango" Example

The inequality in eq. (7) holds for all relevant p if and only if $\lambda_L \geq [\lambda_H^2(v-c)^2]/[4\lambda_S vc]$. Our "thrango" situation arises when this fails, as it does, at a price $p^\ddagger$, for $\lambda_L > 0$ sufficiently small. If so, then the "tango" between firms 2 and 3 ends at the price $p^\ddagger$, which satisfies eq. (7), and here is:

$$p^\ddagger = \frac{v + c - \sqrt{(v+c)^2 - 4\left(1 + [\lambda_L \lambda_S / \lambda_H^2]\right) vc}}{2\left(1 + [\lambda_L \lambda_S / \lambda_H^2]\right)}. \tag{9}$$

At this point, $p^\ddagger$, there is a partner swap: firm 2 shifts all remaining weight to price at v, while firm 1 then begins mixing.[21] Over the interval $[p^\ddagger, v)$ the relevant mixing distributions are

$$1 - F_3(p) = \frac{\lambda_L}{\lambda_H} \frac{v-p}{v-p^\ddagger} \frac{p^\ddagger}{p-c} \quad \text{and} \quad 1 - F_1(p) = \frac{v-p}{v-p^\ddagger} \frac{p^\ddagger}{p}. \tag{10}$$

The equilibrium CDFs of this example are illustrated in Figure 1 for suitable parameter choices.

Changing Costs. The asymmetric solution allows us to vary costs . To ease exposition, we consider local changes that maintain the ranking $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. We also assume (a sufficient condition is $\lambda_{n-1} \leq \min_{i \in \{1,\ldots,n-2\}}\{\lambda_i\}$) that the equilibrium involves mixing by only two firms. Each firm $i \in \{1, \ldots, n-2\}$ sets $p_i = v$, which is unaffected by any local cost changes, and such firms' costs of do not influence prices. The interesting exercises concern firms n and $n-1$.

[21]Because firms 2 and 3 are completely symmetric in this example, there is a second equilibrium in which firm 3 (instead of 2) leaves the dance floor and firms 1 and 2 mix over $[p^\ddagger, v)$. Departures from symmetry leave only one.

Inspecting eq. (5), c_n enters only into the solution for firm $n-1$. The distribution $F_{n-1}(p)$ is increasing in c_n: an increase in the marginal cost of firm n lowers the prices charged by firm $n-1$. This is because firm $n-1$ prices more aggressively to maintain the incentive for the (now more costly) firm n to price at $p^\dagger_{n-1}$ rather than the (now more attractive, given the higher cost) higher prices within $[p^\dagger_{n-1}, v)$. This implies that the captive customers of firm $n-1$, as well as the shoppers, benefit from any cost increase suffered by the most aggressive firm that also most often supplies the shoppers (as we confirmed via claim (iv) of Proposition 2).

The cost c_{n-1} of firm $n-1$ has a more conventional impact. A direct effect of an increase in c_{n-1}, by inspection of eq. (5), is to increase $F_n(p)$ and so to push down the prices charged by firm n. (This follows from the logic discussed just above.) However, an increase in c_{n-1} also raises the lower bound of the interval of sales prices charged by both firms, $p^\dagger_{n-1}$. This lowers both $F_{n-1}(p)$ and $F_n(p)$. There are competing effects on $F_n(p)$, but overall the impact (as the proof of the next proposition confirms) is to push up the prices charged by both competitors.

Proposition 4 (The Effect of Costs on Mixed-Strategy Prices). *Suppose that* $p^\dagger_n < p^\dagger_{n-1} < p^\dagger_{n-2}$. *Further suppose that the equilibrium involves mixing only by firms* $n-1$ *and* n.

(i) No price changes in response to local changes in the cost c_i *of any firm* $i \in \{1, \ldots, n-2\}$.

(ii) A local increase in c_{n-1} *shifts rightward the distributions of prices charged by* $n-1$ *and* n.

(iii) A local increase in c_n *shifts leftward the distribution of prices charged by* $n-1$.

Claim (iii) implies that firm n disproportionately gains from any reduction in its marginal cost. A reduction in c_n has the usual direct effect on its profit. However, it also prompts a price rise from its competitor (in the market for sales to shoppers) firm $n-1$. This is a positive strategic effect. (Notably, a negative strategic effect is more common in pricing games.) In fact,

$$\frac{\partial \pi_i}{\partial c_i} = -\begin{cases} \lambda_n + \lambda_S & \text{if } i = n\text{, and} \\ \lambda_i & \text{otherwise,} \end{cases} \tag{11}$$

and so a given firm gains more from a cost reduction when it is the most aggressive than when it is not. This suggests that there are asymmetric incentives to engage in innovative activity. We develop this observation when we study endogenous technology choice in Section 2.

Efficiency. The only efficiency-relevant question is: who serves the shoppers? The outcome is efficient only if shoppers are served by the lowest cost firm.[22] Here, however, they are served by the two most aggressive firms. These shopper-serving firms may not have low costs (resulting in greater inefficiency) if the aggression is because (for example) low-cost firms have bigger captive audiences. We return to welfare considerations toward the end of Section 2.

2. Process Innovations and Endogenous Asymmetry

From eq. (11), the most aggressive firm gains distinctly more (relative to another firm with a similar captive audience) from a reduction in its marginal cost. Here we explore such cost reductions via the consideration of a prior-stage innovation game followed by model-of-sales pricing.

We have several reasons for pursuing this avenue of inquiry. Firstly, and pragmatically, our analysis presents us with a natural opportunity to do so. Secondly, the asymmetric incentives to reduce costs suggests that asymmetric costs might arise endogenously. This reinforces the requirement for an asymmetric-cost model. Thirdly, we are able to evaluate the likely cross-sectional relationship between captive-audience size, marginal cost, and the use of sales. Finally, we ask: to what extent do the "right" firms serve the shoppers, and is innovation efficient in a model-of-sales world?

We also note here that innovation is equivalent (in a familiar way) to a choice of production technology: a higher fixed cost to achieve a lower marginal cost can be thought of as a shift from a labor-intensive to a capital-intensive production technique. We can ask, therefore, whether firms that use sales (rather than those who are captive-focused) will use different technologies.

A Model of Process Innovation. Prior to the pricing stage firms choose their production technologies via costly innovations. We study the following two-stage game.

(1) Firms simultaneously choose and observe production technologies, denoted by z_i.
(2) Firms play a Nash equilibrium of the pricing game (Section 1) with the profits of eq. (4).

[22] This is always true if costs are symmetric, and there is no welfare analysis in classic papers (Varian, 1980; Baye, Kovenock and De Vries, 1992). Welfare in a duopoly was considered by Golding and Slutsky (2000).

We interpret a technology choice, $z_i \in [0, \bar{z}_i]$, as a fixed-cost expenditure which lowers the marginal cost of production: it is a (costly) process innovation. It is (as usual) equivalent to a product innovation that raises valuations; what really matters is the net surplus, $v - c_i$, created. We assume

$$v - c_i = V_i(z_i), \tag{12}$$

where $V_i(z_i)$ is positive, smoothly increasing, and concave. We also assume the regularity conditions $\lambda_i V_i'(0) > 1 > (\lambda_i + \lambda_S)V_i'(\bar{z}_i)$ so that innovation choices satisfy $z_i \in (0, \bar{z}_i)$.

We have (so far) labeled firms n and $n-1$ as most aggressive. Here that status is endogenous because firms choose their technologies. For now, then, we do not allocate such labels.

Deducting z_i from the gross equilibrium expected profit from eq. (4), a firm's net profit is

$$\pi_i = -z_i + \begin{cases} (\lambda_i + \lambda_S)V_i(z_i) - \lambda_S(\lambda_i + \lambda_S)\max_{j \neq i}\left[\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right] & \frac{V_i(z_i)}{\lambda_i + \lambda_S} > \max_{j \neq i}\left\{\frac{V_j(z_j)}{\lambda_j + \lambda_S}\right\}, \\ \lambda_i V_i(z_i) & \text{otherwise.} \end{cases} \tag{13}$$

The first case applies when firm i is the most aggressive (so that $p_i^\dagger < \min_{j \neq i} p_j^\dagger$) and the second case applies if not. Using these expected profit expressions as the outcomes from stage (2) described above, we study a simultaneous-move "innovation game": stage (1) described above.

In contrast to prices, we think of a firm's technology choice as a longer-term decision subject to adjustment following the choices of others. We look, therefore, for choices from which no firm would deviate ex post: a pure-strategy equilibrium Nash equilibrium of the innovation game. Nevertheless, there are parameters under which there are multiple pure-strategy equilibria and so there will also be mixed-strategy equilibria. Such mixed equilibria are not our focus.[23]

The Shape of Technological Opportunity. The function $V_i(z_i)$ relates firm i's fixed-cost expenditure to the per-unit surplus that it generates. The properties of this innovation production function determine how changes in expected output determine technology choice. Here we flesh out these

[23]This contrasts with the mixed equilibria of our pricing game. In fact, mixing is not crucial for generating the expected payoffs of Proposition 1. In an older working version of this paper we generated pure-strategy play on the equilibrium path of natural pricing games (using the ideas in Myatt and Ronayne, 2023) in the same asymmetric models of sales environment. The expected payoffs to firms in those games coincide with what we report here.

properties given that (as we will show) they matter when we find the relationship between the demand-side characteristics of firms and their aggression in the pricing game.

For this discussion we drop the "i" subscript and consider the socially optimal technology choice z_Q when Q units are supplied. This satisfies the first-order condition $QV'(z_Q) = 1$. As output rises, the optimally higher choice of z_Q raises the per-customer surplus $V(z_Q)$. We say that the technological opportunity for innovation is *strong* if this surplus reacts elastically to output.

Definition. *The* technological opportunity for innovation *is* strong *if the socially optimal per-customer surplus always responds elastically to an increase in output. Similarly, that opportunity is* weak *if the socially optimal surplus always responds inelastically.*

This notion is relevant because a firm with a larger captive audience has a greater incentive to innovate and so lower its marginal cost. These features (more captives but lower marginal cost) have opposing effects on its willingness to charge a lower price. Whether the second factor outweighs the first factor turns on whether the (endogenously chosen) per-customer surplus reacts elastically (that is, more than proportionally) to an increase in output. Checking when this is so,

$$\frac{\partial[V(z_Q)/Q]}{\partial Q} > 0 \quad \Leftrightarrow \quad \underbrace{\frac{z_Q V'(z_Q)}{V(z_Q)}}_{\text{elasticity of } V(z)} > \underbrace{-\frac{z_Q V''(z_Q)}{V'(z_Q)}}_{\text{curvature of } V(z)}, \tag{14}$$

and so per-customer surplus reacts elastically to output if the elasticity of $V(z)$ exceeds its curvature.[24] This is easy to evaluate when $V(z)$ takes a constant-elasticity form.[25] Suppose that

$$V(z) = \beta z^{\bar{\gamma}} \quad \text{where} \quad \bar{\gamma} \in (0, 1). \tag{15}$$

The elasticity and curvature of this function are $\bar{\gamma}$ and $1-\bar{\gamma}$ respectively. Technological opportunity is strong if and only if $\bar{\gamma} > \frac{1}{2}$, which says that $V(z)$ is less concave than $\sqrt{z}$.

[24] The elasticity measures the impact of innovation on per-customer surplus; the curvature is the elasticity of the slope of $V(z)$ and so measures how quickly the incentive to innovate falls as innovation increases. These measures jointly determine how the (endogenously chosen) impact of innovation reacts to output.

[25] This specification is reminiscent (albeit different from) the constant-elasticity relationship between production cost and research and development expenditure specified by Dasgupta and Stiglitz (1980, p. 273).

Asymmetric Equilibrium Innovation. We now consider the possible profiles of equilibrium innovation choices. Inspecting eq. (13), the response of firm i's profit to a local change in z_i depends on whether it expects to be the most aggressive firm. If it does not, then its expected profit $\lambda_i V(z_i) - z_i$ is equal to that obtained from fully exploiting captive customers. For any "wallflower" firms that do not use sales, this is literally true: such a firm expects to act as a monopolist over its captive customers and to extract all surplus from them. This profit expression also applies, however, to other firms (other than the most aggressive firm) that use sales.

