The Impact of Perceived Strength in the War of Attrition

David P. Myatt
London Business School
www.david-p-myatt.org · dmyatt@london.edu

November 2024.[1]

Abstract. In a war of attrition a player's *perceived strength* is the distribution describing beliefs about her valuation. Small asymmetries in strength have a large effect: in the unique equilibrium of a game with a deadline the war ends quickly (instantly, as the deadline becomes infinite) with a concession by the (perceived) weaker player. The ranking of strength compares hazard rates in the upper tails of the distributions of beliefs; greater uncertainty about a player tends to give her more strength. The results also hold if techniques other than a deadline are used to obtain a unique equilibrium.

In a war of attrition, the first player to quit concedes a prize to her opponent. Costly fighting is worthwhile only if an opponent quits in the near future. These features are common to important phenomena, including labour negotiations, the voluntary provision of public goods, macroeconomic stabilisation, the adoption of technological standards, political lobbying, and many others. Natural questions arise. Who will win the war? When will it end?

The context (for this paper) is a game in which players differ in their (privately known) valuations for the prize. A player's *perceived strength* corresponds to the distribution that represents beliefs about her valuation. For the main result the comparison of strength involves the hazard rates in the upper tails of the distributions. The response to "who wins and when?" is this: *the war ends with the immediate concession of the perceived weaker player.*

Of course, the classic war of attrition has many equilibria. This can be resolved by making one of several small (and reasonable) modifications that dissuade a player from fighting forever. Here I do this directly by placing an upper limit on stopping times. The unique equilibrium converges to (and so selects) an equilibrium of a classic game as the time limit (in essence, a deadline) grows large. Other methods for obtaining uniqueness yield the same results.

The unique equilibrium is (of course) symmetric when players' beliefs are symmetric. However, asymmetric players have valuations that are (believed to be) drawn from different distributions. A player is then *perceived to be stronger* if her distribution of valuations stochastically dominates (in some appropriate sense) that of her opponent. I use three inter-related dominance rankings to predict properties of the equilibrium, and I preview three associated results here.

[1]This paper is based upon elements of a long-neglected paper (Myatt, 2005) which was dormant for many years after (embarrassingly) I lost the files for it. Only now have I returned to write a successor to it. Looking back to the ancestor paper, I offer warm thanks to Jon Dworak, Justin P. Johnson, Max Kwiek, Paul Klemperer, Eric Rasmusen, Kevin Roberts, Juuso Välimäki, Chris Wallace, Paddy Wallace, Peyton Young, and the anonymous reviewers of Myatt (2005) for their comments and suggestions. Thank you also to recent commentators, including notably Alex Teytelboym and Ludvig Sinander, for encouraging me to resurrect and to develop the paper. Finally, I thank the editor, associate editor, and two anonymous referees for their careful reviews and guidance.

Firstly, if the distribution of the (perceived) stronger player's valuation hazard-rate dominates that of her weaker opponent then she uses a more aggressive stopping-rule strategy. Secondly, if her distribution first order stochastically dominates (a less stringent condition) and if valuations are strictly positive, then the weaker opponent exits at the beginning of the game with positive probability. Thirdly, suppose that a hazard-rate ranking applies to the upper tails of the distributions. Under this condition: as the limit to players' stopping times increases, *the equilibrium converges to one in which the (perceived) weaker player always exits at the beginning.*

The third result has bold implications. Firstly, true prize valuations do not (at least in the limiting case of a long time horizon) determine the outcome, and so the prize allocation may be inefficient. Secondly, the game ends quickly and so the classic war-of-attrition model might not, by itself, explain delay. Thirdly, any asymmetries in perception are critical to a player's likely success, and so she may pursue activities that enhance (a particular feature of) perceptions of her valuation of the prize rather than develop the valuation itself.

I also offer notes of caution. Firstly, the "instant exit" result applies only as the finite horizon grows large. If players are perceived similarly and the deadline is not too distant, then the symmetric equilibrium of a symmetric specification can offer a reasonable prediction. Secondly, the probability of instant concession rises slowly as the time limit grows: very long horizons are needed for the most extreme result to bite. Thirdly, the desire for the weaker player to fight rather than concede returns if there is the chance of an event that changes players' perceptions.

The most relevant notion of strength depends on an upper-tail hazard-rate comparison. This is readily satisfied if the distribution of the stronger player is an upward mean-shifted version of that of the weaker player. However, the comparison does not require such a difference of means. Instead, the player perceived to be stronger is the one that is relatively more likely to have a very high prize valuation, even if she has a lower expected valuation. This holds (for leading cases) when there is greater uncertainty about a player: a combatant with uncertain real strength enjoys (according to the criterion) greater perceived strength. This final implication feeds into players' incentives to manipulate perceptions. A player wishes to prevent any activity which allows observers to learn about her own valuation (to maintain an "air of mystery") while encouraging anything that allows learning about her opponent.

Related Literature

War-of-attrition games in the literature (Maynard Smith, 1974; Bishop and Cannings, 1978b,a; Bishop, Cannings, and Maynard Smith, 1978; Riley, 1979, 1980; Nalebuff and Riley, 1985) have many equilibria. Researchers have often focused on the symmetric equilibrium of a symmetric game (e.g. Kapur, 1995; Krishna and Morgan, 1997; Bulow and Klemperer, 1999). However, (reasonable) modifications result in a unique equilibrium: a finite time limit (Cannings and Whittaker, 1995; Ponsati, 1995), a chance of players who fight forever (Fudenberg and Tirole, 1986; Kornhauser, Rubinstein, and Wilson, 1989), or a hybrid all-pay auction specification (Güth and Van Damme, 1986; Amann and Leininger, 1996; Riley, 1999). In games with complete information such modifications predict the instant concession of the weaker player (Kornhauser, Rubinstein, and Wilson, 1989; Riley, 1999; Kambe, 1999; Abreu and Gul, 2000).

But what if privately-known valuations are perceived to be from different distributions? The literature establishes (conditions for) uniqueness, but rarely offers a cross-player comparison of strategies; nor does it usually characterise the equilibrium of a classic war of attrition that is selected as any uniqueness-generating modification is removed. I contribute by using the distinction of perceived strength (comparing distributions) to show that it is the key driver of who wins and when. In doing so, the closest work to this one is an insightful analysis by Martinelli and Escorza (2007) which extends the Alesina and Drazen (1991) model of stabilisation to asymmetric players, and shows positive-probability instant concession when players are committed to fight forever with small probability.

I also complement more recent studies of asymmetric all-pay auctions and contests (notably Siegel, 2009, 2010, 2014a,b) including scenarios with spillovers (Xiao, 2018; Betto and Thomas, 2024). Others (including Huangfu, Ghosh, and Liu, 2023) have examined resource constraints, which correspond to player-specific time limits. The selected equilibrium of this paper suggests a war of attrition without attrition. This is also a theme of recent work by Georgiadis, Kim, and Kwon (2022) in which conditions change according to a Brownian motion: Markov pure strategies (interpreted as "no attrition") are played in equilibrium. It and other related papers by Lambrecht (2001), Murto (2004), and Steg (2015) fall within the framework of a recent and comprehensive paper by Décamps, Gensbittel, and Mariotti (2023).

Preliminaries: Model and Equilibrium

Here I specify a finite-horizon war of attrition with privately known prize valuations, I discuss briefly the specification choices, and I confirm the existence of a unique equilibrium.

The Game. Two players indexed by $i \in \{1, 2\}$ simultaneously choose stopping times $t_i \in [0, T]$ where T is a time limit (or deadline). The payoff of player i is

$$\pi_i(t_i, t_j) = u_i \times \left[\mathfrak{I}[t_i > t_j] + \frac{\mathfrak{I}[t_i = t_j]}{2} \right] - \min\{t_1, t_2\} \tag{1}$$

where "$\mathfrak{I}[\cdot]$" is the indicator function. A player's privately known type u_i is her *prize valuation.* Player $j \neq i$ believes that u_i is drawn from the distribution $F_i(\cdot)$ with strictly positive and continuous density $f_i(\cdot)$ on the support $(\underline{u}_i, \infty)$ where $\underline{u}_i < 2T$. $F_i(\cdot)$ represents the *perceived strength* of player i from the perspective of $j \neq i$.

A convenient illustrative example that I use below is obtained when each player $i \in \{1, 2\}$ is believed to have exponentially distributed valuations with hazard rate λ_i. In this case, the distribution describing the beliefs about player i's prize valuation is $F_i(u) = 1 - e^{-\lambda_i(u - \underline{u}_i)}$.

A strategy for player i maps her valuation to a stopping time: $t_i(u_i) : (\underline{u}_i, \infty) \mapsto [0, T]$. I seek Bayesian Nash equilibria, where (without loss) a player chooses the highest payoff-maximising stopping time whenever indifferent. Henceforth I refer to this as an "equilibrium."

Commentary. The specification of eq. (1) awards a prize to the player who fights for longest, with a "coin toss" tie break. Other tie-break rules may be used while retaining most results.

The fighting cost is proportional to the time elapsed (following Maynard Smith, 1974). Allowing "leader" and "follower" payoff outcomes to be functions of time (as in, for example, Bishop and Cannings, 1978b; Hendricks, Weiss, and Wilson, 1988) can offer similar insights.[2] The "linear costs" approach used here conveniently allows the war of attrition to be interpreted as an ascending-price all-pay auction (Klemperer, 1999): the price t rises until a player concedes, and both players pay the exit price.[3] What matters for behaviour is a player's valuation relative to the cost of fighting, and so I normalise marginal costs to unity. A player chooses duration rather than intensity, and so there is no signalling from endogenous costly effort of the kind considered by Hörner and Sahuguet (2007, 2011).

The cost $C_i(t_1, t_2) = \min\{t_1, t_2\}$ incurred by a player depends only on the time of first exit: a player stops fighting immediately following a concession, and so further planned fighting is costless. Other specifications make it costly to plan a later exit. For example, if the cost is

$$C_i(t_1, t_2) = \min\{t_1, t_2\} + \beta \max\{t_i - \min\{t_1, t_2\}, 0\}, \tag{2}$$

then a player pays β per unit of additional planned fighting time.[4] This corresponds to a hybrid all-pay auction (Güth and Van Damme, 1986; Amann and Leininger, 1996; Riley, 1999) in which the winner pays a combination of the winning and losing bids. (The war of attrition corresponds to $\beta = 0$; a plain-vanilla first-price all-pay auction to $\beta = 1$.) Later I note that eq. (2) yields a unique equilibrium even if there is no time limit ($T = \infty$).

This is a finite-horizon war (Cannings and Whittaker, 1995; Ponsati, 1995) in which the deadline T limits a player's aggression. This means that her opponent will fight to the end with positive probability (player j with $u_j > 2T$ prefers to do so rather than exiting earlier). This feature (that an opponent might never quit) means that fighting for longer is always costly for a player. Helpfully, this is just what is needed to pin down a unique equilibrium. For this reason

[2]If those leader and follower payoffs are $L_i(t)$ and $S_i(t)$ respectively then setting $L_i'(t) < 0$ ensures that the leader would rather quit sooner, and $S_i(t) > L_i(t)$ means that a player is willing to wait for the anticipated exit of her opponent. A "fighting cost" can be the delay before the award of a second prize: this might be implemented via $L_i(t) = Be^{-\delta_i t}$ and $S_i(t) = Ae^{-\delta_i t}$, where players differ via δ_i. Alternatively (Ponsati and Sákovics, 1995) it may be implemented via $L_i(t) = B_i e^{-t}$ and $F_i(t) = e^{-t}$ where players differ via B_i.

[3]This contrasts with a "first-price all-pay" auction (Baye, Kovenock, and De Vries, 1993, 1996) in which (unlike the second-price case) it is costly to raise a bid even if that bid is the highest.

[4]For Betto and Thomas (2024, pp. 183–184) eq. (2) specifies a war with costly preparation. Hendricks, Weiss, and Wilson (1988) and Pitchik (1981) referred to wars of attrition as "noisy" games in the sense that a player can "hear" her opponent's exit. (In contrast the "noisy players" of Anderson, Goeree, and Holt (1998a,b) choose quantal best replies in a logit equilibrium à la McKelvey and Palfrey (1995, 1996).) In a "silent" game a player does not observe such an exit; this is an all-pay auction. Equation (2) has this interpretation: a player fails to observe (or "hear") her opponent's exit with probability β and so continues to her planned exit time.