If a firm does expect to be the most aggressive, however, then its expected profit is instead equal to $(\lambda_i + \lambda_S)V_i(z_i) - z_i$ minus a term that does not depend on z_i. Such a firm faces an incentive to act as if it fully exploits a customer base of size $\lambda_i + \lambda_S$, giving it (relative to another firm with a similarly sized actual captive audience) a heightened incentive to innovate. In fact, we find that

$$\frac{\partial \pi_i}{\partial z_i} = -1 + V_i'(z_i) \begin{cases} \lambda_i + \lambda_S & \text{if } \frac{V_i(z_i)}{\lambda_i + \lambda_S} > \max_{j \neq i} \left\{ \frac{V_j(z_j)}{\lambda_j + \lambda_S} \right\} \text{ (so that } i \text{ is most aggressive)} \\ \lambda_i & \text{if } \frac{V_i(z_i)}{\lambda_i + \lambda_S} < \max_{j \neq i} \left\{ \frac{V_j(z_j)}{\lambda_j + \lambda_S} \right\}, \end{cases} \tag{16}$$

implying that the profit of firm i has a (convex) kink when i becomes the most aggressive firm. This kink implies that firm i will never optimally choose z_i such that $p_i^\dagger = \min_{j \neq i}\{p_j^\dagger\}$ and so, in equilibrium, the most aggressive firm is uniquely identified. Furthermore, the expressions for $\partial \pi_i / \partial z_i$ (and so any potential first-order condition solutions) do not depend on z_j for $j \neq i$.

Drawing these observations together, we identify two possible solutions for firm i's innovation choice: z_i^H for when it is the most aggressive firm, and z_i^L for when it is not. These solutions are uniquely determined by the two respective conditions

$$1 = \lambda_i V_i'(z_i^L) \quad \text{and} \quad 1 = (\lambda_i + \lambda_S) V_i'(z_i^H), \tag{17}$$

they satisfy $z_i^H > z_i^L$, and they do not depend on the innovation choices of any other firms.

There are n candidate equilibrium profiles. For each candidate we choose a firm $i \in \{1, \dots, n\}$ to be most aggressive, set $z_i = z_i^H$, and $z_j = z_j^L$ for $j \neq i$. Two checks are then needed. Firstly, firm i must be the most aggressive firm. This requires $p_i^\dagger < \min_{j \neq i}\{p_j^\dagger\}$ or equivalently

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j \neq i} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \quad \text{or equivalently} \quad \frac{V_i(z_i^H)}{\lambda_i + \lambda_S} > \max_{j \in \{1, \dots, n\}} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\}. \tag{18}$$

Secondly, we need to check that firm i does not wish to deviate back to z_i^L, and that no $j \neq i$ wishes to deviate from z_j^L to z_j^H. For example, suppose a firm i uniquely maximizes (across the set of firms) both $V_j(z_j^L)/(\lambda_j+\lambda_S)$ and also $V_j(z_j^H)/(\lambda_j+\lambda_S)$. Such a firm can always take the "most aggressive" role in the innovation game. The proof of Proposition 5 establishes more generally that we can always find at least one firm to take this role.

Proposition 5 (Asymmetric Innovation Equilibria). *Consider the innovation game.*

(i) There is at least one pure-strategy Nash equilibrium, and there are at most n such equilibria.

(ii) In any such equilibrium there is a uniquely most aggressive firm at the pricing stage.

(iii) If $\arg\max_j\{V_j(z_j^L)/\lambda_j\}$ is unique, there is a unique equilibrium if λ_S is sufficiently small.

(iv) If firms are symmetric, then there are exactly n pure-strategy equilibria.

The final claim also holds when firms are sufficiently similar. It is of interest because it implies that exogenously symmetric firms become endogenously asymmetric.

Corollary 1 (Ex Ante Symmetry and Asymmetric Outcomes). *Suppose firms have the same sized captive audiences and technological opportunities. In any pure-strategy equilibrium, innovation and pricing choices are asymmetric. One firm chooses strictly higher innovation, giving it a strictly lower marginal cost; the $n-1$ others choose the same (and lower) innovation.*

This provides a distinct rationale for opening up models of sales to cost asymmetry. It is especially relevant because (with innovation) it arises ex post even when firms are ex ante symmetric.

Claim (iii) of Proposition 5 also identifies a sufficient condition for a unique equilibrium. As λ_S becomes small, z_i^H (the innovation that firm i chooses when it expects to be the most aggressive) becomes close to z_i^L. In the limit as $\lambda_S \to 0$, the inequality of eq. (18) that is required for firm i to take the most aggressive position reduces to

$$\frac{V_i(z_i^L)}{\lambda_i} > \max_{j\neq i}\left\{\frac{V_j(z_j^L)}{\lambda_j}\right\}, \tag{19}$$

which can be satisfied only for the firm that maximizes $V_j(z_j^L)/\lambda_j$.

From Demand-Side to Supply-Side Asymmetry. In the presence of endogenous innovation, which firms ultimately engage in sales? To address this, we specify symmetric technological opportunities, so that $v - c_i = V(z_i)$ for all i, and we place firms in size order, $\lambda_1 < \cdots < \lambda_n$.

Suppose that the opportunity for innovation is strong in the sense defined earlier. If firms had the same (exogenous) marginal cost then firm 1 would be the most aggressive. Here, however, larger (captive audience) firms innovate sufficiently to be able to become more aggressive. In fact,

$$\frac{V(z_1^H)}{\lambda_1 + \lambda_S} < \cdots < \frac{V(z_n^H)}{\lambda_n + \lambda_S} \quad \text{and also} \quad \frac{V(z_1^L)}{\lambda_1 + \lambda_S} < \cdots < \frac{V(z_n^L)}{\lambda_n + \lambda_S}. \tag{20}$$

The first set of equalities is implied by eq. (14), and the second set follows straightforwardly.[26]

Firm i can obtain the most aggressive position if the inequality of eq. (18) holds. Here this is

$$\frac{V(z_i^H)}{\lambda_i + \lambda_S} > \frac{V(z_n^L)}{\lambda_n + \lambda_S}. \tag{21}$$

The left-hand side is increasing in λ_i and so the firms satisfying this inequality are $\{k, \ldots, n\}$ for some $k \in \{1, \ldots, n\}$; the firms with larger captive bases. Moreover, the left-hand side of eq. (21) is increasing in λ_S while the right-hand side is decreasing in λ_S. This means as the population of shoppers falls (λ_S shrinks), k rises. For $i < n$,

$$\lim_{\lambda_S \downarrow 0} \frac{V(z_i^H)}{\lambda_i + \lambda_S} = \frac{V(z_i^L)}{\lambda_i} < \frac{V(z_n^L)}{\lambda_n} = \lim_{\lambda_S \downarrow 0} \frac{V(z_n^L)}{\lambda_n + \lambda_S}, \tag{22}$$

and so if λ_S is sufficiently small then eq. (21) must fail for any firm $i < n$.

Our discussion assumed technological opportunity that is strong. If it is weak then we cannot make all of the statements above.[27] Nevertheless, the firm with the smallest captive audience is uniquely able to take the most-aggressive position when there are sufficiently few shoppers.

For our statements in the following proposition we write $z_i^\star$ and $c_i^\star = v - V(z_i^\star)$ for the equilibrium innovation choices and final marginal costs of firms in equilibrium.

[26]Specifically, for firm i we write $V(z_i^L)/(\lambda_i + \lambda_S) = [\lambda_i/(\lambda_i + \lambda_S)] \times [V(z_i^L)/\lambda_i]$. The first term is (trivially) increasing in λ_i (and so increasing in i given the size ordering of the firms) and the second term is also increasing given that the technological opportunity for innovation is uniformly strong.

[27]For example, we can no longer be sure that the second set of inequalities from eq. (20) hold.

Proposition 6 (The Most Aggressive Firm). *Suppose that firms have the same technology opportunities, that $\lambda_1 < \cdots < \lambda_n$, and that there is a unique equilibrium (e.g. if λ_S is small).*

(i) If the technological opportunity for innovation is strong then larger firms innovate more, so that $z_1^\star < \cdots < z_n^\star$, operate with lower marginal costs, so that $c_1^\star > \cdots > c_n^\star$, and are more aggressive: $p_1^\dagger > \cdots > p_n^\dagger$. The shoppers are served by (at least) the two largest firms. Prices satisfy

$$\mathrm{E}[p_n] < \mathrm{E}[p_{n-1}] < \mathrm{E}[p_{n-2}] \leq \cdots \leq \mathrm{E}[p_1]. \tag{23}$$

(ii) If the technological opportunity for innovation is weak then the smallest firm is the most aggressive, and amongst other firms the larger ones innovate more:

$$p_1^\dagger < \min_{i>1}\{p_i^\dagger\} \quad \textit{and} \quad z_2^\star < \cdots < z_n^\star \quad \textit{or equivalently} \quad c_2^\star > \cdots > c_n^\star, \tag{24}$$

but the smallest (and most aggressive) firm (which competes for shoppers) might innovate more or less than the other firms. If, additionally, λ_S is sufficiently small, then $z_1^\star < z_2^\star$, $c_1^\star > c_2^\star$, and $p_1^\dagger < \cdots < p_n^\dagger$. Furthermore, in the pricing game the two smallest (and highest cost) firms compete for shoppers, whereas other firms charge v to their captive customers.

For case (i), firms with larger captive audiences have disproportionately lower costs, which is sufficient to make them more aggressive than other (smaller, in terms of captives) firms. The condition ($c_n \leq c_{n-1}$) needed for claim (iii) of Proposition 2 is met, and we can rank (expected) prices: shoppers are most often served by the lowest-cost and (in expectation) cheapest firm. However, the condition for claim (iv) of Proposition 2 fails, and so the ingredients are in place for a possible pricing equilibrium in which three (or more) firms engage in randomized sales to win the shoppers.

For case (ii) a smaller captive audience is the dominant factor driving aggression. This brings the situation closer to the classic symmetric-cost model in which the most aggressive firms are the smallest, and so the condition for claim (iv) of Proposition 2 holds. The unique equilibrium of the single-stage pricing game involves the classic "tango" between the two most aggressive firms (and no dancing by any other firm). However, the most aggressive firm does not have the lowest cost, and so claim (iii) of our Proposition 2 does not rank the pricing distributions of the dancers.

These two cases are cleanly illustrated when $V(z)$ takes a constant-elasticity form, as in eq. (15).

Corollary 2 (Technological Opportunity with Constant Elasticity). *Under the conditions of Proposition 6, suppose further that the technological opportunity for innovation corresponds to a function with constant elasticity $\bar{\gamma}$. If $\bar{\gamma} > \frac{1}{2}$ (so that the innovation production function is sufficiently elastic) then the claims of (i) hold, so that the largest firm is the most aggressive; but if $\bar{\gamma} < \frac{1}{2}$ then the claims of (ii) hold, and the smallest firm is the most aggressive.*

This analysis reveals how demand-side asymmetries (captive populations) interact with technological opportunities to determine supply-side asymmetries and which firms serve the shoppers.

Efficiency. In our model all customers are supplied, and captives are restricted to be served by their respective captors. There are, therefore, two requirements for efficiency: to allocate the shoppers to the "right" firm, and to choose each firm's innovation appropriately.

Suppose firms all have access to the same technological opportunities and order them in order of their captive bases so that $\lambda_1 < \cdots < \lambda_n$. The efficient solution is to allocate all shoppers to the largest (captive audience) firm n, and ask firms to innovate optimally in response to this demand.[28] Efficient technology choices are then z_n^H and z_i^L for $i < n$, from eq. (17). Relative to this benchmark, there can be two kinds of inefficiency: shoppers are incorrectly allocated to firms; and the technology choices of (some) firms are not efficient given their expected sales.

To streamline discussion, suppose that technological opportunity is strong, and that an equilibrium is played in which the largest firm takes the most aggressive position.[29] We know that (at least) the two largest firms use sales, and more generally k firms do so. In this situation, $n - k$ firms price at v and innovate efficiently: they focus solely on serving captive customers and use the correct technology. However, shoppers are allocated inefficiently amongst those who offer sales. To nudge output to the lowest-cost (and largest) firm the planner might uses subsidies and taxes. Or, it could seek to increase the innovation of the larger firm and reduce that of smaller firms.

However, any intervention to influence technology choices faces a second problem: those choices are themselves inefficient. The innovation of the largest firm is z_n^H, which is optimal for a firm that serves shoppers with certainty; however, this firm does not. Similarly, the innovation of a firm

[28] We can think of allocating n different production technologies. To exploit the lowest marginal cost, we allocate the best technology to both the shoppers and the largest captive audience, and so firm n serves the shoppers.

[29] From Proposition 5 part (iii) such an equilibrium is unique when λ_S is sufficiently small.

$i < n$ that mixes is z_i^L, which is optimal for a firm that serves only its captives; but such a firm does sometimes sell to the shoppers. Fixing the allocation of shoppers, this means that the largest firm over-innovates and its competitors that mix under-innovate. This presents the planner with a tension: fixing the allocation of shoppers it would be preferable to shift innovation from firm n to firm i; but to induce a better allocation of those shoppers it would be better to induce the opposite.

Discriminatory Pricing. It is also instructive to compare the outcome here to when firms use discriminatory pricing. To do this, we simplify by assuming equal captive-audience sizes so that $\lambda_i = \lambda$ for all i. Dropping the subscript i, we write z^L and z^H for the optimal innovation choices that are associated with serving only captives, so that $1 = \lambda V'(z^L)$, or also shoppers, so that $1 = (\lambda + \lambda_S)V'(z^H)$, respectively. We then allow firms to set different captive and shopper prices.

At the second stage, let us order firms by innovation so that $z_1 \leq \cdots \leq z_n$, and so higher indexed firms have lower marginal cost. All firms charge v to their captive customers, and compete à la Bertrand for shoppers. That Bertrand game has (for $n \geq 3$) many pure-strategy equilibria, but all involve the lowest-cost firm n supplying all of the shoppers at a price equal to the marginal cost of the second-lowest-cost firm $n - 1$. Firms' payoffs are $\lambda_i V(z_i) - z_i$ for $i < n$ and $\lambda_n V(z_n) + \lambda_S(V(z_n) - V(z_{n-1})) - z_n$. These payoffs are equal precisely to the expected payoffs obtained in an innovation game that is followed by uniform pricing.[30]

Three conclusions follow. Firstly, there are n equilibria in which one firm chooses innovation z^H and ultimately serves all shoppers while others choose z^L and serve only captives. Secondly, this is fully efficient: a single (lowest cost) firm supplies the shoppers, and all technologies are optimized. Finally, expected profits equal those obtained in a model of sales with uniform pricing, and so the efficiency gain from a shift to discriminatory pricing results in a higher consumer surplus.

3. The Demand Side: Shifting and Accessing Customers

In a symmetric-cost context, researchers have considered actions related to the demand side. Here we consider three demand-related extensions but within our asymmetric-marginal-cost framework.

[30] For an in-depth duopoly treatment of price discrimination with captive customers, see Armstrong and Vickers (2019).

Acquiring Captive Customers. Asymmetric technologies emerge because the expected profit of the most aggressive firm reacts in a distinctive way to a change in its costs. Chioveanu (2008) found a related effect when costly "persuasive" advertising determines captivity, or "brand loyalty."[31]

Her insights hold when costs are asymmetric. From eq. (13), if a firm i is not the most aggressive then its profit, $\pi_i = \lambda_i V_i(z_i) - z_i$, does not depend on the characteristics of competitors, and the incentive to acquire new captives corresponds exactly to the per-customer surplus that such a captive customer generates. In contrast, if a firm i is the most aggressive, then its expected profit is

$$\pi_i = (\lambda_i + \lambda_S)V_i(z_i) - \frac{\lambda_S(\lambda_i + \lambda_S)V_k(z_k)}{\lambda_k + \lambda_S} - z_i \quad \text{where} \quad k = \arg\max_{j \neq i} \left\{ \frac{V_i(z_j)}{\lambda_j + \lambda_S} \right\}, \tag{25}$$

The first term (just as before) provides an incentive to acquire captive customers equal to the surplus $V_i(z_i)$. However, the second (negative) term is decreasing in λ_i and increasing in λ_k: it encourages the most aggressive firm to push captives to its nearest competitor. These effects were identified by Chioveanu (2008), and mean that a firm's profit as a function of its captive audience has a convex kink at the point where it becomes the most aggressive firm.[32] If we add a formal stage at which firms take actions to acquire captives, then the presence of this kink means that in equilibrium the most aggressive firm will be distinct. This feature is shared with our endogenous-innovation model. A difference between cost-reducing innovations and captive-audience acquisition of is that the distinct firm (the aggressive firm that expects to serve shoppers) faces a stronger incentive to innovate but a weaker incentive to recruit captives. However, the underlying force is the same: that firm faces a stronger incentive to become more aggressive. That aggression is achieved by over-investment in one case, and under-investment in the other.

Firms might also make longer-term decisions to jointly influence captive audiences and costs. To illustrate this idea, suppose that each symmetric firm has a budget, $\bar{z}$, to spend on process innovations via a production function $V(z_i)$ or on captive acquisition so that $\lambda_i = \Lambda(\bar{z} - z_i)$ where $\Lambda(\cdot)$ has natural properties. If firms simultaneously choose z_i at a pre-pricing stage then there are n equilibria in which $n - 1$ firms allocate the same budget, $z^\star$, to cost-reducing innovation,

[31]A similar finding, but in a setting with a comparison site that advertises alongside sellers for its captive base can be found in an earlier version of Ronayne and Taylor (2022): Ronayne and Taylor (2020, Appendix W.3).

[32]The findings of Chioveanu (2008) also resonate with Ireland (1993) and McAfee (1994), which considered firms that send ads that inform customers randomly and independently à la Butters (1977) and Grossman and Shapiro (1984) before choosing prices. There, one firm advertises distinctly more than others, even if they are ex ante symmetric.

where this allocation satisfies $V'(z^\star)\Lambda(\bar{z} - z^\star) = V(z^\star)\Lambda'(\bar{z} - z^\star)$. However, the remaining (and endogenously most aggressive) firm devotes more to cost reduction by diverting resources away from acquiring captive customers; it pursues heightened aggression in two dimensions.

Captive Customers who Switch Firms. The forces pushing toward asymmetry come from the decisions of firms. This contrasts with the extended model of Baye, Kovenock and De Vries (1992, Section V) in which captive customers (interpreted as "uninformed" about the identity of the lowest-price firm, but have correct expectations) switch between firms. If marginal costs are symmetric then the firm with the fewest captives has the lowest expected price, and so captives seeking low prices move toward the smallest firm, which (under a reasonable game form) can result in equal captive-audience sizes. This underpinned an argument from Baye, Kovenock and De Vries (1992, Theorem 3) to focus on the symmetric play of a symmetric game.

If firms endogenously choose their technologies then asymmetry (rather than symmetry) can be reinforced by the endogenous decisions of captive customers to switch suppliers: the argument can be reversed. If the opportunity for innovation is strong then there is an equilibrium in which the firm with the most captives is (endogenously) the most aggressive, and becomes the most frequent supplier of shoppers. If we were to extend our innovation and pricing stages to allow captive customers to choose their captors then an equilibrium outcome (depending on the exact specification) can involve a single dominant firm with a large captive audience.

Clearinghouses. One interpretation of captive customers is that they are the local locked-in clientele of a firm, whereas (remotely located) shoppers compare all prices via a "clearinghouse" that acts as fee-levying "information gatekeeper" (Baye and Morgan, 2001) for firms' prices.[33] We take this up here, supposing that firms must pay a fee to advertise their prices via the clearinghouse. Shoppers buy from the cheapest advertising supplier, but have no other access to the firms' prices.

Specifically, firm i can earn $\lambda_i(v - c_i)$ from restricting to its captive customers. Otherwise, it pays $a_i > 0$ to a clearinghouse to advertise a price $p_i \in [0, v]$. Shoppers buy from a firm advertising the lowest price; if no firm advertises via the clearinghouse then they do not buy at all.

[33]Such informative price advertising is typically modeled as a channel to reach shoppers. This contrasts with the persuasive advertising we studied in our first demand-side application, which typically induces brand loyalty (captivity). Baye and Morgan (2009) bring these ideas together in their study of firms that engage in both forms of advertising.

We write $F_i(p)$ for the distribution of prices chosen by firm i across $p \in [0, v]$ so that $1 - F_i(v)$ is the probability that firm i does not join the clearinghouse. Clearly, if $a_i > \lambda_S(v - c_i)$ then firm i will never advertise, and so we suppose that $a_i \leq \lambda_S(v - c_i)$ for all i.

We can define for firm i the minimum undominated (advertised) price, $p_i^\dagger$, satisfying

$$(p_i^\dagger - c_i)(\lambda_i + \lambda_S) = \lambda_i(v - c_i) + a_i \quad \Leftrightarrow \quad p_i^\dagger = \frac{\lambda_i v + \lambda_S c_i + a_i}{\lambda_i + \lambda_S}. \tag{26}$$

For this extension, we strictly rank the most aggressive firms so that $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$. In Section 1 we discussed a more general notion of aggression via the "minimum win probability" (the required probability of selling to shoppers) for firm i contemplating charging a price p instead of simply exploiting captive customers. That quantity is central to our algorithm that solves for equilibrium (detailed in Appendix B). Here that expression, from eq. (3), becomes

$$\underline{w}_i(p) = \frac{\lambda_i(v - p) + a_i}{\lambda_S(p - c_i)}, \tag{27}$$

where (as before, and by construction) these functions satisfy $\underline{w}_i(p_i^\dagger) = 1$ and $\underline{w}_i(v) > 0$ for all firms. These minimum win probability functions can readily cross. The extra degree of freedom provided by the clearinghouse cost parameter a_i makes it easier to construct examples of this.

In the presence of a clearinghouse, key properties of an equilibrium are maintained. Notably, the profit prediction of Proposition 1 (which itself can be attributed to Siegel, 2009) is maintained.

Proposition 7 (Pricing via a Clearinghouse). *In an equilibrium with a clearinghouse the joint support of firms' pricing strategies is $[p_{n-1}^\dagger, v]$. Suppose that $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$.*

(i) Every firm $i \in \{1, \ldots, n-1\}$ earns its captive-only profit. With positive probability it chooses not to use the clearinghouse and charges v to its captive-only customers. Any such firm that does use the clearinghouse advertises prices that are strictly below v.

(ii) Firm n earns $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ more than its captive-only profit. This (most aggressive) firm always joins the clearinghouse, and advertises the price v with strictly positive probability.

It remains for us to sketch the steps that can be used to construct a (generically, unique) equilibrium. We set the required win probability for each firm $i \in \{1, \ldots, n-1\}$ to equal its minimum win probability: $w_i(p) = \underline{w}_i(p)$. We then set a different required win probability for firm n to reflect its additional expected profit of $\Delta_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. These win probabilities satisfy $w_n(p^\dagger_{n-1}) = w_{n-1}(p^\dagger_{n-1}) < w_i(p^\dagger_{n-1})$ for $i \in \{1, \ldots, n-2\}$.

We now follow a procedure that we used for our main model (in Section 1) and which is described in full detail in Appendix B: firms n and $n-1$ continuously mix from $p^\dagger_{n-1}$ upwards using distribution functions $1 - F_n(p) = w_{n-1}(p)$ and $1 - F_{n-1}(p) = w_n(p)$, so that the distribution $F_S(p)$ of the cheapest price satisfies $1 - F_S(p) = w_n(p)w_{n-1}(p)$. One possibility (and a leading case of interest) is that this solution satisfies $w_n(p)w_{n-1}(p) < w_i(p)$ for all $i \notin \{n-1, n\}$ and $p \in [p^\dagger_{n-1}, v)$. If so, then firms $n-1$ and n mix over the whole interval: this is a (classic) de facto duopoly. Firm n then places remaining mass at v, while firm $n-1$ places remaining mass on the act of not advertising. All other firms refrain from joining the clearinghouse. This is an equilibrium.