I require the upper bound of the supports of each distribution $F_i(\cdot)$ extend beyond $2T$. This is sufficient for many results to hold, including existence and uniqueness.[5] However, to allow for my central "large T" result (Proposition 3) and to simplify exposition I specify full support.[6] The assumption on the lower-bound of support ($\underline{u}_i/2 < T$) allows for players who optimally exit before time T. (With other suitable model specifications, such as those that are outlined in Proposition 4 toward the end of the paper, I can easily allow for bounded valuations.)

A player's (true) strength here is her valuation for a prize. A related formulation is where a player's type (and so strength) is determined by her cost of fighting.[7]

Equilibrium. An equilibrium has familiar properties: players fight over some time interval $(0, \bar{t})$ for $\bar{t} \in (0, T)$ using monotonic stopping rules. However, some players with high valuations fight until T, and others with low valuations may concede immediately. I note these properties here. (The literature-standard proofs of Lemmas 1 to 3 are reported in Appendix A.)

Lemma 1 (Basic Properties of Equilibrium Stopping Rules). *For both players:*

(i) There is an upper-bound $\bar{u}^\star$ at and above which a player fights until T.

(ii) There is a lower-bound $\underline{u}_i^\star \in [\underline{u}_i, \infty)$ at or below which a player $i \in \{1, 2\}$ exits at time zero. If $\underline{u}_i^\star > \underline{u}_i$ then she "instantly exits" with strictly positive probability at the beginning.

(iii) There is a unique time $\bar{t} \in (0, T)$ at or after which neither player exits.

(iv) The stopping rule $t_i(u)$ is strictly and continuously increasing from 0 to $\bar{t}$ for $u \in (\underline{u}_i^\star, \bar{u}^\star)$.

Claim (iv) ensures that the inverses of the stopping rules are well defined for $t \in (0, \bar{t})$.[8] I write $v_i(t)$ for such (differentiable, as Lemma 2 below will show) inverses. The distribution $G_i(\cdot) \equiv \Pr[t_i \leq t]$ of the stopping time for player i and its hazard rate are

$$G_i(t) = F_i(v_i(t)) \quad \text{and so} \quad \frac{g_i(t)}{1 - G_i(t)} = \frac{v_i'(t) f_i(v_i(t))}{1 - F_i(v_i(t))} \quad \text{for} \quad t \in (0, \bar{t}). \tag{3}$$

The optimal behaviour of player j for a stopping time $t \in (0, \bar{t})$ is straightforward: the marginal benefit of fighting is equal to her valuation $v_j(t)$ multiplied by the hazard rate $g_i(t)/[1 - G_i(t)]$ of her opponent's exit. The first-order conditions for the two players are

$$\frac{v_2(t) v_1'(t) f_1(v_1(t))}{1 - F_1(v_1(t))} = \frac{v_1(t) v_2'(t) f_2(v_2(t))}{1 - F_2(v_2(t))} = 1. \tag{4}$$

This pair of differential equations can be solved for $v_1(t)$ and $v_2(t)$, and then inverted to obtain the equilibrium stopping rules. However, (a pair of) boundary conditions are needed.

---

[5] Ponsati and Sákovics (1995) offered this summary: "There are a continuum of equilibria [of the classic war of attrition] characterised by a system of ordinary differential equations. Uniqueness may be achieved by perturbing the game, imposing that for a positive measure of types it is a dominant strategy not to concede."

[6] If that result is not required then it is sufficient to assume that the distributions extend above $2T$.

[7] Another interpretation of strength is that a weaker player faces a risk of a random forced exit (Asako, 2015).

[8] We can also define the left and right limits $v_i(0) = \lim_{t \downarrow 0} v_i(t)$ and $v_i(\bar{t}) = \lim_{t \uparrow \bar{t}} v_i(t)$.

Consider behaviour at $t = 0$ and (to streamline discussion here) suppose that valuations are strictly positive: $\min\{\underline{u}_1, \underline{u}_2\} > 0$. If a player exits with positive probability at the beginning then her opponent always find it profitable to fight for some time: there cannot be (the positive probability of) "instant exit" by both players. This means that $v_i(0) = \underline{u}_i^\star = \underline{u}_i$ for one of $i \in \{1, 2\}$, while the other player $j \neq i$ must satisfy $v_j(0) = \underline{u}_j^\star \geq \underline{u}_j$. This leaves a free choice for $\underline{u}_j^\star$, and so the need for a second boundary condition.[9]

That condition is obtained by considering the end of active play, where $v_i(\bar{t}) = \bar{u}^\star$ for both players. A player with valuation $\bar{u}^\star$ exiting at time $\bar{t}$ is indifferent to staying until the deadline T, and so $\frac{\bar{u}^\star}{2} = T - \bar{t}$. This is sufficient to pin down unique solutions for $v_i(t)$.

Lemma 2 (Uniqueness, Existence, and Properties of Inverse Stopping Rules). *There is a unique equilibrium. Each equilibrium inverse $v_i(t)$ is differentiable for $t \in (0, \bar{t})$. Define*

$$\Lambda_i(u) = \int_u^{\bar{u}^\star} \frac{1}{x} \frac{f_i(x)}{1 - F_i(x)}\, dx \quad \textit{where} \quad \bar{u}^\star = 2(T - \bar{t}). \tag{5}$$

For any $t \in (0, \bar{t})$ the pair of inverse stopping rules satisfy $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$.

The crucial step needed for uniqueness is the boundary condition at $\bar{t}$. This is generated by the existence of the deadline T. In its absence, there is only a single boundary of condition at $t = 0$, and so there is a family of equilibrium solutions each determined by the identity of the player $j \in \{1, 2\}$ and the probability $F_j(\underline{u}_j^\star)$ with which that player instantly exits at time zero. If $T = \infty$ but there are other mild modifications to the game (these are discussed as extensions in a later section) then a suitable boundary condition for $t \to \infty$ can still be obtained.

Illustration. I noted earlier that a convenient illustrative example is when valuations are (believed to be) exponentially distributed, so that $F_i(u) = 1 - e^{-\lambda_i(u - \underline{u}_i)}$. In this case the solution to eq. (5) is straightforwardly $\Lambda_i(u) = \lambda_i \log[\bar{u}^\star/u]$. Applying Lemma 2, the inverse-stopping rules are related by $\lambda_2 \log v_2(t) = (\lambda_2 - \lambda_1) \log \bar{u}^\star + \lambda_1 \log v_1(t)$. Equivalently,

$$\log\left(\frac{v_2(t)}{v_1(t)}\right) = \left(1 - \frac{\lambda_1}{\lambda_2}\right) \log\left(\frac{\bar{u}^\star}{v_1(t)}\right) \quad \text{or also} \quad v_2(t) = (\bar{u}^\star)^{1-(\lambda_1/\lambda_2)} (v_1(t))^{(\lambda_1/\lambda_2)}. \tag{6}$$

Suppose now that $\lambda_1 < \lambda_2$ and set $\underline{u}_1 > \underline{u}_2 > 0$. Under this specification the expected valuations of the players satisfy $\mathrm{E}[u_1] = \bar{u}_1 + (1/\lambda_1) > \bar{u}_2 + (1/\lambda_2) = \mathrm{E}[u_2]$, and so there is a robust sense in which the first player has greater perceived strength than than the second player. (I discuss further different notions of perceived strength in the next section.)

By inspection of eq. (6), the inverse-stopping rules satisfy $v_1(t) < v_2(t)$. This means that the first player is more willing to fight until time t than the second player, and so in equilibrium fights more aggressively. This holds at the beginning of the war: $\underline{u}_1^\star = v_1(0) < v_2(0) = \underline{u}_2^\star$. This implies that $\underline{u}_2^\star > \underline{u}_2$: the second player exits at time zero with positive probability.

[9]If valuations extend below zero then with probability $F_i(0)$ a player prefers to lose and so exits at $t = 0$. An opponent $j \neq i$ with $u_j > 0$ always fights for some period of time: $\underline{u}_1^\star = \underline{u}_2^\star = 0$. This pair of boundary conditions hints at a unique solution. Unfortunately, there is a failure of Lipschitz continuity at $t = 0$ (as explained by Fudenberg and Tirole, 1986): writing a first-order condition from eq. (4) as $v_i'(t) = [1 - F_i(v_i(t))]/[v_j(t) f_i(v_i(t))]$, the denominator of the right-hand side tends to zero as $t \to 0$. A terminal boundary condition is still required.

The exponential specification also allows a convenient illustration of how a unique equilibrium is obtained. The two first-order conditions for the competing players are

$$v_2(t)v_1'(t)\lambda_1 = 1 \quad \text{and} \quad v_1(t)v_2'(t)\lambda_2 = 1. \tag{7}$$

Working with the condition for the first player and substituting in the solution for $v_2(t)$ in terms of $v_1(t)$ taken from eq. (6), that condition becomes

$$(\bar{u}^\star)^{1-\lambda_1/\lambda_2} (v_1(t))^{(\lambda_1/\lambda_2)} v_1'(t)\lambda_1 = 1 \quad \text{or equivalently} \quad t_1'(u) = \lambda_1 (\bar{u}^\star)^{1-\lambda_1/\lambda_2} (u)^{(\lambda_1/\lambda_2)}. \tag{8}$$

This integrates straightforwardly to give the stopping rule $t_1(u)$:

$$t_1(u) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} (\bar{u}^\star)^{1-(\lambda_1/\lambda_2)} (u)^{1+(\lambda_1/\lambda_2)} + \text{constant}. \tag{9}$$

The constant of integration can be obtained by setting $t_1(\underline{u}_1) = 0$, and then $\bar{u}^\star$ is obtained by setting $t_1(\bar{u}^\star) = \bar{t} = T - (\bar{u}^\star/2)$. The general solution for both equilibrium stopping-time function is particularly easy to state when $\underline{u}_1 = \underline{u}_2 = 0$. In this case, for $i \neq j$,

$$t_i(u) = \frac{\lambda_1\lambda_2}{\lambda_1+\lambda_2} (\bar{u}^\star)^{1-(\lambda_i/\lambda_j)} (u)^{1+(\lambda_i/\lambda_j)} \quad \text{where} \quad \bar{u}^\star = \left(-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{4\lambda_1\lambda_2 T}{\lambda_1+\lambda_2}}\right)\frac{\lambda_1+\lambda_2}{2\lambda_1\lambda_2}. \tag{10}$$

The proof of Lemma 2 follows a procedure similar to the one used here for this specific case.

Main Results: Perceived Strength and Instant Concession

I now ask how the unique equilibrium responds to the players' perceptions of their opponents' prize valuations. I use three notions of strength that compare the distributions $F_1(\cdot)$ and $F_2(\cdot)$ to develop three associated results that form the core contribution of the paper.

Notions of Perceived Strength. A player i is perceived to be stronger than her opponent j (from now on, simply "stronger") if $F_i(\cdot)$ stochastically dominates $F_j(\cdot)$ in some sense. Of course, different notions of stochastic dominance are possible. I describe three here.[10]

Definition (Notions of Comparative Strength). *Player 1 is stronger than Player 2*

(i) in the sense of hazard-rate dominance, $F_1 \succ_{HRD} F_2$, *if* $\underline{u}_1 \geq \underline{u}_2$ *and for all* $u \in (\underline{u}_1, \infty)$,

$$\frac{f_2(u)}{1-F_2(u)} > \frac{f_1(u)}{1-F_1(u)}; \tag{11}$$

(ii) in the sense of first-order dominance, $F_1 \succ_{FOSD} F_2$, *if for all* $u \in (\max\{\underline{u}_1, \underline{u}_2\}, \infty)$,

$$F_2(u) > F_1(u); \tag{12}$$

(iii) and in the sense of asymptotic hazard-rate dominance, $F_1 \succ_{AHRD} F_2$, *if*

$$\liminf_{u\to\infty} \left[\frac{f_2(u)}{1-F_2(u)} - \frac{f_1(u)}{1-F_1(u)}\right] > 0. \tag{13}$$

[10] Definitions (i) and (ii) use strict inequalities, but are readily modified. A standard definition of first-order dominance would ask eq. (12) to apply only weakly in general, but strictly for some values.