The other possibility (just as in Appendix B, and as we showed by our "thrango" example in Section 1) is that there is some price, $p^\ddagger < v$, at which some other firm j satisfies $w_j(p^\ddagger) = w_n(p^\ddagger)w_{n-1}(p^\ddagger)$. If so, then we execute a "partner swap" at this price, just as we have done in our earlier constructions. It is straightforward to confirm (again, as we do in Appendix B) that there are situations in which such partner-swapping necessarily occurs, and the equilibria in such circumstances involve active mixing (in this case, this means the active use of the clearinghouse) by more than two firms. Generically, we only see two firms dancing within any interval of prices.

4. Concluding remarks

Our general treatment of models of sales also allows us to bring together existing insights in the literature, shed light on which generalize and obtain new insights. The classic model predicts that sale prices will be offered by a de facto duopoly consisting of the two smallest (captive audience) firms. Our work shows how this picture now changes in three ways. Firstly, those using sales may be larger or established firms with lower marginal costs. Secondly, if this is so then a "thrango" of three or more (even all) firms can use sales; although we offer a sufficient condition (satisfied if the two most aggressive firms have the smallest captive audiences) for a traditional "tango" to be danced. Thirdly, the pattern of sales prices (for almost all parameter choices) consists of multiple distinct pairwise battles. A single firm (the most "aggressive" in our language) mixes with a succession of partners, one at a time, i.e., over a distinct sub-interval of its support, for each partner. This predicts marked dispersion when many firms engage in randomized sales: those firms' prices can be starkly spread out because each mixes over a distinct range of prices. Those not engaging in sales sit back and charge a high "regular" or monopoly price.

We also showed that asymmetric costs endogenously arise (even if firms are ex ante symmetric). The endogenous supply-side asymmetry resonates with papers that find endogenously asymmetric demand structures. A common feature to all this work is that there is one firm with distinct incentives. We predicted a uniquely most aggressive firm emerges. With strong technological opportunities, we revealed that firms with more captive customers innovate more, obtain lower marginal costs and set lower prices on average. In terms of pricing patterns, an inverse relation between captive base sizes and marginal costs is in fact that most conducive to (a necessary condition for) multiple firms to engage in randomized sales in equilibrium of a model of sales pricing game.

Appendix A. Core Omitted Proofs

Proof of Proposition 1. Theorem 5 of Dasgupta and Maskin (1986, p.14) applies. Specifically, that theorem asks for the sum of players' payoffs (the total industry profit here) to be upper semi-continuous in actions (here, the profile of prices) and this holds if ties are broken in favor of lower-cost firms. (From our work later on, the equilibria are the same for any tie-break rule.)

Turning to payoffs, if $p^{\dagger}_{n} < p^{\dagger}_{n-1} < p^{\dagger}_{n-2}$ then claim (v) of Lemma B1 in Appendix B shows that any Nash equilibrium gives the payoffs specified in the proposition. In Appendix B we provide an algorithm to construct an equilibrium with the required payoffs for any parameter values. □

Proof of Proposition 2. Standard properties of any equilibrium (see Lemma B1 in Appendix B) are that all firms mix continuously up to v with any atoms at v. Firm n earns more than its captive-only profit, requiring others to play atoms at v while firm n does not. This completes claim (i).

If the lower bound of all prices were to strictly exceed $p^{\dagger}_{n-1}$ then (at least) firms n and $n-1$ could (by pricing just above $p^{\dagger}_{n-1}$) sell to all shoppers and strictly exceed their equilibrium profit. We conclude that the joint support of firms' mixed strategies extends down to $\min_i \underline{p}_i = p^{\dagger}_{n-1}$. Prices below $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ are strictly dominated for firms $i \in \{1,\ldots,n-2\}$, and so (given the absence of gaps; Lemma B1) firms $n-1$ and n must mix continuously over $[p_{n-1}^\dagger, \min_{j\in\{1,\ldots,n-2\}} p_j^\dagger)$. Given that they both price below $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ with strictly positive probability, a price at or just above $\min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ will not be chosen by any firm $i \in \{1,\ldots,n-2\}$, and so there is some $p^\ddagger > \min_{j\in\{1,\ldots,n-2\}} p_j^\dagger$ such that firms $n-1$ and n mix on the interval $[p_{n-1}^\dagger, p^\ddagger)$.

The expected profit earned by firm $n-1$ from charging a price $p \in [p_{n-1}^\dagger, p^\ddagger)$ is

$$\pi_{n-1}(p) = (p - c_{n-1})\left(\lambda_{n-1} + \lambda_S\left(1 - F_n(p)\right)\right) = \lambda_{n-1}(v - c_{n-1}), \tag{A1}$$

where the final term is its captive-only profit. The expected profit of firm n from $p \in [p_{n-1}^\dagger, p^\ddagger)$ is

$$\pi_n(p) = (p - c_n)\left(\lambda_n + \lambda_S\left(1 - F_{n-1}(p)\right)\right) = \lambda_n(v - c_n) + (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger), \tag{A2}$$

where the final expression is the profit of firm n from (v) of Lemma B1. These equations solve:

$$F_n(v) = 1 - \frac{\lambda_{n-1}(v-p)}{\lambda_S(p - c_{n-1})} \quad \text{and} \quad F_{n-1}(v) = 1 - \frac{\lambda_n(v-p)}{\lambda_S(p-c_n)} - \frac{(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)}{\lambda_S(p - c_n)}. \tag{A3}$$

These are valid distribution functions that strictly and continuously increase from $F_{n-1}(p_{n-1}^\dagger) = F_{n-1}(p_{n-1}^\dagger) = 0$, and they can be rewritten to obtain eq. (5), and completing claim (ii).

For claim (iii), we use the properties $c_n \le c_{n-1}$ and $p_n^\dagger < p_{n-1}^\dagger$. From rearrangement, $\lambda_i + \lambda_S = \lambda_S(v-c_i)/(v-p_i^\dagger)$ and so substituting for $\lambda_{n-1}+\lambda_S$ and $\lambda_n+\lambda_S$ in $F_n(p)$ and $F_{n-1}(p)$ respectively,

$$F_{n-1}(p) < F_n(p) \quad \Leftrightarrow \quad \frac{v - c_n}{(v - p_n^\dagger)(p - c_n)} < \frac{v - c_{n-1}}{(v - p_{n-1}^\dagger)(p - c_{n-1})}, \tag{A4}$$

which holds if $p_n^\dagger < p_{n-1}^\dagger$ and (given that $v \ge p$) $c_n \le c_{n-1}$.

For claim (iv), any equilibrium involves mixing by $n-1$ and n up to some $p^\ddagger$. One possibility is that $p^\ddagger = v$, so that all other firms choose $p_i = v$. By inspection, $\lim_{p\uparrow v} F_n(p) = 1$ and $\lim_{p\uparrow v} F_{n-1}(p) \le 1$ (the latter inequality is strict if $p_{n-1}^\dagger > p_n^\dagger$) which means that we have valid distributions. Firms $n-1$ and n cannot improve by deviating. If there is an equilibrium in which only these two firms mix then (by construction) it is unique. We must check to see if some other firm i might wish to deviate to $p_i < v$. Firm $i \in \{1,\ldots,n-2\}$ earns $\lambda_i(v - c_i)$. By deviating to

$p \in [p^\dagger_{n-1}, v)$, and assuming that $\lambda_{n-1} \le \lambda_i$ (the condition in the proposition) it earns

$$\pi_i(p) = (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_{n-1}(p)\right)\left(1 - F_n(p)\right)\right) \tag{A5}$$

$$< (p - c_i)\left(\lambda_i + \lambda_S\left(1 - F_n(p)\right)\right) \tag{A6}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_{n-1}(v-p)}{(p - c_{n-1})}\right) \tag{A7}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_{n-1}(v-p)}{\lambda_{n-1}(v - p^\dagger_{n-1}) + \lambda_S(p - p^\dagger_{n-1})}\right) \tag{A8}$$

$$\le (p - c_i)\left(\lambda_i + \frac{\lambda_S \lambda_i (v-p)}{\lambda_i(v - p^\dagger_i) + \lambda_S(p - p^\dagger_i)}\right) \tag{A9}$$

$$= (p - c_i)\left(\lambda_i + \frac{\lambda_i(v-p)}{(p - c_i)}\right) = \lambda_i(v - c_i). \tag{A10}$$

The third line is from substitution of $F_n(p)$. The fourth line is obtained by writing c_{n-1} in terms of λ_{n-1} and $p^\dagger_{n-1}$ and rearranging. The fifth line holds because $\lambda_{n-1} \le \lambda_i$ and $p^\dagger_{n-1} \le p^\dagger_i$. The final line is from substituting $p^\dagger_i$ and rearranging. Firm i performs strictly worse by deviating.

We have a (unique, within this class) "tango" equilibrium. Any other equilibrium requires another firm "step onto the dance floor" at some $p^\ddagger < v$. The argument above demonstrates (given the lack of atoms below v, and continuity properties) that this would be strictly suboptimal. □

Proof of Proposition 3. This follows from the preceding argument in the main text. To see a specific example of this, consider a "two to tango" equilibrium in which (for simplicity of exposition) firms n and $n-1$ satisfy $c_n = c_{n-1} = 0$ and $\lambda_n = \lambda_{n-1} = \lambda$. This means that

$$p^\dagger_n = p^\dagger_{n-1} = \frac{\lambda v}{\lambda + \lambda_S} \quad \text{and} \quad F_n(p) = F_{n-1}(p) = \frac{(p - p^\dagger_{n-1})(\lambda + \lambda_S)}{p\lambda_S} = 1 - \frac{(v-p)\lambda}{p\lambda_S}. \tag{A11}$$

The condition required for this to be an equilibrium is that there is no price at which another lower-indexed firm wishes to join. Equation (7) from the main text here requires

$$(v-p)\lambda_i \ge (p - c_i)\lambda_S(1 - F_n(p))(1 - F_{n-1}(p)) = (p - c_i)\lambda_S\left(\frac{(v-p)\lambda}{p\lambda_S}\right)^2$$

$$\Leftrightarrow \quad \lambda_i \ge \frac{(p - c_i)(v-p)\lambda^2}{p^2\lambda_S} \quad \forall\, p \in \left(\frac{\lambda v}{\lambda + \lambda_S}, v\right). \tag{A12}$$

By inspection, if $p > c_i$ then this fails if λ_i is small. Pushing further, suppose that this holds for all $i \in \{1, \ldots, n-3\}$, but let us choose λ_{n-2} so that this fails for some p. For firm $n-2$ set

$$c_{n-2} = p^{\dagger}_{n-1} = \frac{\lambda v}{\lambda + \lambda_S}, \tag{A13}$$

which guarantees that $p^{\dagger}_{n-2} > p^{\dagger}_{n-1}$ for any $\lambda_{n-2} > 0$, no matter how small. Firm $n-2$ will wish to step on to the dance floor at the lowest price $p^{\ddagger}$ which satisfies

$$\lambda_{n-2} = \frac{(p^{\ddagger} - c_{n-2})(v - p^{\ddagger})\lambda^2}{(p^{\ddagger})^2 \lambda_S} = \frac{\big(p^{\ddagger}(\lambda + \lambda_S) - \lambda v\big)\,(v - p^{\ddagger})\lambda^2}{(p^{\ddagger})^2 \lambda_S (\lambda + \lambda_S)}. \tag{A14}$$

Explicitly, this is the lower solution to

$$(\lambda + \lambda_S)\left(1 + \frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)(p^{\ddagger})^2 - (2\lambda + \lambda_S) v p^{\ddagger} + \lambda v^2 = 0$$

$$\Rightarrow \quad p^{\ddagger} = \frac{v(2\lambda + \lambda_S) - v\sqrt{\lambda_S^2 - \frac{4\lambda_{n-2}\lambda_S(\lambda+\lambda_S)}{\lambda}}}{2(\lambda + \lambda_S)\left(1 + \frac{\lambda_{n-2}\lambda_S}{\lambda^2}\right)}, \tag{A15}$$

where this solution satisfies $p^{\ddagger} \downarrow \lambda v/(\lambda + \lambda_S)$ as $\lambda_{n-2} \downarrow 0$.

The argument given is that there can be parameters under which a third firm must participate. In Appendix B we provide a characterization of equilibria. The same approach taken here applies: if there are k firms that engaged in randomized sales then we can add an additional firm with relatively high marginal cost, but few captives, that wishes to participate in sales. □

Proof of Proposition 4. Claim (i) holds because firms $i \in \{1, \ldots, n-2\}$ play $p_i = v$ as pure strategies, and their costs do not enter the solutions reported in eq. (5). For claim (ii), $F_{n-1}(p)$ is decreasing in $p^{\dagger}_{n-1}$ which itself is increasing in c_{n-1}. For $F_n(p)$,

$$\frac{\partial F_n(p)}{\partial c_{n-1}} = \frac{\lambda_{n-1} + \lambda_S}{\lambda_S}\left(\frac{(p - p^{\dagger}_{n-1})}{(p - c_{n-1})^2} - \frac{1}{p - c_{n-1}}\frac{\partial p^{\dagger}_{n-1}}{\partial c_{n-1}}\right) \tag{A16}$$

$$= \frac{\lambda_{n-1} + \lambda_S}{\lambda_S(p - c_{n-1})}\left(\frac{(p - p^{\dagger}_{n-1})}{(p - c_{n-1})} - \frac{\lambda_S}{\lambda_{n-1} + \lambda_S}\right) = -\frac{(v - p)\lambda_{n-1}}{\lambda_S(p - c_{n-1})^2} < 0 \tag{A17}$$

The CDFs are both decreasing in c_{n-1}, which means an increase in c_{n-1} pushes rightward the distributions of prices. Claim (iii) follows an inspection of $F_{n-1}(p)$. □

Proof of Proposition 5. We begin by re-writing the profit of firm i from eq. (13) as

$$\pi_i = \lambda_i V_i(z_i) + \lambda_S \max\left\{0, V_i(z_i) - (\lambda_i + \lambda_S) \max_{j \neq i} \left\{ \frac{V_j(z_j)}{\lambda_j + \lambda_S} \right\} \right\} - z_i. \tag{A18}$$

As noted in the text, this is maximized by either high or low innovation choices $z_i \in \{z_i^L, z_i^H\}$ which satisfy the first-order conditions from eq. (17) and where $z_i^H > z_i^L$. In essence, the innovation game is a binary-action game where each firm chooses either high or low innovation.

Some firms will always choose low innovation. For example, any firm i where

$$\frac{V_i(z_i^H)}{\lambda_i + \lambda_S} \leq \max_{j \neq i} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \tag{A19}$$

will not choose z_i^H because it would not be the most aggressive firm (even if every other firm $j \neq i$ chooses low innovation) and so it would earn $\lambda_i V_i(z_i^H) - z_i^H < \lambda_i V_i(z_i^L) - z_i^L$. Hence we restrict to firms that do not satisfy eq. (A19), and then further restrict attention to those firms who would choose high innovation if all of their competitors choose low innovation. These firms satisfy

$$\lambda_i V_i(z_i^H) + \lambda_S \max\left\{0, V_i(z_i^H) - (\lambda_i + \lambda_S) \max_{j \neq i} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \right\} - z_i^H \geq$$
$$\lambda_i V_i(z_i^L) + \lambda_S \max\left\{0, V_i(z_i^L) - (\lambda_i + \lambda_S) \max_{j \neq i} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \right\} - z_i^L. \tag{A20}$$

For firms where the inequality of eq. (A19) fails (these are firms that are able to take the most aggressive position by choosing z_i^H while $j \neq i$ choose z_j^L), the inequality eq. (A20) is

$$(\lambda_i + \lambda_S) \left[V_i(z_i^H) - \max_{j \neq i} \left\{ \frac{\lambda_S V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \right] - z_i^H \geq$$
$$\lambda_i V_i(z_i^L) + \lambda_S \max\left\{0, V_i(z_i^L) - (\lambda_i + \lambda_S) \max_{j \neq i} \left\{ \frac{V_j(z_j^L)}{\lambda_j + \lambda_S} \right\} \right\} - z_i^L. \tag{A21}$$

Consider the set of firms for which eq. (A20) holds. This set is non-empty. To see why, consider a firm $i \in \arg\max_{j \in \{1,\ldots,n\}} V_j(z_j^L)/(\lambda_j + \lambda_S)$. (This is a firm that would be the most aggressive (or jointly most aggressive) firm if all firms chose z_j^L.) For this firm eq. (A20) reduces to

$$(\lambda_i + \lambda_S) V_i(z_i^H) - z_i^H \geq (\lambda_i + \lambda_S) V_i(z_i^L) + \lambda_S V_i(z_i^L) - z_i^L, \tag{A22}$$

which holds strictly because z_i^H is the unique maximizer of $(\lambda_i + \lambda_S)V_i(z_i) - z_i$. Amongst the non-empty set of firms that satisfy eq. (A20), find a firm that maximizes $V_i(z_i^H)/(\lambda_i + \lambda_S)$.

We now label this firm as firm n. There is a pure-strategy Nash equilibrium of the innovation game in which $z_n = z_n^H$ and $z_j = z_j^L$ for $j \in \{1, \ldots, n-1\}$. Firm n satisfies the inequality of eq. (A20) and so does indeed wish to choose $z_n = z_n^H$. Any other firm $i \neq n$ that satisfies eq. (A20) also satisfies $V_i(z_i^H)/(\lambda_i + \lambda_H) \leq V_n(z_n^H)/(\lambda_n + \lambda_H)$, and so cannot achieve the position of the most aggressive firm by deviating to $z_i = z_i^H$. This proves claim (i).

Claim (ii) follows from the argument after eq. (16): the profit of firm i has an upward kink as the firm ties to be the most aggressive, and so cannot be the optimal choice of z_i.

For claim (iii), we note that $\lim_{\lambda_S \downarrow 0} z_i^H = z_i^L$. Suppose that $n = \arg\max_{j \in \{1,\ldots,n\}} V_j(z_j^L)/\lambda_j$. Then for $i \neq n$ and λ_S sufficiently small we can guarantee that $V_i(z_i^H)/(\lambda_i + \lambda_S) < V_n(z_n^L)/\lambda_n + \lambda_S)$, and so firm n must always be the most aggressive firm.

Claim (iv) is straightforward: any firm can be firm n when they are symmetric. □

Proof of Proposition 6. First suppose technological opportunities are strong. It remains to show which firm becomes most aggressive. Equations (21) and (22) (and the surrounding text) explain that when there are sufficiently few shoppers, only n can be the most aggressive. The order of the $z_i^\star$ terms follows because firms are indexed in order of how many consumers they serve in total in equilibrium. The rankings of $c_i^\star$ and $p_i^\dagger$ terms follow automatically. This proves claim (i).

Now suppose technological opportunities are weak. The key difference from strong opportunities is that $V(z_i^L)/\lambda_i$ is decreasing in λ_i. The relevant version of eq. (22) is that for firms $i > 1$,

$$\lim_{\lambda_S \downarrow 0} \frac{V(z_i^H)}{\lambda_i + \lambda_S} = \frac{V(z_i^L)}{\lambda_i} < \frac{V(z_1^L)}{\lambda_1} = \lim_{\lambda_S \downarrow 0} \frac{V(z_1^L)}{\lambda_1 + \lambda_S}, \tag{A23}$$

meaning that eq. (21) fails for $i > 1$ when λ_S is sufficiently small so that only firm 1 can take the most aggressive role. Firms $i > 1$ are left serving their captives only, and so their investments (and costs) are ordered by their index. We cannot yet determine the position firm 1 takes in that ranking (the fewer total customers it serves, the less it will invest). If λ_S is sufficiently small, then a firm will serve more customers than another in total if it has more captives. When that is the case, the order of innovation intensities will follow the ranking of the size of captive audiences, such that for any two firms i and j, $\lambda_i < \lambda_j \Leftrightarrow z_i^\star < z_j^\star$. The rankings of $c_i^\star$ and $p_i^\dagger$ terms follow automatically. Because $p_1^\dagger < p_2^\dagger < p_3^\dagger$ and $\lambda_2 < \min_{i\in\{3,\dots,n\}}\{\lambda_i\}$, claim (iv) of Proposition 2 holds and firms 1 and 2 price in mixed strategies. This proves claim (ii). □

Proof of Proposition 7. The properties of any equilibrium with a clearinghouse are reported in Lemma B2 of Appendix B. Firm n guarantees $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ more than its captive-only profit by choosing $p_n = p_{n-1}^\dagger$. From claim (v) of Lemma B2, other firms earn captive-only profits. If n were to earn strictly more, then the lower bound of its support would exceed $p_{n-1}^\dagger$. This would give $n-1$ an opportunity to earn strictly more than its own captive-only profit; a contradiction. □

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APPENDIX B. FURTHER PROOFS AND EXTENDED RESULTS

In this appendix, we report the basic properties of equilibria; fully construct equilibria; illustrate pathological equilibria in knife-edge cases; document previously the properties of an equilibrium with a clearinghouse; and, finally, consider the simultaneous choices of prices and technologies.

B.1. **Equilibrium Properties of Simultaneous Pricing.** We now document several (relatively standard) properties that must hold for any (Nash) equilibrium. We write $F_i(p)$ for the mixed strategy of firm i. As usual, by an atom we mean a price at which $F_i(p)$ discontinuously increases. We write $\underline{p}_i$ for the infimum of the support of prices played by firm i in equilibrium.

Lemma B1 (Equilibrium Properties). *An equilibrium of a model of sales has these properties.*

(i) There are no atoms strictly below v and at most $n-1$ atoms at v.

(ii) The upper bound of the support of prices for all firms is v.

(iii) There is no gap in the joint support of firms' strategies. Relatedly, if any interval of prices is in the support for some firm i then it is in the support for some other firm $j \neq i$.

(iv) At least $n-1$ of the firms earn their captive-only profit.

(v) Profits satisfy: (a) firm n earns at least $(\lambda_n+\lambda_S)(p^\dagger_{n-1}-c_n) \geq \lambda_n(v-c_n)$ with strict inequality if $p^\dagger_n < p^\dagger_{n-1}$; (b) if $p^\dagger_n < p^\dagger_{n-1}$, then firm $i<n$ earns its captive-only profit and places an atom at v; and (c) if $p^\dagger_{n-1} < p^\dagger_{n-2}$, then firm n earns exactly $(\lambda_n+\lambda_S)(p^\dagger_{n-1}-c_n)$.

Proof. Claims (i)–(iii) are relatively standard for games of this kind, and so the proofs are omitted. For completeness, they are retained in Appendix C of our working paper.

Claim (iv). All firms earn weakly more than their captive-only profit. Suppose that two or more firms earn strictly more. If all place an atom at v, the necessarily all have strictly positive probability of winning the shoppers at this price. There would be a strict incentive for at least one firm to undercut. We conclude that at least one firm j that earns more strictly more than its captive-only profit does play an atom at v, but is willing price arbitrarily close v.

Consider another firm $i \neq j$. If that firm plays an atom at v, then that atom will lose to the firm j so will sell only to captives, and so will earn its captive-only profit. If instead firm i mixes up to (but do not place an atom at) v then it is willing to price arbitrarily close to v and doing so almost always loses all shopper sales. Once again it must earn its captive-only profit.

Claim (v). Part (a) follows from an argument in the main text. Turning to part (b), by claim (iv) $n-1$ firms earn captive-only profits. Given that n earns strictly more, this must apply to all $i \in \{1, \ldots, n-1\}$. For n to earn strictly more than captive-only profits and yet be willing to price arbitrarily close to v, it must win the shoppers with probability bounded away from zero, which implies that each $i < n$ places an atom at v. Finally, for part (c), if firm n were to earn strictly more than $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$, then $\underline{p}_n > p^\dagger_{n-1}$. Firm $n-1$ could then set a price $p_{n-1} \in (p^\dagger_{n-1}, p^\dagger_{n-2})$ which would capture shoppers with certainty and earn strictly more than its captive-only profit; a contradiction of claim (iv). □

We note here the properties of profits in claim (v). Firm n earning $(\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n)$ is the profit level implied by the characterization of equilibrium strategies we give below, for any model parameters. That characterization also gives a unique equilibrium profile for generic parameter values. This leaves open special cases with $p^\dagger_n = p^\dagger_{n-1}$ or $p^\dagger_{n-1} = p^\dagger_{n-2}$ or both. A continuum of other equilibria can exist, in which exactly one of $\{n, n-1, n-2\}$ does strictly better than the profits given by Proposition 1. These "pathological" payoffs are mentioned in the main text, and then set aside. In this appendix we also describe such (pathological) equilibria.