The illustrative specification using exponential distributions can be readily configured to meet all three criteria. Recall (from the previous section) that such exponential distributions satisfy $F_i(u) = 1 - e^{-\lambda_i(u-\underline{u}_i)}$. Setting $\underline{u}_1 \geq \bar{u}_2$ and $\lambda_1 < \lambda_2$, criterion (i) to (iii) all hold.

More generally, the first criterion implies the second criterion, and also

$$\frac{\partial}{\partial u} \log \left[\frac{1 - F_1(u)}{1 - F_2(u)}\right] = \frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} > 0, \tag{14}$$

and so the odds ratio of $u_1 > u$ versus $u_2 > u$ is increasing in u.[11] Furthermore, and given $u_1 > u$ and $u_2 > u$, the conditional distribution of u_1 stochastically dominates (in the first-order sense) that of u_2. This is conditional stochastic dominance (Maskin and Riley, 2000).

The first criterion does not necessarily imply the third criterion: even if players' hazard rates are ordered it may be true that the difference between those hazard rates disappears in the upper tails. A player that is stronger in the asymptotic hazard-rate sense is much more likely (in a relative sense) to experience very high valuations:[12]

$$F_1 \succ_{\text{AHRD}} F_2 \quad \Rightarrow \quad \lim_{u \to \infty} \frac{1 - F_1(u)}{1 - F_2(u)} = \infty. \tag{15}$$

The second and third criteria are not nested. However, if a player is stronger in the asymptotic hazard-rate sense then she cannot be weaker in the first-order sense.

Sometimes the first two criteria do not apply, but the third one does. This is naturally so when a player's valuation is riskier in the spirit of Rothschild and Stiglitz (1970). To illustrate, suppose that players are characterised by mean μ_i and variance σ_i^2 parameters, and $F_i(u) = ((u - \mu_i)/\sigma_i)$ where $F(\cdot)$ has full support, zero mean, unit variance, and a hazard rate with a derivative that is bounded away from zero in the upper tails. (The standard normal, for example.)

If $\sigma_1 = \sigma_2$ and $\mu_1 > \mu_2$ then Player 1 is stronger than Player 2 according to all three of the notions here; her distribution is a mean-shift upward of that of her opponent. However, if $\sigma_1 > \sigma_2$ then Player 1 is stronger than Player 2 in the asymptotic hazard-rate sense, but not necessarily in the other senses. Her distribution is a spread of that of her opponent, which makes it more likely (relative to her opponent) that she has a very high valuation.

If the valuation distributions of the two players have increasing hazard rates then for reasonable specifications the players are likely to be ranked in the sense of asymptotic hazard-rate dominance. For such a ranking to fail then hazard rates need to cross repeatedly in the upper tails of the distributions or to converge. They can readily converge, of course, if hazard rates are decreasing. An increasing hazard rate is a standard regularity assumption in many settings, of course, but many standard textbook distributions have decreasing hazard rates.

[11]When evaluating $\Pr[i = 1 \mid u_i > u]$ an increase in u is "good news" in the sense of Milgrom (1981).
[12]This is related to the unbounded likelihood-ratio property required for Mirrlees (1999) contracts.

One specification with a decreasing hazard rate is the Pareto distribution. For some α_i

$$F_i(u) = 1 - \left(\frac{\underline{u}_i}{u}\right)^{\alpha_i} \quad \Rightarrow \quad \frac{f_i(u)}{1 - F_i(u)} = \frac{\alpha_i}{u}. \tag{16}$$

If $\alpha_1 < \alpha_2$ and $\underline{u}_1 \geq \underline{u}_2$ then criteria (i) and (ii) are met, but (give that upper-tail hazard rates converge to zero) criterion (iii) of the comparative-strength definitions does not hold.

Instant Exit. I now relate notions of strength to the equilibrium stopping rules.

Suppose that $F_1 \succ_{\text{HRD}} F_2$, so that Player 1 is stronger in the hazard-rate sense.

Other things equal (when stopping rules intersect) Player 2's higher hazard rate means that she exits more quickly. To maintain an equilibrium, a wider interval of types of Player 1 must exit within that interval of time. More formally, if the stopping rules cross at some valuation u and at time $t = t_1(u) = t_2(u)$ then the conditions from eq. (4) reduce to

$$\frac{v_2'(t) f_2(u)}{1 - F_2(u)} = \frac{v_1'(t) f_1(u)}{1 - F_1(u)} \quad \Rightarrow \quad v_2'(t) < v_1'(t) \quad \Rightarrow \quad t_2'(u) > t_1'(u), \tag{17}$$

which says that the stopping time of a weaker player rises more quickly, and so her stopping rule crosses that of the stronger player from below to above. This means that the players' stopping rules can intersect only once.[13] For the two stopping rules to intersect at the end-of-attrition time $\bar{t}$, as is required by claim (iv) of Lemma 1, they must begin where the weaker player's stopping time is below that of the stronger player.

Proposition 1 (Hazard-Rate Dominance ⇒ Quicker Exit by a Weaker Player). *If Player 1 is stronger than Player 2 in the sense of hazard-rate dominance, then Player 1 plays more aggressively than Player 2, and so Player 2 exits more quickly from the war of attrition.*

Formally, if $F_1 \succ_{HRD} F_2$ *then* $t_1(u) > t_2(u)$ *for all* $u \in (\max\{\underline{u}_1, 0\}, \bar{u}^\star)$.

Proof. If $F_1 \succ_{\text{HRD}} F_2$ then the integrand in the expression for $\Lambda_i(u)$ from Lemma 2 is lower for $i = 1$ than for $i = 2$, and so $v_1(t) < v_2(t)$ for $t < \bar{t}$ in order to satisfy $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$. □

An implication is that the prize allocation is biased toward the stronger player.

Another implication is obtained by tracing the stopping rules back to time zero: the ranking of those rules means (if the lower bounds of valuations are strictly positive) that $\underline{u}_1^\star < \underline{u}_2^\star$. The hazard-rate ranking of the players implies that $\underline{u}_2 \leq \underline{u}_1$, and from this I conclude that $\underline{u}_2^\star > \underline{u}_2$: the weaker player must concede at time $t = 0$ with positive probability. In fact, this result holds under the weaker condition of first-order dominance.

[13] This follows from a familiar argument: if they intersected more than once then at one of the intersections $t_2(u)$ would need to cross $t_1(u)$ from above to below, and this would contradict eq. (17).

Proposition 2 (First-Order Dominance ⇒ Some Instant Exit). *If Player 1 is stronger than Player 2 in the sense of first-order dominance, and if valuations are strictly positive, then the weaker Player 2 exits at time zero with positive probability.*

Formally, if $F_1 \succ_{FOSD} F_2$ *and* $\min\{\underline{u}_1, \underline{u}_2\} > 0$ *then* $\underline{u}_2^\star > \underline{u}_2$ *and so* $\Pr[t_2(\cdot) = 0] > 0$.

Proof. $F_1 \succ_{\text{FOSD}} F_2$ implies that $\underline{u}_2 \leq \underline{u}_1$. Consider an equilibrium in which Player 2 does not exit at time zero: $\underline{u}_2^\star = \underline{u}_2 \leq \underline{u}_1 \leq \underline{u}_1^\star$. Taking eq. (5), integrating parts, and applying Lemma 2,

$$\Lambda_i(\underline{u}_i^\star) = \frac{\log[1 - F_i(\underline{u}_i^\star)]}{\underline{u}_i^\star} - \frac{\log[1 - F_i(\bar{u}^\star)]}{\bar{u}^\star} - \int_{\underline{u}_i^\star}^{\bar{u}^\star} \frac{\log[1 - F_i(u)]}{u^2}\, du$$

$$\Rightarrow \quad 0 = \Lambda_2(\underline{u}_2^\star) - \Lambda_1(\underline{u}_1^\star) = \underbrace{\frac{\log[1 - F_2(\underline{u}_2^\star)]}{\underline{u}_2^\star}}_{\text{zero if } \underline{u}_2^\star = \underline{u}_2} - \underbrace{\int_{\underline{u}_2^\star}^{\underline{u}_1^\star} \frac{\log[1 - F_2(u)]}{u^2}\, du}_{\text{strictly negative}} - \underbrace{\frac{\log[1 - F_1(\underline{u}_1^\star)]}{\underline{u}_1^\star}}_{\text{strictly negative}}$$
$$+ \underbrace{\frac{1}{\bar{u}^\star} \log \frac{1 - F_1(\bar{u}^\star)}{1 - F_2(\bar{u}^\star)}}_{\text{strictly positive if } F_1 \succ_{\text{FOSD}} F_2} + \underbrace{\int_{\underline{u}_1^\star}^{\bar{u}^\star} \frac{1}{u^2} \log \frac{1 - F_1(u)}{1 - F_2(u)}\, du}_{\text{strictly positive if } F_1 \succ_{\text{FOSD}} F_2} > 0. \quad (18)$$

The right-hand side is strictly positive. This contradicts the supposition that $\underline{u}_2^\star = \underline{u}_2$. □

I turn to the third (and main) result, which uses only asymptotic hazard-rate dominance.

As the deadline T increases, the fraction of types who have a dominant strategy to fight to the deadline (with valuations $u_i \geq 2T$) falls. This can drive up the time $\bar{t}$ at which attrition stops but, importantly, it also (as the proof of Lemma 3 confirms) increases the length $T - \bar{t}$ of the "no more concessions" period, which increases the upper bound $\bar{u}^\star = 2(T - \bar{t})$ to the set of player types who fight through this period.

Lemma 3 (Receding Deadline). *The unique equilibrium satisfies* $\lim_{T\to\infty} \bar{u}^\star = \infty$*, and so as the deadline recedes the probability that players fight to that deadline falls to zero.*

Now suppose that Player 1 is stronger than Player 2 in the asymptotic hazard-rate sense.

If the deadline is distant (T is large) then hazard rates are bounded apart at the end-of-attrition time $\bar{t}$ when both marginal players share the same valuation $v_1(\bar{t}) = v_2(\bar{t}) = \bar{u}^\star$. Working back from that deadline valuation, the stopping rule of the weaker (in the "$\succ_{\text{AHRD}}$" sense) player is steeper than that of the stronger player. If that deadline is sufficiently distant and the difference in hazard rates is maintained, then the backward-induction process means that the stopping rule of the weaker player hits a time of zero for some relatively large valuation.

Proposition 3 (Asymptotic Hazard-Rate Dominance $\Rightarrow$ Complete Instant Exit). *If Player 1 is stronger than Player 2 in the sense of asymptotic hazard-rate dominance, then in the limit with a long time horizon Player 2 always exits at time zero, and Player 1 always wins.*

Formally, if $F_1 \succ_{AHRD} F_2$ and $\underline{u}_1 > 0$ then $\lim_{T\to\infty} \underline{u}_2^\star = \infty$. Equivalently, $t_2(u) = 0$ for any $u \in (\underline{u}_2, \infty)$ if T is sufficiently large. If $\underline{u}_1 < 0$ then $\lim_{T\to\infty} t_2(u) = 0$ for any $u \in (\underline{u}_2, \infty)$.

In both of these subcases, $\lim_{T\to\infty} t_1(u) > 0$ for any $u \in (\max\{\underline{u}_1, 0\}, \infty)$

Proof. To shorten the exposition, suppose that valuations are strictly positive: $\min\{\bar{u}_1, \bar{u}_2\} > 0$. (The proof readily extends to other cases.) Fix $\varepsilon > 0$, a valuation $u_\varepsilon^\dagger$ above which the hazard rates differ by at least ε, and T large enough so that $\bar{u}^\star > u_\varepsilon^\dagger$.