B.2. **Equilibrium Construction.** A construction here delivers equilibria with non-pathological profits (as in Proposition 1) for any parameters. We use the following notation and terminology.

Definition (Required and Minimum Win Probabilities). *The* required win probability $w_i(p)$ *is the probability with which firm i must win the business of shoppers for it to earn its equilibrium profit from the price $p \in (0, v)$. Relatedly, the* minimum win probability $\underline{w}_i(p)$ *is the probability that gives the firm its captive-only profit $\lambda_i(v - c_i)$ from charging the price p.*

In this construction we look only for non-pathological payoffs. Each firm $i < n$ earns its captive-only profit and so, $w_i(p) = \underline{w}_i(p)$. For the remaining firm, $w_n(p) \geq \underline{w}_n(p)$. We will express equilibrium mixtures in terms of required win probabilities: if p is in the support of firm i then it must sell to the shoppers with probability $w_i(p)$. If p is not in the support, then it captures that business with probability weakly less than $w_i(p)$. Recall that $\underline{w}_i(p)$ satisfies

$$\lambda_i(v - c_i) = (p - c_i)\left(\lambda_i + \lambda_S \underline{w}_i(p)\right) \quad \Rightarrow \quad \underline{w}_i(p) = \frac{\lambda_i(v-p)}{\lambda_S(p-c_i)}. \tag{B1}$$

This is decreasing in p and satisfies $\underline{w}_i(v) = 0$.

We pause briefly to relate minimum win probabilities to the ordering of firms by aggressiveness. A firm's lowest undominated price $p_i^\dagger$ satisfies $\underline{w}_i(p_i^\dagger) = 1$, which solves to give eq. (1) from the main text. We defined firm i to be (strictly) more aggressive than firm j if $p_i^\dagger < p_j^\dagger$, so firm i is willing to choose a lower price to capture shoppers. Equivalently, this holds if

$$\frac{v - c_i}{v - c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}. \tag{B2}$$

By construction $\underline{w}_j(p) > 1 \geq \underline{w}_i(p)$ for $p \in [p_i^\dagger, p_j^\dagger)$. However, a stronger aggression ranking entails $\underline{w}_j(p) > \underline{w}_i(p)$ for all $p \in [p_j^\dagger, v)$. This (partial ordering of firms) requires

$$\frac{\lambda_i}{p - c_i} < \frac{\lambda_j}{p - c_j}, \tag{B3}$$

which holds for all relevant p if and only if

$$\frac{v - c_i}{v - c_j} > \max\left\{\frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \frac{\lambda_i}{\lambda_j}\right\}. \tag{B4}$$

(We can use this to derive a weaker condition than that stated in Proposition 2 to establish the uniqueness of a "two to tango" equilibrium.) If this does not hold so that

$$\frac{\lambda_i}{\lambda_j} \geq \frac{v - c_i}{v - c_j} > \frac{\lambda_i + \lambda_S}{\lambda_j + \lambda_S}, \tag{B5}$$

then i is more aggressive ($p_i^\dagger < p_j^\dagger$) and so is more willing to charge a lower price, but for sufficiently high prices firm j has a lower minimum win probability, which means that j is relatively more enthusiastic about offering a higher price. In this situation there is a unique price $p_{ij}^\dagger$ at which $\underline{w}_i(p_{ij}^\dagger) = \underline{w}_j(p_{ij}^\dagger)$, and as p rises through this point $\underline{w}_j(p)$ crosses $\underline{w}_i(p)$ from above to below. We will use this property (which corresponds to an effective change in "aggression" ranking as price increases) when we fully characterize an equilibrium below.

We now consider the required win probability for firm n. It can earn more than its captive-only profit. We have already discussed conditions under which it earns $\pi_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - c_n) = \lambda_n(v - c_n) + \Delta_n$ where $\Delta_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. Its required win probability is

$$w_n(p) = \frac{\lambda_n(v-p) + \Delta_n}{\lambda_S(p - c_n)} = \underline{w}_n(p) + \frac{\Delta_n}{\lambda_S(p-c_n)}. \tag{B6}$$

We now characterize the distributions used in firms' mixed strategies. In $F_i(p)$ for firm i is continuously increasing and satisfies $F_i(p) < 1$ for $p < v$ (from (i) and (ii) of Lemma B1). We write $I(p) \subseteq \{1, \ldots, n\}$ for the firms that are on the "dance floor" at price $p \in (p^\dagger_{n-1}, v)$. These are firms where $F_i(p)$ is strictly increasing at that price.[34] If a firm is on the dance floor then at that price its expected profit must equal its equilibrium profit, or equivalently its probability of winning the shoppers must equal its required win probability $w_i(p)$. It wins the shoppers only if all other firms $j \neq i$ price above it, which happens with probability $\prod_{j\neq i}(1 - F_j(p))$. That is,

$$w_i(p) = \prod\nolimits_{j\neq i}(1 - F_j(p)) \quad \Leftrightarrow \quad 1 - F_i(p) = \frac{1 - F_S(p)}{w_i(p)}$$
$$\text{where} \quad 1 - F_S(p) \equiv \prod\nolimits_{j=1}^{n}(1 - F_j(p)), \tag{B7}$$

where we obtained the second equality after multiplying and dividing by $1 - F_i(p)$, which we are able to do given that $F_i(p) < 1$ and so $1 - F_i(p) > 0$ for $p < v$. Here $F_S(p)$ is the distribution of the cheapest price and so the distribution of prices paid by the shoppers. Relatedly, if any new firm were to charge a price p then it would win the sales of shoppers with probability $1 - F_S(p)$.

We can substitute the expression for $F_i(p)$ back into the expression for $F_S(p)$, and obtain

$$1 - F_S(p) = \prod\nolimits_{j=1}^{n}(1 - F_j(p)) = \prod\nolimits_{i\in I(p)} \frac{1 - F_S(p)}{w_i(p)} \prod\nolimits_{j\notin I(p)}(1 - F_j(p))$$
$$\Leftrightarrow \quad 1 - F_S(p) = \left(\frac{\prod_{j\in I(p)} w_j(p)}{\prod_{j\notin I(p)}(1 - F_j(p))}\right)^{1/(|I(p)|-1)}$$
$$\Rightarrow \quad 1 - F_i(p) = \frac{1}{w_i(p)}\left(\frac{\prod_{j\in I(p)} w_j(p)}{\prod_{j\notin I(p)}(1 - F_j(p))}\right)^{1/(|I(p)|-1)}, \tag{B8}$$

where $|I(p)|$ is the number of firms actively mixing ("dancing") at p. The term $\prod_{j\notin I(p)}(1 - F_j(p))$ corresponds to firms who do not dance, and so is (locally) constant. For those firms $i \in I(p)$

[34] By which we mean that $F_i(p_L) < F_i(p_H)$ for all p_H and p_L satisfying $p_H > p > p_L$.

who dance, we need the solutions $F_i(p)$ to be valid distribution functions that are (given that firms actively mix) strictly increasing. The density $f_i(p)$ is

$$f_i(p) = (1 - F_i(p)) \left(\frac{w_i'(p)}{w_i(p)} - \frac{1}{|I(p)| - 1} \sum_{j \in I(p)} \frac{w_j'(p)}{w_j(p)} \right), \tag{B9}$$

and this is strictly positive for all $i \in I(p)$ (as required) if and only if

$$(|I(p)| - 1) \max_{i \in I(p)} \left\{ \frac{-w_i'(p)}{w_i(p)} \right\} < \sum_{j \in I(p)} \frac{-w_j'(p)}{w_j(p)}. \tag{B10}$$

If $|I(p)| = 2$ (so that there is a "tango" between two firms) then this is always satisfied. However, it can fail (and, as we show, it will fail for asymmetric firms) if $|I(p)| > 2$.

We now construct an equilibrium. Such an equilibrium partitions the full "dance floor" $[p_{n-1}^{\dagger}, v)$ into at most $n-1$ subintervals. In each subinterval one firm $i \in \{1, \ldots, n-1\}$ continuously mixes (or "dances") together with firm n, and then at the top of the subinterval firm i shifts all remaining mass to v and is replaced by a substitute firm j mixing in the next subinterval. Firm n mixes over the entire interval $[p_{n-1}^{\dagger}, v)$ but (in essence) swaps dance partners at various points so that the "two to tango" property (Baye, Kovenock and De Vries, 1992) holds within each subinterval, but more than two firms can participate in randomized sales overall.

Suppose that the two most aggressive firms are distinct: $p_n^{\dagger} < p_{n-1}^{\dagger} < p_{n-2}^{\dagger}$. We have found exact equilibrium profits in this case (claim (v) of Lemma B1) and firms n and $n-1$ must mix down to $p_{n-1}^{\dagger}$ (if $p_n^{\dagger} = p_{n-1}^{\dagger}$ or $p_{n-1}^{\dagger} = p_{n-2}^{\dagger}$ we can also proceed with the profits implied by claim (v) of Lemma B1). We set $I(p) = \{n-1, n\}$ and so $|I(p)| = 2$ for all $p \in [p_{n-1}^{\dagger}, p_{n-2}^{\dagger})$, and use the solutions for the mixing distributions reported in eq. (B8), which simplify to $F_n(p) = 1 - w_{n-1}(p)$ and $F_{n-1}(p) = 1 - w_n(p)$ and where $F_S(p) = 1 - w_{n-1}(p)w_n(p)$. We note that

$$1 - F_S(p) = w_{n-1}(p)w_n(p) < w_{n-1}(p) < 1 \leq w_i(p) \tag{B11}$$

for all $p \in (p_{n-1}^{\dagger}, p_{n-2}^{\dagger}]$ and all $i \in \{1, \ldots, n-2\}$. This means that no other firm wishes to "join the dance floor" at a price $p_{n-2}^{\dagger}$ and just above. Thus we continue to apply the solutions here as p increases through $p_{n-2}^{\dagger}$. One possibility is that $1 - F_S(p) < w_i(p)$ for all $p \in [p_{n-2}^{\dagger}, v)$ and all $i \in \{1, \ldots, n-2\}$. If so, then we have constructed a unique equilibrium in which firms $n-1$ and

n "tango" over $[p^\dagger_{n-1}, v)$ while all other firms strictly prefer to maintain $p_i = v$. We note that the solutions reported here satisfy $\lim_{p\uparrow v} F_n(p) = 1$.

The other possibility is that we reach a price at which $1 - F_S(p) = w_i(p)$ for some firm $i \in \{1, \dots, n-2\}$ so that firm i wishes to "step on to the dance floor" to join the tango. Without loss of generality, we label the firms so that it is firm $n-2$ that wishes to join the dance floor and we write $p^\ddagger_{n-2}$ for the (lowest) price at which this happens. For generic parameter choices, firm $n-2$ is uniquely defined and so our construction will be unique. If there is more than one firm that wishes to "join in" then we pick a firm for which $w_j(p)$ is falling most rapidly, so that $w'_{n-2}(p^\ddagger_{n-2}) \le w'_j(p^\ddagger_{n-2})$ for any other firm $j \in \{1, \dots, n-3\}$ where $w_j(p^\ddagger_{n-2}) = w_{n-2}(p^\ddagger_{n-2})$. There can be (non-generic) circumstances in which we have multiple choices available. One such situation is when two firms i and j are pairwise symmetric in the sense that $\lambda_i = \lambda_j$ and $c_i = c_j$, and in this circumstance our choice of firm that "steps in" is arbitrary; there are multiple equilibria in this case. For our chosen (generically unique) firm $n-2$,

$$w_{n-2}(p^\ddagger_{n-2}) = 1 - F_S(p^\ddagger_{n-2}) < w_{n-1}(p^\ddagger_{n-2}). \tag{B12}$$

This means that $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below within the interval $(p^\dagger_{n-1}, p^\ddagger_{n-2})$. Given that the minimum win probability functions can cross only once (as established earlier) we can conclude that $w_{n-1}(p) > w_{n-2}(p)$ for all $p \in (p^\ddagger_{n-2}, v)$. This means (as we will confirm) that once firm $n-2$ joins the dance floor, firm $n-1$ will strictly prefer to stay off it.

We continue the construction for prices above $p^\ddagger_{n-2}$. We set $F_{n-1}(p) = F_{n-1}(p^\ddagger_{n-2})$ for all $p \in [p^\ddagger_{n-2}, v)$ so that firm $n-1$ leaves the dance floor and places remaining mass at v. We then set $I(p) = \{n-2, n\}$ (and so we maintain $|I(p)| = 2$) for prices at and (at least locally) above $p^\ddagger_{n-2}$. These firms then mix according to eq. (B8) where these solutions satisfy

$$F_n(p) = 1 - \frac{w_{n-2}(p)}{w_n(p^\ddagger_{n-2})} \quad \text{and} \quad F_{n-2}(p) = 1 - \frac{w_n(p)}{w_n(p^\ddagger_{n-2})} \quad \Rightarrow$$

$$1 - F_S(p) = (1 - F_n(p))(1 - F_{n-1}(p^\ddagger_{n-2}))(1 - F_{n-2}(p)) = \frac{w_n(p) w_{n-2}(p)}{w_n(p^\ddagger_{n-2})}. \tag{B13}$$

We apply these solutions for prices rising above $p^\ddagger_{n-2}$ until a price (discussed below) at which we see another "partner swapping event." Before we do this, however, we perform two checks.

Firstly, we consider whether firm $n-2$ could join the dance floor to form a threesome rather than replacing firm $n-1$, so that $I(p) = \{n-2, n-1, n\}$ at $p^{\ddagger}_{n-2}$ and just above. Given that $|I(p)| = 3$, then inequality of eq. (B10) required for positive densities is

$$2 \min_{i \in \{n-2,n-1,n\}} \left\{ \frac{w_i'(p)}{w_i(p)} \right\} > \sum_{j \in \{n-2,n-1,n\}} \frac{w_j'(p)}{w_j(p)}. \tag{B14}$$

A necessary condition for this to hold is

$$\frac{w_{n-2}'(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} \geq \sum_{j \in \{n-1,n\}} \frac{w_j'(p^{\ddagger}_{n-2})}{w_j(p^{\ddagger}_{n-2})}. \tag{B15}$$

However, we know that $w_{n-2} > 1 - F_S(p) = w_n(p)w_{n-1}(p)$ for $p < p^{\ddagger}_{n-2}$ but with equality at $p = p^{\ddagger}_{n-2}$, and so $w_{n-2}(p) - w_n(p)w_{n-1}(p)$ is decreasing at $p^{\ddagger}_{n-2}$. That is,

$$w_{n-2}'(p^{\ddagger}_{n-2}) < w_n(p^{\ddagger}_{n-2})w_{n-1}'(p^{\ddagger}_{n-2}) + w_n'(p^{\ddagger}_{n-2})w_{n-1}(p^{\ddagger}_{n-2}). \tag{B16}$$

Dividing through by $w_{n-2}(p^{\ddagger}_{n-2}) = w_{n-1}(p^{\ddagger}_{n-2})w_n(p^{\ddagger}_{n-2})$, this inequality is

$$\frac{w_{n-2}'(p^{\ddagger}_{n-2})}{w_{n-2}(p^{\ddagger}_{n-2})} < \frac{w_{n-1}'(p^{\ddagger}_{n-2})}{w_{n-1}(p^{\ddagger}_{n-2})} + \frac{w_n'(p^{\ddagger}_{n-2})}{w_n(p^{\ddagger}_{n-2})}, \tag{B17}$$

a contradiction. This means that we cannot have firm $n-1$ remaining on the dance floor.

Secondly, we need to check that firm $n-1$ does not wish to return to the dance floor:

$$w_{n-1}(p) \geq (1 - F_n(p))(1 - F_{n-2}(p)) = \frac{w_{n-2}(p)w_n(p)}{(w_n(p^{\ddagger}_{n-2}))^2} \tag{B18}$$

This holds as an equality at $p^{\ddagger}_{n-2}$. It holds strictly for all higher p if, taking derivatives,

$$\frac{w_{n-1}'(p)}{w_{n-1}(p)} > \frac{w_{n-2}'(p)}{w_{n-2}(p)} + \frac{w_n'(p)}{w_n(p)}, \tag{B19}$$

and given that $w_n'(p) < 0$ a sufficient condition for this to hold is

$$\frac{w_{n-1}'(p)}{w_{n-1}(p)} \geq \frac{w_{n-2}'(p)}{w_{n-2}(p)} \quad \Leftrightarrow \quad c_{n-1} \leq c_{n-2}, \tag{B20}$$

which follows from differentiation of the expression for the minimum win probability. This holds because the left-hand inequality held at some price $p < p^{\ddagger}_{n-2}$. We know this because $w_{n-2}(p)$ crossed $w_{n-1}(p)$ from above to below as it descended at some point within $(p^{\dagger}_{n-1}, p^{\ddagger}_{n-2})$ and at the point where it crossed $w_{n-2}(p) = w_{n-1}(p)$. As an aside, this also tells us that a firm that joins the dance floor is always a firm with a higher marginal cost, and so (given that it then has a lower minimum win probability) a smaller captive audience.

Having completed these checks, we maintain our new solutions for the mixing of firms n and $n-2$. Just as before, a possibility is that $1-F_S(p) < w_i(p)$ for all $p \in [p^{\ddagger}_{n-2}, v)$ and all $i \in \{1, \dots, n-3\}$ and, if so, we have constructed an equilibrium. Otherwise, another firm $n-3$ wishes to step in at price $p^{\ddagger}_{n-3} \in (p^{\ddagger}_{n-2}, v)$ where $1 - F_S(p^{\ddagger}_{n-3}) = w_{n-3}(p^{\ddagger}_{n-3})$. We execute another partner swap so that firm $n-3$ replaces firm $n-2$, and firm $n-2$ shifts all remaining mass to $p_{n-2} = v$.

This construction continues iteratively until we reach the upper bound v.

For generic parameter values (by which we mean that no two firms wish to join the dance floor at the same price) this construction is unique. For other knife-edge cases (including, for example, $p^{\dagger}_n = p^{\dagger}_{n-1}$ or $p^{\dagger}_{n-1} = p^{\dagger}_{n-2}$) the construction also works, but there can be multiple equilibria.

B.3. **Summary of the Equilibrium Construction for Generic Parameters.** In summary, there is a Nash equilibrium of the single-stage game in which the profit of firm $i \in \{1, \dots, n\}$ is

$$\pi_i = \lambda_i(v - c_i) + \begin{cases} (\lambda_n + \lambda_S)(p^{\dagger}_{n-1} - p^{\dagger}_n) & \text{if } i = n\text{, and} \\ 0 & \text{otherwise.} \end{cases} \tag{B21}$$

This means $w_i = \underline{w}_i$ as in eq. (B1) for $i < n$, while w_n is given by eq. (B6). In this equilibrium:

- Firm n plays a mixed strategy with support $[p^{\dagger}_{n-1}, v]$.
- Firm $n-1$ plays a mixed strategy with support $[p^{\dagger}_{n-1}, p^{\ddagger}_j]$ with $p^{\ddagger}_j \in (p^{\dagger}_{n-1}, v]$ where $p^{\ddagger}_j$ is the lowest such price that solves $w_j(p) = 1 - F_S(p) = w_{n-1}(p)w_n(p)$ for any $j \le n-2$.
- No firm $i \le n-2$ has $[p^{\dagger}_{n-1}, p^{\ddagger}_j)$ in their support.
- If $p^{\ddagger}_j = v$, then each $i \le n-2$ plays the pure strategy $p_i = v$.
- If $p^{\ddagger}_j < v$, then j (effectively takes over the role of $n-1$ and) plays a mixed strategy with support $[p^{\ddagger}_j, p^{\ddagger}_k]$ with $p^{\ddagger}_k \in (p^{\ddagger}_j, v]$ where $p^{\ddagger}_k$ is the lowest such price that solves $w_k(p) = 1 - F_S(p) = w_j(p)w_n(p)/(w_n(p^{\ddagger}_j)^2)$ for any $k \notin \{n, n-1, j\}$.
- If $p^{\ddagger}_k = v$, then each $i \notin \{n, n-1, j, k\}$ plays the pure strategy $p_i = v$.
- If $p^{\ddagger}_k < v$, then k (effectively takes over the role of j and) the construction continues exactly as it did above for j with $p^{\ddagger}_j < v$.

The procedure above ends when either (i) all firms have been assigned mixed strategies or (ii) we find that for each firm yet to be assigned a mixed strategy, l, there is no $p < v$ such that $w_l(p) = 1 - F_S(p)$. In case (ii), each of those remaining firms, l, plays the pure strategy $p_l = v$.

Expressions for the CDFs for firms that play mixed strategies are recovered from eq. (B8).

This equilibrium is unique for (generic) parameters which satisfy: (i) $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$ and (ii) for any two values $p_i^\ddagger, p_j^\ddagger < v$ encountered during the iterative procedure, $p_i^\ddagger \neq p_j^\ddagger$.

Our algorithm constructs an equilibrium (for all parameter values) in which $n-1$ firms earn their captive-only profits, while firm n earns exactly $(\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ more than its captive-only profit. Moreover, the equilibrium is unique for generic parameter choices, and the equilibrium profits are uniquely defined when $p_n^\dagger < p_{n-1}^\dagger < p_{n-2}^\dagger$.

For the remaining knife-edge cases when either $p_n^\dagger = p_{n-1}^\dagger$ or $p_{n-1}^\dagger = p_{n-2}^\dagger$ (or possibly both), we have the existence of an equilibrium with the stated profits. However, there could also be an equilibrium in which (for example) firm n earns strictly more than the stated profit. We deemed such equilibria as "pathological" in the main text and excluded them from analysis.

B.4. **Knife-Edge Cases.** Suppose that $p_n^\dagger < p_{n-1}^\dagger = p_{n-2}^\dagger$ so that the second-most aggressive firm is not uniquely defined, and further suppose (for the simplicity of exposition) that all other firms $i \in \{1, \ldots, n-3\}$ choose $p_i = v$, leaving an effective three-player game between firms $i \in \{n-2, n-1, n\}$. We know that all firms other than n earn their captive-only profits, and that firm n earns Δ_n more than its captive-only profit where $\Delta_n \geq (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. Suppose that this holds as a strict inequality. This implies that $\underline{p}_n > p_{n-1}^\dagger$, and so $w_n(p_{n-1}^\dagger) > 1$. Firms $n-1$ and $n-2$ must mix together down to $p_{n-1}^\dagger$, do so according to distributions $1 - F_{n-1}(p) = w_{n-2}(p)$ and $1 - F_{n-2}(p) = w_{n-1}(p)$, and so $F_S(p) = 1 - w_{n-1}(p)w_{n-2}(p)$. We know that $1 - F_S(p_{n-1}^\dagger) = 1 < w_n(p_{n-1}^\dagger)$, and that $1 - F_S(v) = 0 < w_n(v)$. Moreover, the second (strict) inequality holds even if $\Delta_n = (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$. This means that $w_n(p)$ lies (strictly) above $1 - F_S(p) = w_{n-1}(p)w_{n-2}(p)$ at both the beginning and the end of the interval $[p_{n-1}^\dagger, v]$. We also know that firm n must join the dance floor at some point. This implies that there exists some $p^\ddagger$ where $w_n(p^\ddagger) = 1 - F_S(p^\ddagger) = w_{n-1}(p^\ddagger)w_{n-2}(p^\ddagger)$, which implies that $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below and then subsequently crosses from below to above within the interval $[p^\dagger_{n-1}, v]$. It is also true that $w_n(p)$ crosses $w_i(p)$ in this way for each $i \in \{n-2, n-1\}$. To proceed, we note that $w_n(p)$ is below $w_i(p)$ whenever

$$\frac{\lambda_n(v-p)+\Delta_n}{\lambda_S(p-c_n)} \le \frac{\lambda_i(v-p)}{\lambda_S(p-c_i)} \quad \Leftrightarrow \quad \lambda_n + \frac{\Delta_n}{v-p} \le \lambda_i \frac{p-c_n}{p-c_i}. \tag{B22}$$

The left-hand side is increasing in p. The right-hand side is decreasing in p if $c_n < c_i$. This means that $w_n(p)$ crosses $w_i(p)$ at most once from below to above. This is a contradiction. From this we conclude that if firm n has lower costs than others then there cannot be an equilibrium in which $\Delta_n > (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. Equivalently, we have unique equilibrium profits. An equilibrium with $\underline{p}_n > p^\dagger_{n-1}$ must entail $c_n > c_i$, or in this case $c_n > \max\{c_{n-1}, c_{n-2})$. Firm n is the most aggressive firm, and so necessarily this also implies that $\lambda_n < \min\{\lambda_{n-1}, \lambda_{n-2}\}$.