The first step in the proof is to show that $\underline{u}_1^\star < \underline{u}_2^\star$ if T sufficiently large. To show this, begin by supposing that $\underline{u}_1^\star \geq \max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}$. The equality $\Lambda_1(\underline{u}_1^\star) = \Lambda_2(\underline{u}_2^\star)$ implies that

$$\underbrace{\int_{\underline{u}_2^\star}^{\underline{u}_1^\star} \frac{1}{u} \frac{f_2(u)}{1 - F_2(u)}\, du}_{\text{weakly positive}} = \int_{\underline{u}_1^\star}^{\bar{u}^\star} \frac{1}{u} \left[\frac{f_1(u)}{1 - F_1(u)} - \frac{f_2(u)}{1 - F_2(u)} \right] du \leq -\varepsilon \int_{\underline{u}_1^\star}^{\bar{u}^\star} \frac{1}{u}\, du < 0, \tag{19}$$

which is a contradiction. Next suppose that $u_\varepsilon^\dagger > \underline{u}_1^\star \geq \underline{u}_2^\star$. Using $\Lambda_1(\underline{u}_1^\star) = \Lambda_2(\underline{u}_2^\star)$ again,

$$\underbrace{\int_{\underline{u}_2^\star}^{\underline{u}_1^\star} \frac{1}{u} \frac{f_2(u)}{1 - F_2(u)}\, du}_{\text{weakly positive}} \leq \underbrace{\int_{\underline{u}_1^\star}^{u_\varepsilon^\dagger} \frac{1}{u} \left[\frac{f_1(u)}{1 - F_1(u)} - \frac{f_2(u)}{1 - F_2(u)} \right] du}_{\text{bounded}} - \varepsilon \underbrace{\int_{u_\varepsilon^\dagger}^{\bar{u}^\star} \frac{1}{u}\, du}_{\text{diverges as } \bar{u}^\star \to \infty} . \tag{20}$$

The right-hand side diverges to $-\infty$ as $T \to \infty$ (and so $\bar{u}^\star \to \infty$, from Lemma 3) and so the inequality must fail (a contradiction) for T sufficiently large.

The first step is complete: if T is sufficiently large then $\underline{u}_1^\star < \underline{u}_2^\star$. There are two possibilities. One is that $\underline{u}_2^\star$ diverges to ∞, which is the desired result. The other is that $\underline{u}_2^\star$ remains bounded. To investigate (and contradict) the latter case, construct a sequence of deadlines and associated equilibria for which $\underline{u}_2^\star$ remains bounded. Using $\Lambda_1(\underline{u}_1^\star) = \Lambda_2(\underline{u}_2^\star)$ once more,

$$\begin{aligned} \underbrace{\int_{\underline{u}_1^\star}^{\underline{u}_2^\star} \frac{1}{u} \frac{f_1(u)}{1 - F_1(u)}\, du}_{\text{bounded if } \underline{u}_2^\star \text{ remains bounded}} &= \int_{\underline{u}_2^\star}^{\bar{u}^\star} \frac{1}{u} \left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] du \\ &\geq \underbrace{\int_{\underline{u}_2^\star}^{\max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}} \frac{1}{u} \left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] du}_{\text{bounded}} + \varepsilon \underbrace{\int_{\max\{\underline{u}_2^\star, u_\varepsilon^\dagger\}}^{\bar{u}^\star} \frac{1}{u}\, du}_{\text{diverges as } \bar{u}^\star \to \infty}, \end{aligned} \tag{21}$$

which again fails if T (and so $\bar{u}^\star$) is sufficiently large; the desired contradiction. ☐

This proof uses the asymptotic hazard-rate dominance property (the hazard rates of the players are bounded apart in the upper tails) as a sufficient condition to obtain the desired result. This is, of course, not a necessary condition. The exact condition that is used is that

$$\lim_{\bar{u}^\star \to \infty} \int_{u_\varepsilon^\dagger}^{\bar{u}^\star} \frac{1}{u} \left[\frac{f_2(u)}{1 - F_2(u)} - \frac{f_1(u)}{1 - F_1(u)} \right] du = \infty. \tag{22}$$

In an illustration below (using Pareto specification for players' beliefs) I show an example where the hazard rates of players are ranked (so that the stronger player exhibits hazard-rate dominance over the weaker player) but where eq. (22) fails. In this case, the perceived weaker player does not always exist at the beginning of play.

Unlike Propositions 1 and 2, Proposition 3 takes a parameter to the limit by extending the deadline T and so "selects" an equilibrium from those of the classic war of attrition. (That classic war of attrition corresponds to the limiting case where there is no deadline.)

This paper asks "who wins and when?" For the selected equilibrium the "who?" is the player that is perceived as stronger in the sense of asymptotic hazard-rate dominance. This player might well (be likely to) have a low valuation. What it takes to win is a belief (held by an opponent) that a player is relatively more likely to have a very high valuation.

Turning to the "when?" the answer is "immediately!" This is a somewhat uncomfortable prediction that wars of attrition will not be fought. However, if the deadline is present but not too large then there is only partial early concession.

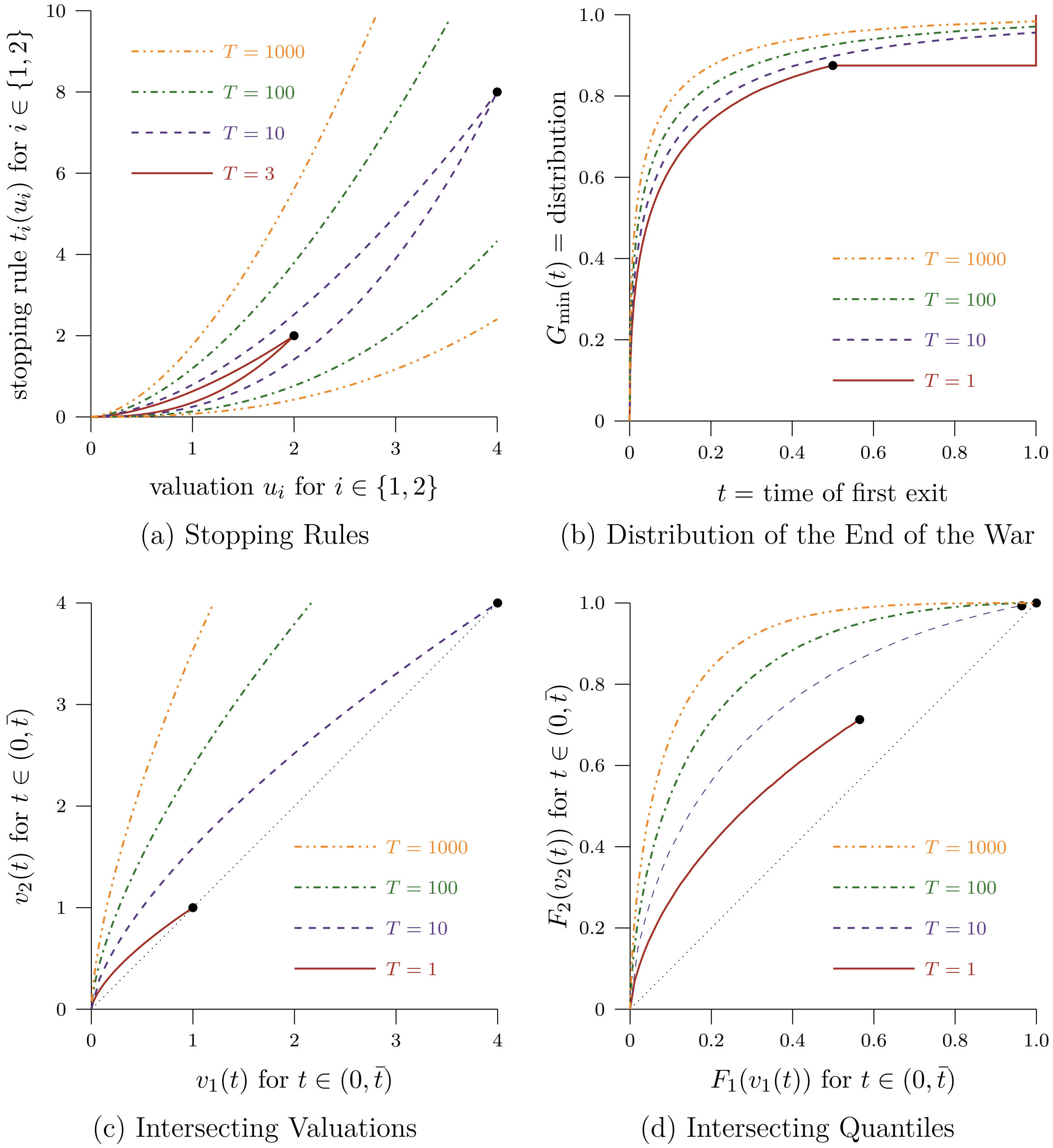
Illustrations. I have used the exponential distribution as an illustrative example for the specification of players' beliefs and so their perceived strengths: $F_i(u) = 1 - e^{-\lambda_i(u-\underline{u}_i)}$ for each $i \in \{1,2\}$. For illustration here I set $\underline{u}_1 = \underline{u}_2 = 0$ and $\lambda_1 < \lambda_2$ so that the first player has greater perceived strength. I reported the equilibrium stopping rules in eq. (10). Here I express these solutions in terms of the expected valuations of the two players, $\mu_i \equiv \mathrm{E}[u_i] = 1/\lambda_i$ for $i \in \{1,2\}$, so that $\mu_1 > \mu_2$, and I also normalise the average expected valuation across players so that $(\mu_1 + \mu_2)/2$. This implies that the deadline T is measured relative to the expected valuation of a prize that is randomly awarded to one of the combatants.

$$t_1(u) = \frac{(\bar{u}^\star)^{1-(\mu_2/\mu_1)}\,(u)^{1+(\mu_2/\mu_1)}}{2} \quad \text{and} \quad t_2(u) = \frac{(\bar{u}^\star)^{1-(\mu_1/\mu_2)}\,(u)^{1+(\mu_1/\mu_2)}}{2}$$
$$\text{where} \quad \bar{u}^\star = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2T}. \quad (23)$$

A player's stopping time (for a particular valuation) is increasing (without bound) in the deadline T for the stronger player, but decreasing (in the limit, to zero) for the weaker player.

The prediction of rapid concession (Proposition 3) is obtained in the limit as $T \to \infty$, based on an argument that uses the final term of eq. (21). This diverges with $\log \bar{u}^\star$, and so at a slow rate. Very long deadlines might be required for near-instant exit.

I evaluate this hypothesis using an exponential specification with hazard rates $\lambda_1 = \frac{5}{6}$ and $\lambda_2 = \frac{5}{4}$; the expectation $\mu_1 = \frac{6}{5}$ is 50% higher than $\mu_2 = \frac{4}{5}$. I set $\underline{u}_1 = \underline{u}_2 = 0$, so the valuation supports extend down to zero. This means that $\underline{u}_1^\star = \underline{u}_2^\star = 0$, and so there is no literal instant exit (Proposition 2 has no bite). Nevertheless, there can be relatively rapid exit.



(a) Stopping Rules

(b) Distribution of the End of the War

(c) Intersecting Valuations

(d) Intersecting Quantiles

These four panels illustrate the equilibria of a war-of-attrition game where players' valuations are exponentially distributed with hazards $\lambda_1 = \frac{5}{6}$ and $\lambda_2 = \frac{4}{5}$.

Any bullet points in the plots correspond to $t = \bar{t}$ or $u_1 = u_2 = \bar{u}^\star$.

Panel (a) illustrates the equilibrium stopping rules for the two players for different values of the time-horizon deadline T. The higher stopping rule is Player 1.

Panel (b) illustrates the distribution of the earliest stopping time, and so the distribution over the length of the war of attrition. This is a distribution $G_{\min}(t)$ which satisfies $1 - G_{\min}(t) = (1 - G_1(t))(1 - G_2(t))$ where $G_i(t) = F_i(v_i(t))$.

Panel (c) illustrates the valuation pairs $\{v_1(t), v_2(t)\}$ for which players both fight to time t. Any point above and to the left of the relevant line corresponds to a valuation combination leading to a win for Player 2; similarly points below and to the right generate wins for Player 1. Any combination "north east" of a bullet point corresponds to valuation pairs that are both above $\bar{u}^\star$ and so both players fight to the deadline. Panel (d) translates those valuation pairs into quantiles for the two players.

Figure 1. Equilibrium Properties for an Exponential Specification

In Figure 1 I illustrate equilibria. From panel (a), the stopping rules are ranked (from Proposition 1) and move apart as the T grows, while panel (b) shows how the game ends more quickly. Panels (c) and (d) compare the valuation pairs (and quantiles) at which the two players fight to the same time. This illustrates the valuation combinations which result in wins for the different players. As the time limit grows the prize shifts toward the perceived stronger player and the war ends more quickly. However, fighting may persist for some time with a chance that the perceived weaker player wins. For example, with $T = 100$, there is a 15% chance that the war lasts longer than 25% of the cross-player expected valuation. Also, if the stronger player has a median valuation then she still faces a 7% chance of losing to the weaker player.

A nuanced message is that a finite-horizon war of attrition (with a reasonably long deadline) is strongly (but not completely) biased towards an early win by a perceived-stronger player. However, if the deadline is not too distant and players are not too dissimilar then the equilibrium may be closer to a symmetric-equilibrium-style prediction.

Finally, I return to a second illustrative specification to which Proposition 3 does not apply. Suppose that players' valuations are Pareto distributed and so follow the specification of eq. (16). I set $\alpha_1 < \alpha_2$ and for simplicity I equate the lower-bounds of the supports of the players' valuation distributions so that $\underline{u}_1 = \underline{u}_2 = \underline{u}$. Under this specification Proposition 2 does apply and so $\bar{u}_1^\star = \bar{u}$ (the stronger player always fights) and $\bar{u}_2^\star > \bar{u}$ (the weaker player exits at time zero with positive probability). In fact, the valuation $\bar{u}_2^\star$ below which instant exit occurs satisfies $\Lambda_2(\bar{u}_2^\star) = \Lambda_1(\bar{u})$ where the solution for $\Lambda_i(u))$ using eq. (5) is

$$\Lambda_i(u) = \alpha_i \left(\frac{1}{u} - \frac{1}{\bar{u}^\star} \right). \tag{24}$$

Allowing the deadline (and so the threshold valuation $\bar{u}^\star$ above which players fight to that deadline) to grow large: $\lim_{\bar{u}^\star \to \infty} \Lambda_i(u) = \alpha_i / u$. Omitting the details, this implies that

$$\lim_{T \to \infty} \underline{u}_2^\star = \frac{\alpha_2}{\alpha_1} \underline{u} \quad \text{and so} \quad \lim_{T \to \infty} \Pr[t_2 = 0] = 1 - \left(\frac{\alpha_1}{\alpha_2} \right)^{\alpha_2}. \tag{25}$$

In the "selected" equilibrium of a classic war of attrition there is the positive probability of instant concession, but also the weaker player may fight for some period of time.

Extensions and Implications

In this section I consider briefly specifications (other than the use of a deadline) that can be used to obtain a unique equilibrium, I discuss possible extensions that include both direct learning about players' valuations and also pre-play tactics, and I describe applications of the results

Other Specifications. There are game modifications (other than a deadline) that yield uniqueness. Set $T = \infty$ (players can fight forever) but adopt the cost function $C(t_i, t_j)$ from eq. (2): a player pays β per unit of planned fighting time beyond her opponent's exit. Equivalently,

$$\pi_i(t_i, t_j) = u_i \times \left[\mathfrak{I}[t_i > t_j] + \frac{\mathfrak{I}[t_i = t_j]}{2} \right] - \min\{t_1, t_2\} - \beta \max\{t_i - \min\{t_1, t_2\}, 0\}. \tag{26}$$

This is equivalent to a hybrid all-pay auction.[14] It is now always costly to plan to fight for longer. This rules out equilibria in which a player plans to fight forever.

The first-order condition for a player i with valuation $v_i(t)$ planning to quit at time t becomes

$$v_i(t)f_j(v_j(t))v_j'(t) = \underbrace{1-F_j(v_j(t))}_{\text{marginal cost if loser}} + \underbrace{\beta F_j(v_j(t))}_{\text{marginal cost if winner}} \quad\Leftrightarrow\quad 1 = \frac{v_i(t)f_j(v_j(t))v_j'(t)}{1-(1-\beta)F_j(v_j(t))}. \tag{27}$$

A variant of Lemma 2 holds, including $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$, where I re-define $\Lambda_i(u)$ as

$$\Lambda_i(u) = \int_u^\infty \frac{1}{x}\frac{f_i(x)}{1-(1-\beta)F_i(x)}\,dx. \tag{28}$$

In essence, a boundary condition at time $\bar{t}$ (when players stay to the deadline T) is replaced by a condition as $t \to \infty$. However, for $\Lambda_i(u)$ to be well defined requires $\beta > 0$.[15]

A specification with the same effect is obtained when each player believes her opponent to be "crazy" and exogenously fight forever with probability $\xi > 0$.[16] Equivalently, a player may suffer exit failure with this probability. In either case, each first-order condition becomes

$$v_i(t)(1-\xi)f_j(v_j(t))v_j'(t) = 1-(1-\xi)F_j(v_j(t)). \tag{29}$$

To proceed, change the definition of $\Lambda_i(u)$ by switching "β" to "ξ" and continue as before.

Although the full details are omitted, the key findings are maintained (see Appendix B).[17] In particular, as $\max\{\beta,\xi\} \to 0$ the perceived weaker player concedes immediately.

Proposition 4 (Other Specifications). *Consider a hybrid all-pay auction with parameter β. Equivalently, either (i) players pay a cost β for planned fighting beyond the exit of an opponent; or (ii) a player fails to see the exit of an opponent with probability β. Suppose also that an opponent is believed to be crazy (and so exogenously fight forever) with probability ξ.*

There is a unique equilibrium if $\max\{\beta,\xi\} > 0$. *Proposition 1 holds if* $\max\{\beta,\xi\}$ *is sufficiently small. Proposition 2 holds. Proposition 3 holds if "$T \to \infty$" is replaced by "*$\max\{\beta,\xi\} \to 0$*".*

This specification can also allow for upper bounds to valuations, so that $u \in (\underline{u}_i, \bar{u}_i)$. In such a case, the asymptotic hazard-rate ranking extends so that $F_1 \succ_{\text{AHRD}} F_2$ if $\bar{u}_1 > \bar{u}_2$.

[14] Related features are present in auction-theoretic treatments of wars of attrition. Bulow and Klemperer (1999), for example, considered $N+K$ symmetric bidders competing for N prizes. When $K > 1$ such a game has no symmetric equilibrium (Haigh and Cannings, 1989). To circumvent this, Bulow and Klemperer (1999) perturbed the game: a conceding player continues to pay a (perhaps small) fraction of her fighting costs until the game ends. The perturbed game has a symmetric equilibrium involving rapid exit of $K-1$ players until the $N+1$ highest-valued players remain. In contrast, I suggest the instant exit of K players. Bulow and Klemperer (1999, p. 178, note 15) acknowledged this possibility, noting that "[a]symmetric perfect-Bayesian equilibria include those in which K (pre-identified) firms quit in zero time ... [e]quilibria of this kind seem particularly natural if (in contrast to our model) there any asymmetries between players."

[15] If $\beta = 0$ then $\Lambda_i(u)$ can fail to be well defined, and there are multiple equilibria.

[16] This corresponds to the approach of Kornhauser, Rubinstein, and Wilson (1989) and (more recently) Martinelli and Escorza (2007) and Kambe (2019) who, following Kreps and Wilson (1982), Milgrom and Roberts (1982), and Kreps, Milgrom, Roberts, and Wilson (1982), introduced a kind of "irrationality" into the model.

[17] These specifications can be further generalised by specifying player-specific parameters β_i and ξ_i for $i \in \{1,2\}$. The relative rates at which these parameters vanish to zero then matters for results analogous to Proposition 3.

Direct Learning. An uncomfortable conclusion (in the $T = \infty$ limit) is that there will be little if any fighting in a war of attrition: the (perceived) weaker player concedes immediately.

What else might explain fighting? One possibility is that perceptions can change, and so players may stick around in anticipation of this. Specifically, over time players may directly learn about each other (a player already updates her beliefs as her opponent stays in the game) and such direct learning may change the strength ranking.

To explore this idea, suppose that there is an arrival rate ρ of a public revelation of players' valuations. If this happens, then players go on to play a complete-information game in which (when the deadline is distant) the lower-valuation player exits immediately.

Consider two players at time t and suppose that $v_1(t) < v_2(t)$. The potential arrival of a day-of-revelation has no effect on the incentives of Player 1 with valuation $v_1(t)$: she knows that if valuations are revealed then she will be unmasked as the lower valuation player (given that $u_2 \geq v_2(t) > v_1(t)$.) However, Player 2 (with valuation $v_2(t)$) knows that such an arrival might prompt a win: if $u_1 \in [v_1(t), v_2(t))$ then the (perceived stronger, but perhaps actually weaker) Player 1 exits immediately. The first-order conditions for the players become

$$1 = \frac{v_1(t)v_2'(t)f_2(v_2(t))}{1 - F_2(v_2(t))} = \frac{v_2(t)v_1'(t)f_1(v_1(t))}{1 - F_1(v_1(t))} + \underbrace{\frac{\rho v_2(t)(F_1(v_2(t)) - F_1(v_1(t)))}{1 - F_1(v_1(t))}}_{\text{extra incentive for weaker player}}. \tag{30}$$

Working back from the end-of-attrition time $\bar{t}$ the incorporation of the second term brings the inverse stopping rules closer together, and pushes against "instant exit" results.[18]

Pre-Play Tactics. The importance of perceived strength implies that players may wish to influence such perceptions.[19] Suppose, for example beliefs are symmetric and that players have yet to discover their true valuations. Both players have distinct preferences over what might happen before they observe their valuations and the war-of-attrition game begins.

Consider, for example, a pre-play public signal of a player's valuation. The posterior over that valuation will be more concentrated, and so (in the asymptotic hazard rate sense) such a player will be perceived as weaker. A player wishes there to be a public signal of her opponent's valuation, but not of her own. This suggests a pre-play tactic via which a player strives to maintain an air of mystery about herself, while encouraging learning about her opponent.

[18]This idea resonates with a recent paper by Gieczewski (2023) who considered a model in the spirit of Smith (1998) and (more distantly) other work in international relations (Slantchev, 2003a,b; Powell, 2004), as well as more recent theory (Ortner, 2013, 2017). Players' costs evolve over time, and so a motive to stay in the game is that conditions may (exogenously) move in a player's favour.

[19]Relatedly, players may seek to change player-specific deadlines in a pre-play stage (Foster, 2018).

Applications. A war of attrition is a stylised representation of many applied scenarios.

Workers and employers may prolong a strike in order to obtain a preferred resolution (Kennan and Wilson, 1989). Potential suppliers may delay public-good provision in an effort to free ride (Bliss and Nalebuff, 1984; Bilodeau and Slivinski, 1996). Oligopolists may incur losses in anticipation of profitability following a competitor's exit (Fudenberg and Tirole, 1986; Ghemawat and Nalebuff, 1985, 1990). Macroeconomic stabilization may be delayed in order to push the burden of an agreement towards others (Alesina and Drazen, 1991; Casella and Eichengreen, 1996). The sponsor of a technological standard may continue its promotion in the hope that a competitor will adopt it (Farrell and Saloner, 1988; David and Monroe, 1994; Farrell and Simcoe, 2012). Political actors may expend lobbying costs to gain political influence (Hillman and Samet, 1987; Hillman and Riley, 1989) while (a conclave of) voters or candidates themselves may need to concede to achieve consensus (Indridason, 2008; Kwiek, 2014; Kwiek, Marreiros, and Vlassopoulos, 2016, 2019). In a bargaining game with fixed proposals, agreement requires the acquiescence of one participant (Osborne, 1985; Ordover and Rubinstein, 1986; Chatterjee and Samuelson, 1987; Abreu and Gul, 2000; Kambe, 1999). Of course, such a war-of-attrition-style negotiation can also take place rather literally in a conflict environment (for example Fearon, 2004; Powell, 2004, 2017; Leventoğlu and Slantchev, 2007; Krainin, 2014).