Working with such a configuration (so that firm n has high costs but few captives), let us begin by specifying $w_n(p)$ such that $\Delta_n = (\lambda_n + \lambda_S)(p^\dagger_{n-1} - p^\dagger_n)$. We have already constructed a non-pathological equilibrium for this case. To construct a pathological equilibrium, we need

$$w'_n(p^\dagger_{n-1}) < w'_{n-1}(p^\dagger_{n-1}) + w'_{n-2}(p^\dagger_{n-1}). \tag{B23}$$

Noticing that $w_{n-2}(p^\dagger_{n-1}) = w_{n-2}(p^\dagger_{n-1}) = 1$ for this case, this says that $w_n(p)$ declines more quickly than $w_{n-1}(p)w_{n-2}(p)$ when evaluated at $p^\dagger_{n-1}$. (This inequality also stops the construction of an equilibrium in which all three firms $\{n-2, n-1, n\}$ mix as a threesome.) This means that we can raise Δ_n, so that $w_n(p^\dagger_{n-1}) > 1$, but still guarantee (so long as we don't increase Δ_n too much) that there is some larger $p^\ddagger > p^\dagger_{n-1}$ at which $w_n(p)$ crosses $w_{n-1}(p)w_{n-2}(p)$ from above to below. We then construct an equilibrium by allowing $n-1$ and $n-2$ to "dance" until $p^\ddagger$ when n joins for a "partner swap" at $\underline{p}_n = p^\ddagger$. We next illustrate with a specific example.

B.5. **Construction of Pathological Equilibria.** Consider a triopoly (so setting $n = 3$) in which two pairwise-symmetric firms have low costs but many captives, whereas the third firm has high cost and few captives. Costs satisfy $c_1 = c_2 = 0$ and $c_3 = c > 0$ while the sizes of captive audiences satisfy $\lambda_1 = \lambda_2 = \lambda_H$ and $\lambda_3 = \lambda_L$ where $\lambda_H > \lambda_L$.

We choose parameters so that firms share the same lowest undominated price $p^\dagger_1 = p^\dagger_2 = p^\dagger_3 = p^\dagger$:

$$p^\dagger = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \frac{\lambda_L v + \lambda_S c}{\lambda_L + \lambda_S} = \frac{\lambda_H v}{\lambda_H + \lambda_S} = \quad \Leftrightarrow \quad c = \frac{(\lambda_H - \lambda_L)v}{\lambda_H + \lambda_S}. \tag{B24}$$

Henceforth when we vary the λ parameters we adjust c so that it satisfies this equation.

For this example the non-pathological equilibrium profits are captive-only for all three firms. For such profits, the required win probabilities are the minimum win probabilities. They are:

$$w_1(p) = w_2(p) = \frac{\lambda_H(v-p)}{\lambda_S p} \quad \text{and} \quad w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} \tag{B25}$$

where of course these satisfy $w_i(p^\dagger) = 1$ for all i. Also

$$w_1'(p^\dagger) = w_2'(p^\dagger) = -\frac{(\lambda_H+\lambda_S)^2}{\lambda_H\lambda_S v} \quad \text{and} \quad w_3'(p^\dagger) = -\frac{(\lambda_H+\lambda_S)(\lambda_L+\lambda_S)}{\lambda_L\lambda_S v}. \tag{B26}$$

The minimum win probability functions intersect (by construction) at $p^\dagger$. However, that function for the third firm (which has higher costs but fewer captives) declines more quickly:

$$w_3'(p^\dagger) < w_i'(p^\dagger) \text{ for } i \in \{1,2\} \;\Leftrightarrow\; -\frac{(\lambda_H+\lambda_S)(\lambda_L+\lambda_S)}{\lambda_L\lambda_S v} < -\frac{(\lambda_H+\lambda_S)^2}{\lambda_H\lambda_S v} \;\Leftrightarrow\; \lambda_L < \lambda_H. \tag{B27}$$

They key requirement to construct an equilibrium with pathological profits is that $w_3(p)$ declines more quickly than $w_1(p)w_2(p)$ when evaluated at $p^\dagger$. In this case,

$$w_3'(p^\dagger) < w_1'(p^\dagger) + w_2'(p^\dagger) \;\Leftrightarrow\; \lambda_L < \frac{\lambda_H\lambda_S}{\lambda_H + 2\lambda_S}. \tag{B28}$$

We construct an equilibrium in which firm 3 earns Δ above its captive-only profit, by setting

$$w_3(p) = \frac{\lambda_L(v-p)}{\lambda_S\left(p - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} + \frac{\Delta}{\lambda_S(p-c)}, \tag{B29}$$

which for $\Delta > 0$ now satisfies $w_3(p^\dagger) > 1 = w_1(p^\dagger) = w_2(p^\dagger)$. This means firm 3 does not wish "to dance" at $p^\dagger$. Instead, we construct an equilibrium in which firms 1 and 2 mix over $[p^\dagger, p^\ddagger]$ according to $1 - F_1(p) = w_2(p)$ and $1 - F_2(p) = w_1(p)$ which (from pairwise symmetry) reduces to $1 - F_i(p) = w_i(p)$ for $i \in \{1,2\}$ and $1 - F_S(p) = (w_i(p))^2$. By construction, $w_3(p) > 1 - F_S(p)$ for prices rising above $p^\dagger$. The key threshold is $p^\ddagger$ which satisfies $w_3(p^\ddagger) = 1 - F_S(p^\ddagger) = w_1(p^\ddagger)w_2(p^\ddagger)$. Explicitly, $p^\ddagger$ (which exists so long as $\Delta > 0$ is not chosen to be too large) satisfies

$$\frac{\lambda_L(v-p^\ddagger) + \Delta}{\lambda_S\left(p^\ddagger - \frac{(\lambda_H-\lambda_L)v}{\lambda_H+\lambda_S}\right)} = \left(\frac{\lambda_H(v-p^\ddagger)}{\lambda_S p^\ddagger}\right)^2. \tag{B30}$$

At $p^\ddagger$ there is a partner swap. Firm 2 (for example; this could be firm 3) shifts all further mass to v, which is an atom of size $w_1(p^\ddagger)$. Firms 1 and 3 then mix over the interval $[p^\ddagger, v)$ where

$$F_1(p) = 1 - \frac{w_3(p)}{w_1(p^\ddagger)} \quad \text{and} \quad F_3(p) = 1 - \frac{w_1(p)}{w_1(p^\ddagger)}. \tag{B31}$$

These firms earn their claimed equilibrium profits over this interval. The solutions satisfy $F_3(v) = 1$ (so that firm 3 does not play an atom) but $\lim_{p\uparrow v} F_1(p) < 1$ (so that firm 1 does play an atom). We need only check that firm 2 does not wish to rejoin the dance floor within this interval. We note that the probability that firm 2 wins the shoppers if it were to join is

$$(1 - F_1(p))(1 - F_3(p)) = \frac{w_3(p)}{w_1(p^\ddagger)}\frac{w_1(p)}{w_1(p^\ddagger)} < \frac{w_3(p^\ddagger)}{(w_1(p^\ddagger))^2} w_1(p) = w_1(p). \tag{B32}$$

B.6. **Equilibria with a Clearinghouse.** In Section 3 we consider a model in which firms must pay a fee to reach shoppers via a clearinghouse. Here we report some basic equilibrium features.

Lemma B2 (Equilibrium with a Clearinghouse). *In an equilibrium with a clearinghouse:*

(i) Any atom in a firm's mixed strategy is placed at v or at the "no participation" decision.

(ii) The upper bound of the support for a firm that always joins the clearinghouse is v.

(iii) There are no gaps in the joint support of firms' pricing strategies below v.

(iv) At most one firm places an atom at the advertised price v.

(v) At least $n-1$ of the firms earn their captive-only profit obtained from non-participation.

Proof. Claims (i) to (iii) follow from the arguments used for claims (i)–(iii) of Lemma B1. Claim (iv) also follows from standard arguments: firms only advertise a price v if that can win the business of shoppers, and if two or more do so with positive probability that at least one of those firms has the incentive to undercut the atom played by the others.

For claim (v), we note that any firm that earns strictly more than its captive-only profit must always advertise a price. (Choosing not to advertise gives a firm its captive-only profit.) If two (or more) firms earn strictly more than their captive-only profits then, from claim (ii), they use a support extending up to v. At least one of those firms does not play atom at v. This means other such firms know that pricing at, or close to, v results in arbitrarily few sales to shoppers, and so their profit is arbitrarily close to the captive-only profit. This is a contradiction. □

B.7. **Simultaneous Choice of Price and Technology.** In Section 2 we studied a two-stage model in which firms make technology choices (via marginal-cost-reducing process innovations) and then engage in a model-of-sales pricing game. Here we describe briefly a signal-stage version in which firms simultaneously choose both their prices and production technologies.

We use the notation from Section 2 so that z_i^L is the innovation choice of a firm i that expects to sell only to captives. This generates a captive-only profit of $\lambda_i V_i(z_i^L)$. Similarly, z_i^H is the innovation choice when expecting to sell to shoppers as well as captives. At firm i's minimum undominated price $p_i^\dagger$ it expects to serve shoppers, and so would choose z_i^H, where the corresponding marginal cost is $c_i^\dagger = v - V_i(z_i^H)$. It follows that the minimum undominated price for firm i is

$$\lambda_i V_i(z_i^L) = (\lambda_i + \lambda_S)(p_i^\dagger - v + V_i(z_i^H)) \quad \text{and so} \quad p_i^\dagger = v - V_i(z_i^H) + \frac{\lambda_i V_i(z_i^L)}{\lambda_i + \lambda_S}. \tag{B33}$$

We label firms according to their minimum undominated prices, just as in the main paper.

We can also find minimum win probabilities for a firm. Recall that the minimum win probability $\underline{w}_i(p)$ for firm i at price p is the probability of winning shoppers that makes the firm indifferent between charging p and instead earning its captive-only profit. The adjustment needed here is that we need to specify the technology choice that a firm would make if it expects to win the shoppers with probability $\underline{w}_i(p)$. Such a choice, which we label as $z_i^{(p)}$, maximizes $V_i(z_i)(\lambda_i + \underline{w}_i(p)) - z_i$. It follows that $\underline{w}_i(p)$ and $z_i^{(p)}$ are jointly determined by solving simultaneously

$$V_i'(z_i^{(p)})(\lambda_i + \underline{w}_i(p)) = 1 \quad \text{and} \quad \lambda_i V_i(z_i^L) = (\lambda_i + \underline{w}_i(p)\lambda_S)(p - v + V_i(z_i^{(p)})). \tag{B34}$$

We can similarly construct equilibrium win probabilities. These are $w_i(p) = \underline{w}_i(p)$ for $i < n$. For the most aggressive firm n, we replace the left-hand side term $\lambda_n V_n(z_n^L)$ in the second equation displayed just above with $\lambda_n V_n(z_n^L) + (\lambda_n + \lambda_S)(p_{n-1}^\dagger - p_n^\dagger)$ as before. Using these required win probabilities are repeat our equilibrium construction. Each price within the support of a firm's mixed strategy is paired with a technology (or innovation) choice.