In some applications, it seems reasonable to specify some kind of perceived asymmetry. In the case of a labor dispute, for example, workers and employers may hold different perceptions. Similarly, competing technologies in a standards war are likely heterogeneous, and so perceptions of valuations may differ. This point was made by Martinelli and Escorza (2007) in the context of the Alesina and Drazen (1991) macroeconomic stabilisation model. They noted (p. 1224) that ex ante asymmetry is a strong assumption and concluded (in a message common to that delivered here) that "a political group more exposed to inflation costs will be likely to cave in immediately, leading to immediate reform."[20]

The results of this paper suggest how asymmetries can be important in predicting a relatively early winner. Any prize allocation might also be inefficient, albeit with lessened rent dissipation. However, this leaves open the scope (as noted above) for players to influence perceptions. A hypothesis, then, is that a combatant in a war of attrition may be waiting for (public) perceptions to change rather than (solely) waiting for an opponent to concede.

[20]Their main result (Theorem 2, p. 1231) corresponds to Proposition 2 in an environment with commitment types and where player types correspond to fighting costs rather than prize valuations. The result is equivalent to Proposition 1 of Myatt (2005), which also applies to a model with commitment types and with bounded valuations (or bounded fighting costs). Their specification of common support for player types prevented them from obtaining the "instant exit" result of Proposition 3 or the corresponding result in Myatt (2005).

Appendix A. Omitted Proofs of Lemmas 1 to 3

Lemma 1 (concerning basic equilibrium properties) is assembled from the following claims.

Claim 1. *Sufficiently high valuation types fight until the deadline: $t_i(u_i) = T$ if $u_i > 2T$.*

Proof. A player $u_i > 2T$ will exit at time $t < T$ only if guaranteed to win: $\Pr[t_j(u_j) < t] = 1$ for $j \neq i$. This means that for $j \neq i$ all types $u_j > 2T$ stop before t, which (repeating the argument) means that $\Pr[t_i(u_i) < t] = 1$. This is a contradiction. □

Claim 2. *It is always costly to fight for longer.*

Proof. This holds because at least some opposing types fight until the deadline. □

Claim 3. *A player's stopping time is at least weakly increasing in her valuation.*

Proof. If this is not true then there is a pair of valuations $u_L < u_H$ such that $t_i(u_H) < t_i(u_L)$. Player i with valuation u_L weakly prefers the higher stopping time. Fighting until $t_i(u_L)$ rather than $t_i(u_H)$ involves a strict increase in expected costs (Claim 2) and must be outweighed (at least weakly) by a strictly positive expected benefit. Now consider the same player with a higher valuation u_H. The (strictly positive) expected benefit is strictly higher (given that the prize valuation is strictly higher) and so now strictly exceeds the increase in expected costs. This means that choosing the higher stopping time is now strictly preferred. This contradicts the original claim that the player with valuation u_H would choose the lower stopping time. □

Given that some types fight until the deadline (Claim 1) and that stopping rules are monotonic (Claim 3) I write $\bar{u}_i^\star = \inf\{u : t_i(u) = T\}$ for the threshold at which i switches to fight until T.

Claim 4. *Not all players fight forever: $\bar{u}_i^\star > \underline{u}_i$ for each $i \in \{1, 2\}$.*

Proof. If $\bar{u}_i^\star = \underline{u}_i$ then every type of player i will fight until T. Player j will either fight to T or not fight at all: $t_j(u) \in \{0, T\}$. In particular, $\bar{u}_j^\star = 2T$, so that $t_j(u) = T$ if $u_j > 2T$ but $t_j(u) = 0$ if $u_j < 2T$. (By assumption $\underline{u}_j < 2T$ and so there is positive probability of both $t_j = 0$ and $t_j = T$.) In best reply types $u_i \in (\underline{u}_i, 2T)$ will not fight until T; a contradiction. □

Claim 5. *Any discontinuity in a stopping rule must be a jump up to the deadline T.*

Proof. Suppose that there is a jump at $u^\dagger \in (\underline{u}_i, \infty)$ from $t_L = \lim_{u \uparrow u^\dagger} t_i(u)$ to $t_H = \lim_{u \downarrow u^\dagger} t_i(u)$.

Player $j \neq i$ will not quit in (t_L, t_H): a small reduction in her stopping time will strictly reduce her expected fighting cost (from Claim 2) without reducing her chance of winning.

Suppose that $t_H < T$. For any $\varepsilon \in (0, T - t_H)$ there is some $u \geq u^\dagger$ close to $u^\dagger$ satisfying $t_H \leq t_i(u) \leq t_H + \varepsilon < T$. Player i with valuation u incurs a cost to fight from t_L to $t_i(u)$

and so must expect a positive probability of exit by $j \neq i$. Taking $\varepsilon \to 0$, j must exit at t_H with strictly positive probability. This means that i will never exit at time t_H, which in turn means that j could reduce her stopping time from t_H (strictly saving costs) without reducing her chance of winning. This is a contradiction. □

Claim 6. *A player will not exit with positive probability at time $t \in (0, T)$.*

Proof. If i exits at $t \in (0, T)$ with positive probability, then $j \neq i$ will never exit at or shortly before: she would wait until after t to "capture the atom" at time t. This means that i stopping at t could instead stop earlier, save costs, and yet not reduce her chance of winning. □

Claim 7. *Players begin exiting at $t = 0$: stopping rules satisfy $\lim_{u \downarrow \underline{u}_i} t_i(u) = 0$.*

Proof. $\underline{t}_i = \lim_{u \downarrow \underline{u}_i} t_i(u)$ is well defined given that $t_i(u)$ is at least weakly increasing (Claim 3). If $\underline{t}_i > 0$ then $j \neq i$ will never exit in $(0, \underline{t}_i)$. Since i is always prepared to incur a non-negligible fighting cost to get to $\underline{t}_i$, the logic employed in the proof of Claim 6 implies that j exits at $\underline{t}_i$ with positive probability. This contradicts Claim 6. Hence $\underline{t}_i = 0$. □

I defined an upper threshold $\bar{u}_i^\star$ above which a player fights until the deadline; similarly I define a lower threshold below which a player does not fight: $\underline{u}_i^\star = \inf\{u : t_i(u) > 0\}$. If $\underline{u}_i^\star = \underline{u}_i$ then a player always fights (and so the stopping rule $t_i(u)$ is strictly increasing at the lower bound of a player's valuations), but if $\underline{u}_i^\star > \underline{u}_i$ then there is positive probability of instant exit at the beginning. Naturally, $\underline{u}_i^\star \geq 0$ and so there is always (at least some) instant exit if $\underline{u}_i < 0$.

Claim 8. *There is at least some attrition: $\underline{u}_i^\star < \bar{u}_i^\star$ for each $i \in \{1, 2\}$.*

Proof. If $\underline{u}_i^\star = \bar{u}_i^\star$ then player i chooses $t \in \{0, T\}$, placing an atom at both times. ($t = 0$ is played with positive probability, because playing only T would contradict Claim 7.) If u_j is high then her best reply is $t_j = T$. However, if u_j is low then she prefers to exit just after $t = 0$. No best reply exists, of course, which itself means that this cannot be an equilibrium. Such a technicality can be avoided if j is allowed to achieve a stopping time "just after" zero by, for example, breaking any tie in her favour when choosing $t_j = 0$. However, if this were allowed then then j would place an atom at $t_j = 0$, which would then imply that player i (when choosing $t_i = 0$ and when $u_i > 0$) would no longer be choosing a best reply. □

Claim 9. *Attrition stops (for both players) before the deadline: $\lim_{u \uparrow \bar{u}_i^\star} t_i(u) = \bar{t} \in (0, T)$.*

Proof. Claim 8 implies that $\bar{u}_i^\star > 0$ (types in $(\underline{u}_i^\star, \bar{u}_i^\star)$ fight for some positive time, and so have positive valuations). Write $\bar{t}_i = \lim_{u \uparrow \bar{u}_i^\star} t_i(u)$. If $\bar{t}_i = T$ then a type u_i close to $\bar{u}_i^\star$ chooses to exit at a time t close to T. However, one can find such a type and time such that $u_i > 2(T - t)$. Such a type would prefer instead to stay until the deadline. Hence $\bar{t}_i < T$. Now suppose that $\bar{t}_i < \bar{t}_j$. If this were so then types of player j for whom $t_j(u) \in (\bar{t}_i, \bar{t}_j)$ could strictly save costs (with no effect on their chance of winning) by reducing their exit times. Hence $\bar{t}_i = \bar{t}_j = \bar{t}$. □

Claim 10. *The threshold valuation at which players fight to T is $\bar{u}_1^\star = \bar{u}_2^\star = \bar{u}^\star = 2(T - \bar{t})$.*

Proof. Indifference to paying a cost $T - \bar{t}$ for the equal chance of receiving the prize $\bar{u}_i^\star$. □

Proof of Lemma 1. The statements (i) to (iv) follow from the claims proven above. □

Proof of Lemma 2. For $t \in (0, \bar{t})$ the distribution of i's stopping time is $G_i(t) = F_i(v_i(t))$. The inverse-stopping rule $v_i(t)$ is strictly and continuously increasing, and so differentiable almost everywhere. Consider a time t at which $v_i(t)$ is differentiable, and player j with valuation $v_j(t)$. Her expected payoff from stopping time t_j and its derivative evaluated at $t_j = t$ is

$$\begin{aligned}\pi_j(t_j \mid u_j = v_j(t)) = v_j(t)G_i(t_j) - (1 - G_i(t_j))t_j - \int_0^{t_j} x\, dG_i(x) \\ \Rightarrow \quad \left.\frac{\partial \pi_j(t_j \mid u_j = v_j(t))}{\partial t_j}\right|_{t_j=t} &= v_j(t)g_i(t) - (1 - G_i(t)) \\ &= v_j(t)f_i(v_i(t))v_i'(t) - (1 - F_i(v_i(t))). \quad (31)\end{aligned}$$

Setting this to zero yields the first-order conditions which apply for almost all t:

$$v_i'(t) = \frac{1 - F_i(v_i(t))}{v_j(t)f_i(v_i(t))} \quad \text{for} \quad i, j \in \{1, 2\} \quad \text{and} \quad j \neq i. \tag{32}$$

I will argue that $v_i(t)$ is differentiable everywhere. I begin by establishing Lipschitz continuity.

Take a closed interval $C \subseteq (0, \bar{t})$. $v_j(t)$ is continuous and strictly positive on the compact set C, and so $\min_{t \in C}\{v_j(t)\}$ exists and is strictly positive. Set $K = 1/\min_{t \in C}\{v_j(t)\}$ as a Lipschitz constant on C. Suppose j with valuation $v_j(t_L)$ is considering delaying exit until $t_H > t_L$ where $\{t_L, t_H\} \subseteq C$. Given that t_L is an optimal stopping time,

$$G_i(t_H) - G_i(t_L) \leq \frac{t_H - t_L}{v_j(t_L)} \leq \frac{t_H - t_L}{\min_{t \in C}\{v_j(t)\}} \quad \Rightarrow \quad |G_i(t_H) - G_i(t_L)| \leq K|t_H - t_L|. \tag{33}$$

This means that $G_i(t)$ is Lipschitz continuous in t. Next note that $\min_{t \in C}\{f_i(v_i(t))\}$ exists and is strictly positive. Set $\hat{K} = K/\min_{t \in C}\{f_i(v_i(t))\}$ as a new Lipschitz constant. Then:

$$\begin{aligned}v_i(t_H) - v_i(t_L) = F_i^{-1}(G_i(t_H)) - F_i^{-1}(G_i(t_L)) \leq \frac{G_i(t_H) - G_i(t_L)}{\min_{t \in C}\{f_i(v_i(t))\}} \\ \leq \frac{t_H - t_L}{\min_{t \in C}\{f_i(v_i(t))\} \times \min_{t \in C}\{v_j(t)\}} \quad \Rightarrow \quad |v_i(t_H) - v_i(t_L)| \leq \hat{K}|t_H - t_L|. \quad (34)\end{aligned}$$

This means that $v_i(t)$ is (locally) Lipschitz continuous in t.

With this established, the right-hand side of eq. (32) is (at least locally) Lipschitz continuous in t. This implies that $v_i(t)$ is the integral of a Lipschitz continuous function, and so is differentiable everywhere. Specifically, this supports the claim that $v_i(t)$ is differentiable for $t \in (0, \bar{t})$.

With basic properties established, I now show the existence and uniqueness of the equilibrium. I write $\hat{\Lambda}_i(t) \equiv \Lambda_i(v_i(t))$ for $t \in (0, \bar{t})$. Using Claim 10, $\hat{\Lambda}_i(\bar{t}) \equiv \lim_{t \uparrow \bar{t}} \hat{\Lambda}_i(t) = 0$. Now:

$$\hat{\Lambda}_1'(t) = -\frac{v_1'(t) f_1(v_1(t))}{v_1(t)[1 - F_1(v_1(t))]} = -\frac{v_2'(t) f_2(v_2(t))}{v_2(t)[1 - F_2(v_2(t))]} = \hat{\Lambda}_2'(t), \tag{35}$$

where the central equality is from the players' first-order conditions. I conclude that $\hat{\Lambda}_1(t) = \hat{\Lambda}_2(t)$ or equivalently $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$ as stated in the lemma.

I now show existence and uniqueness, and I begin with the case where $\max\{\underline{u}_1, \underline{u}_2\} < 0$. For this case, the lower bounds of the stopping rules of the two players are $\underline{u}_1^\star = \underline{u}_2^\star = 0$.

To do this I begin by fixing an arbitrary threshold prize valuation $\bar{u}^\star$ and an arbitrary time $\bar{t}$, and I then set $\bar{u}^\star = v_1(\bar{t}) = v_2(\bar{t})$. Taking the pair of differential equations,

$$v_1'(t) = \frac{1 - F_1(v_1(t))}{v_2(t) f_1(v_1(t))} \quad \text{and} \quad v_2'(t) = \frac{1 - F_2(v_2(t))}{v_1(t) f_2(v_2(t))}, \tag{36}$$

I apply the Picard-Lindelöf Theorem to find a unique solution $\{v_1(t), v_2(t)\}$ that intersects $\bar{u}^\star$ at time $\bar{t}$, and then trace that unique solution back until the unique time $\underline{t}$ at which those solutions simultaneously hit zero. More formally, this is $\underline{t} = \inf\{t : v_i(t) > 0\}$ where the same solution is obtained for either $i \in \{1, 2\}$, and where that time is well-defined given that $v_1'(t)$ and $v_2'(t)$ are both bounded away from zero.[21] I now define $\bar{\tau}(\bar{u}^\star) = \bar{t} - \underline{t}$.

In summary: I have found a unique pair of inverse stopping rules satisfying the differential equations eq. (37) that intersect each other at time $\bar{t}$ and valuation $\bar{u}^\star$, and those inverse stopping rules also intersect at the valuation zero at a uniquely defined time $\bar{\tau}(\bar{u}^\star)$ before time $\bar{t}$. Without loss of generality, I now re-index time so that $\underline{t} = 0$ and so $\bar{t} = \bar{\tau}(\bar{u}^\star)$. I also invert the inverse stopping rules to to obtain the candidate equilibriu stopping rules $t_i(u) : [0, \bar{u}^\star) \mapsto [0, \bar{t})$ for $i \in \{1, 2\}$. For this to be an equilibrium, a player with valuation $\bar{u}^\star$ needs to be indifferent between quitting at time $\bar{t} = \bar{\tau}(\bar{u}^\star)$ and fighting until the deadline T. That is, I need

$$T - \frac{\bar{u}^\star}{2} = \bar{\tau}(u^\star). \tag{38}$$

The left-hand side is strictly decreasing from T down to zero as $\bar{u}^\star$ increases from zero up to $2T$. The right-hand side begins at $\bar{\tau}(0) = 0$, is continuous, and is strictly positive for $\bar{u}^\star > 0$. Hence there is at least one solution $\bar{u}^\star$. This shows the existence of an equilibrium: (i) find a solution $\bar{u}^\star$ to eq. (38); (ii) set $\bar{t} = \bar{\tau}(\bar{u}^\star)$; (iii) construct the inverse stopping rules from the unique solution to eq. (37); and then (iv) invert to obtain equilibrium stopping rules.

I now consider uniqueness. To do so I begin by noting that

$$\Lambda_i(u; \bar{u}^\star) \equiv \int_u^{\bar{u}^\star} \frac{1}{x} \frac{f_i(x)}{1 - F_i(x)} \, dx \tag{39}$$

---

[21] A lower bound for $v_i'(t)$ is readily obtained by noting that for $t \leq \bar{t}$,

$$v_i'(t) = \frac{1 - F_i(v_i(t))}{v_j(t) f_i(v_i(t))} \geq \frac{1 - F_i(v_i(t))}{\bar{u}^\star f_i(v_i(t))} \geq \bar{u}^\star \inf_{u \in [0, \bar{u}^\star]} \left\{ \frac{f_i(u)}{1 - F_i(u)} \right\} > 0. \tag{37}$$

The solutions $v_1(t)$ and $v_2(t)$ are linked (as noted) by $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$ and the properties of these "Λ" functions are such that (i) $v_1(t) > 0 \Leftrightarrow v_2(t) > 0$; and (ii) if $\lim_{t \downarrow \underline{t}} v_i(t) = 0$ then $\lim_{t \downarrow \underline{t}} v_j(t) = 0$ for $j \neq i$.

from eq. (5) in the lemma (where I now explicitly note the dependence on $\bar{u}^\star$) is strictly and continuously decreasing in u, with $\Lambda_i(\bar{u}^\star; \bar{u}^\star) = 0$ and $\lim_{u \downarrow 0} \Lambda_i(u; \bar{u}^\star) = \infty$. Its inverse $\gamma_i(\Lambda; \bar{u}^\star)$ (conditional on $\bar{u}^\star$) is well defined and satisfies $\Lambda = \Lambda_i(\gamma_i(\Lambda; \bar{u}^\star); \bar{u}^\star)$. The pair of solutions to eq. (37) and so equilibrium inverse stopping rules satisfy $\Lambda_1(v_1(t); \bar{u}^\star) = \Lambda_2(v_2(t); \bar{u}^\star)$, or equivalently $v_i(t) = \gamma_i(v_j(t); \underline{u}^\star)$ for $i \neq j$. From the first-order condition for player j,

$$t_j'(u) = \frac{\gamma_i(u; \bar{u}^\star) f_j(u)}{1 - F_j(u)} \quad \text{and so} \quad \bar{\tau}(\bar{u}^\star) = \int_0^{\underline{u}^\star} \frac{\gamma_i(u; \bar{u}^\star) f_j(u)}{1 - F_j(u)}\, du. \tag{40}$$

This applies for both $i \in \{1, 2\}$ and $j \neq i$. That is,

$$\bar{\tau}(\bar{u}^\star) = \int_0^{\underline{u}^\star} \frac{\gamma_1(u; \bar{u}^\star) f_2(u)}{1 - F_2(u)}\, du = \int_0^{\underline{u}^\star} \frac{\gamma_2(u; \bar{u}^\star) f_1(u)}{1 - F_1(u)}\, du. \tag{41}$$

Suppose that the hazard rate of the second player exceeds that of the first player at $\bar{u}^\star$, so that

$$\frac{f_2(\bar{u}^\star)}{1 - F_2(\bar{u}^\star)} \geq \frac{f_1(\bar{u}^\star)}{1 - F_1(\bar{u}^\star)}. \tag{42}$$

I will work with the second of the two equations in eq. (41). I note that

$$\bar{\tau}'(\bar{u}^\star) = \frac{\bar{u}^\star f_1(\bar{u}^\star)}{1 - F_1(\bar{u}^\star)} + \int_0^{\bar{u}^\star} \frac{\partial \gamma_2(u; \bar{u}^\star)}{\partial \bar{u}^\star} \frac{f_1(u)}{1 - F_1(u)}\, du. \tag{43}$$

The first term is strictly positive. To evaluate the second term, note that

$$\frac{\partial \gamma_2(u; \bar{u}^\star)}{\partial \bar{u}^\star} = \frac{\gamma_2(u; \bar{u}^\star)}{\bar{u}^\star} \frac{1 - F_2(\gamma_2(u; \bar{u}^\star))}{f_2(\gamma_2(u; \bar{u}^\star))} \left[\frac{f_2(\bar{u}^\star)}{1 - F_2(\bar{u}^\star)} - \frac{f_1(\bar{u}^\star)}{1 - F_1(\bar{u}^\star)} \right] \geq 0, \tag{44}$$

where the inequality follows from eq. (42). I conclude that $\bar{\tau}'(\bar{u}^\star) > 0$. If the inequality eq. (42) did not hold, then I instead work with first of the two equations in eq. (41) and reach the same conclusion: $\bar{\tau}(\bar{u}^\star)$ is strictly increasing.

Returning to the condition of eq. (38), I noted that the left-hand side is strictly decreasing in $\bar{u}^\star$ while the right-hand side is strictly increasing. This means that there is a unique solution.

The proof deals with cases in which the prize valuation supports of both player extend down to or below. For $\min\{\underline{u}_1, \underline{u}_2\} > 0$ and other cases, similar proofs may be used. □

Proof of Lemma 3. Using first-order condition of player j from eq. (4),

$$\begin{aligned} t_i'(v_i(t)) = \frac{1}{v_i'(t)} = \frac{v_j(t) f_i(v_i(t))}{1 - F_i(v_i(t))} &\leq \frac{\bar{u}^\star f_i(v_i(t))}{1 - F_i(v_i(t))} \\ \Rightarrow \quad \bar{t} = t_i(\bar{u}^\star) \leq \int_{\underline{u}_i^\star}^{\bar{u}^\star} \frac{\bar{u}^\star f_i(u)}{1 - F_i(u)}\, du &= \bar{u}^\star \log\left[\frac{1 - F(\underline{u}_i^\star)}{1 - F(\bar{u}^\star)}\right] \leq -\bar{u}^\star \log[1 - F(\bar{u}^\star)]. \end{aligned} \tag{45}$$

If the claim that $\lim_{T \to \infty} \bar{u}^\star = \infty$ is false then a sequence of deadlines can be constructed such that $\bar{u}^\star$ remains bounded. The inequality above would imply that $\bar{t}$ remains bounded. But if that is true the $\bar{u}^\star = 2(T - \bar{t}) \to \infty$; this is a contradiction. □

Appendix B. Other Specifications

Proposition 4 in the main text is concerned with other specifications that can be used to obtain a unique equilibrium: a per-period cost $\beta > 0$ for any planned fighting beyond the exit of an opponent (which itself can be interpreted as either a failure to see the exit of an opponent, or the use of a hybrid all-pay auction) or the belief that an opponent is crazy with probability $\xi > 0$. I can include both of these features as well as a time limit within the same model. To simplify exposition, however, I now focus on the case where $T = \infty$ (there is no deadline), $\xi = 0$ (players are not crazy), but $\beta > 0$ (so that planned fighting is costly).

The proof of Lemma 1 in Appendix A assembles together several claims. Here I discuss how these claims are modified for this new environment.

Claim 1 does not apply because there is no deadline.

Claim 2 holds by assumption given that $\beta > 0$.

Claim 4 is replaced by the claim that no player fights forever; to do so is infinitely costly.

The proof of Claim 5 supports a replacement claim that there are no discontinuities.

Claim 3 and Claims 6–10 all hold as stated.

These (modified) claims can be assembled to form this corresponding replacement for Lemma 1.

Lemma 4 (Basic Properties of Equilibrium Stopping Rules). *Consider a game with no deadline but where there is a cost of planned fighting $\beta > 0$. For both players:*

(i) There is a lower-bound $\underline{u}_i^\star \in [\underline{u}_i, \infty)$ at or below which a player $i \in \{1, 2\}$ exits at time zero.

(ii) There is a unique time $\bar{t} \in (0, \infty) \cup \{\infty\}$ such that the stopping rule $t_i(u)$ is strictly and continuously increasing from 0 to $\bar{t}$ for $u \in (\underline{u}_i^\star, \infty)$.

In the text I explain how the first-order conditions are modified under this new specification. Following steps used in the proof of Lemma 2, the following equivalent result is readily obtained.

Lemma 5 (Uniqueness, Existence, and Properties of Inverse Stopping Rules). *Consider a game with no deadline but where there is a cost of planned fighting $\beta > 0$. There is a unique equilibrium. Each equilibrium inverse $v_i(t)$ is differentiable for $t \in (0, \bar{t})$. Define*

$$\Lambda_i(u) = \int_u^\infty \frac{1}{x} \frac{f_i(x)}{1 - (1-\beta) F_i(x)}\, dx. \tag{46}$$

For any $t \in (0, \bar{t})$ the pair of inverse stopping rules satisfy $\Lambda_1(v_1(t)) = \Lambda_2(v_2(t))$.

Notice in particular that $\Lambda_i(u)$ is well defined for $\beta > 0$, but it is not defined if the distribution $F_i(\cdot)$ has a hazard rate bounded away from and $\beta = 0$. This is a (technical) reason why setting $\beta > 0$ is required to obtain a unique equilibrium when there is no deadline.

Proposition 4 states that variations of Propositions 1 to 3 hold. Here I prove Proposition 2; the other results are straightforward but more cumbersome. Note that, using integration by parts,

$$(1-\beta)\Lambda_i(u) = \frac{\log[1-(1-\beta)F_i(u)}{u} - \int_u^\infty \frac{\log[1-(1-\beta)F_i(x)]}{x^2}\,dx. \tag{47}$$

The steps in the proof of Proposition 2 apply here. Suppose that $F_1 \succ_{\text{FOSD}} F_2$ and so $\underline{u}_2 \leq \underline{u}_1$. Consider an equilibrium in which Player 2 does not exit at time zero: $\underline{u}_2^\star = \underline{u}_2 \leq \underline{u}_1 \leq \underline{u}_1^\star$.

$$\begin{aligned} 0 = (1-\beta)(\Lambda_2(\underline{u}_2^\star) - \Lambda_1(\underline{u}_1^\star)) = \underbrace{\frac{\log[1-(1-\beta)F_2(\underline{u}_2^\star)]}{\underline{u}_2^\star}}_{\text{zero if } \underline{u}_2^\star = \underline{u}_2} - \underbrace{\int_{\underline{u}_2^\star}^{\bar{u}_1^\star} \frac{\log[1-(1-\beta)F_2(u)]}{u^2}\,du}_{\text{strictly negative}} \\ - \underbrace{\frac{\log[1-(1-\beta)F_1(\underline{u}_1^\star)]}{\underline{u}_1^\star}}_{\text{strictly negative}} + \underbrace{\int_{\underline{u}_1^\star}^{\bar{u}^\star} \frac{1}{u^2} \log\frac{1-(1-\beta)F_1(u)}{1-(1-\beta)F_2(u)}\,du}_{\text{strictly positive if } F_1 \succ_{\text{FOSD}} F_2} > 0. \end{aligned} \tag{48}$$

The right-hand side is strictly positive. This contradicts the supposition that $\underline{u}_2^\star = \underline{u}_2$.

References

Abreu, D., and F. Gul (2000): "Bargaining and Reputation," *Econometrica*, 68(1), 85–117.

Alesina, A., and A. Drazen (1991): "Why are Stabilizations Delayed?," *American Economic Review*, 81(5), 1170–1188.

Amann, E., and W. Leininger (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two-Player Case," *Games and Economic Behavior*, 14(1), 1–18.

Anderson, S. P., J. K. Goeree, and C. A. Holt (1998a): "Rent Seeking with Bounded Rationality: An Analysis of the All-Pay Auction," *Journal of Political Economy*, 106(4), 828–853.

——— (1998b): "The War of Attrition with Noisy Players," in *Advances in Applied Microeconomics*, ed. by M. R. Baye. JAI Press, Greenwich.

Asako, Y. (2015): "One-Sided Games in a War of Attrition," *The B.E. Journal of Theoretical Economics*, 15(2), 313–331.

Baye, M. R., D. Kovenock, and C. G. De Vries (1993): "Rigging the Lobbying Process: An Application of the All-Pay Auction," *American Economic Review*, 83(1), 289–294.

——— (1996): "The All-Pay Auction with Complete Information," *Economic Theory*, 8(2), 291–305.

Betto, M., and M. W. Thomas (2024): "Asymmetric All-Pay Auctions with Spillovers," *Theoretical Economics*, 19(1), 169–206.

Bilodeau, M., and A. Slivinski (1996): "Toilet Cleaning and Department Chairing: Volunteering a Public Service," *Journal of Public Economics*, 59(2), 299–308.

Bishop, D. T., and C. Cannings (1978a): "The Generalised War of Attrition," *Advances in Applied Probability*, 10(1), 6–7.

——— (1978b): "A Generalized War of Attrition," *Journal of Theoretical Biology*, 70(1), 85–124.

Bishop, D. T., C. Cannings, and J. Maynard Smith (1978): "The War of Attrition with Random Rewards," *Journal of Theoretical Biology*, 74(3), 377–388.

Bliss, C., and B. Nalebuff (1984): "Dragon-Slaying and Ballroom Dancing: The Private Supply of a Public Good," *Journal of Public Economics*, 25(1–2), 1–12.

Bulow, J. I., and P. D. Klemperer (1999): "The Generalized War of Attrition," *American Economic Review*, 89(1), 175–189.

Cannings, C., and J. C. Whittaker (1995): "The Finite Horizon War of Attrition," *Games and Economic Behavior*, 11(2), 193–236.

Casella, A., and B. Eichengreen (1996): "Can Foreign Aid Accelerate Stabilisation," *Economic Journal*, 106(436), 605.

Chatterjee, K., and L. Samuelson (1987): "Bargaining with Two-sided Incomplete Information: An Infinite Horizon Model with Alternating Offers," *Review of Economic Studies*, 54(2), 175.

David, P. A., and H. K. Monroe (1994): "Standards Development Strategies Under Incomplete Information—Isn't the "Battle of the Sexes" Really a Revelation Game?," All Souls College and IMF Working Paper.

Décamps, J.-P., F. Gensbittel, and T. Mariotti (2023): "The War of Attrition under Uncertainty: Theory and Robust Testable Implications," Toulouse School of Economics Working Paper.

Farrell, J., and G. Saloner (1988): "Coordination Through Committees and Markets," *RAND Journal of Economics*, 19(2), 235–252.

Farrell, J., and T. Simcoe (2012): "Choosing the Rules for Consensus Standardization," *RAND Journal of Economics*, 43(2), 235–252.

Fearon, J. D. (2004): "Why Do Some Civil Wars Last So Much Longer than Others?," *Journal of Peace Research*, 41(3), 275–301.

Foster, J. (2018): "Wars of Attrition with Endogenously Determined Budget Constraints," *BE Journal of Theoretical Economics*, 18(2).

Fudenberg, D., and J. Tirole (1986): "A Theory of Exit in Duopoly," *Econometrica*, 54(4), 943.

Georgiadis, G., Y. Kim, and H. D. Kwon (2022): "The Absence of Attrition in a War of Attrition Under Complete Information," *Games and Economic Behavior*, 131(1), 171–185.

Ghemawat, P., and B. Nalebuff (1985): "Exit," *RAND Journal of Economics*, 16(2), 184–194.

——— (1990): "The Devolution of Declining Industries," *Quarterly Journal of Economics*, 105(1), 167–186.

Gieczewski, G. (2023): "Evolving Wars of Attrition," Princeton University Working Paper.

Güth, W., and E. Van Damme (1986): "A Comparison of Pricing Rules for Auctions and Fair Division games," *Social Choice and Welfare*, 3(3), 177–198.

Haigh, J., and C. Cannings (1989): "The N-Person War of Attrition," *Acta Applicandae Mathematicae*, 14(1-2), 59–74.

Hendricks, K., A. Weiss, and C. Wilson (1988): "The War of Attrition in Continuous Time with Complete Information," *International Economic Review*, 29(4), 663–680.

Hillman, A. L., and J. G. Riley (1989): "Politically Contestable Rents and Transfers," *Economics and Politics*, 1(1), 17–39.

Hillman, A. L., and D. Samet (1987): "Dissipation of Contestable Rents by Small Numbers of Contenders," *Public Choice*, 54(1), 63–82.

Hörner, J., and N. Sahuguet (2007): "Costly Signalling in Auctions," *Review of Economic Studies*, 74(1), 173–206.

——— (2011): "A War of Attrition with Endogenous Effort Levels," *Economic Theory*, 47, 1–27.

Huangfu, B., G. Ghosh, and H. Liu (2023): "Resource Inequality in the War of Attrition," *International Journal of Game Theory*, 52(1), 33–61.

Indridason, I. H. (2008): "When to Run and When to Hide: Electoral Coordination and Exit," *Economics and Politics*, 20(1), 80–105.

Kambe, S. (1999): "Bargaining with Imperfect Commitment," *Games and Economic Behavior*, 28(2), 217–237.

——— (2019): "An N-Person War of Attrition with the Possibility of a Noncompromising Type," *Theoretical Economics*, 14(3), 849–886.

Kapur, S. (1995): "Markov Perfect Equilibria in an N-player War of Attrition," *Economics Letters*, 47(2), 149–154.

Kennan, J., and R. Wilson (1989): "Strategic Bargaining Models and Interpretation of Strike Data," *Journal of Applied Econometrics*, 4(S1), S87–S130.

Klemperer, P. (1999): "Auction Theory: A Guide to the Literature," *Journal of Economic Surveys*, 13(3), 227–286.

Kornhauser, L., A. Rubinstein, and C. Wilson (1989): "Reputation and Patience in the 'War of Attrition'," *Economica*, 56(221), 15–24.

Krainin, C. (2014): "Multilateral War Bargaining," *Quarterly Journal of Political Science*, 9(4), 407–439.

Kreps, D. M., P. Milgrom, J. Roberts, and R. Wilson (1982): "Rational Cooperation in the Finitely Repeated Prisoners' Dilemma," *Journal of Economic Theory*, 27(2), 245–252.

Kreps, D. M., and R. Wilson (1982): "Reputation and imperfect information," *Journal of Economic Theory*, 27(2), 253–279.

Krishna, V., and J. Morgan (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 72(2), 343–362.

Kwiek, M. (2014): "Conclave," *European Economic Review*, 70, 258–275.

Kwiek, M., H. Marreiros, and M. Vlassopoulos (2016): "An Experimental Study of Voting with Costly Delay," *Economics Letters*, 140(1–2), 23–26.

——— (2019): "Voting as a War of Attrition," *Journal of Economic Behavior & Organization*, 167(1), 104–121.

Lambrecht, B. M. (2001): "The Impact of Debt Financing on Entry and Exit in a Duopoly," *Review of Financial Studies*, 14(3), 765–804.

Leventoğlu, B., and B. L. Slantchev (2007): "The Armed Peace: A Punctuated Equilibrium Theory of War," *American Journal of Political Science*, 51(4), 755–771.

Martinelli, C., and R. Escorza (2007): "When are Stabilizations Delayed? Alesina-Drazen Revisited," *European Economic Review*, 51(5), 1223–1245.

Maskin, E., and J. Riley (2000): "Asymmetric Auctions," *Review of Economic Studies*, 67(3), 413–438.

Maynard Smith, J. (1974): "The Theory of Games and the Evolution of Animal Conflicts," *Journal of Theoretical Biology*, 47(1), 209–221.

McKelvey, R. D., and T. R. Palfrey (1995): "Quantal Response Equilibria for Normal Form Games," *Games and Economic Behavior*, 10(1), 6–38.

——— (1996): "A Statistical Theory of Equilibrium in Games," *Japanese Economic Review*, 47(2), 186–209.

Milgrom, P. R. (1981): "Good News and Bad News: Representation Theorems and Applications," *Bell Journal of Economics*, 12(2), 380.

Milgrom, P. R., and J. Roberts (1982): "Predation, Reputation, and Entry Deterrence," *Journal of Economic Theory*, 27(2), 280–312.

Mirrlees, J. A. (1999): "The Theory of Moral Hazard and Unobservable Behaviour: Part I," *Review of Economic Studies*, 66(1), 3–21.

Murto, P. (2004): "Exit in Duopoly under Uncertainty," *RAND Journal of Economics*, 35(1), 111.

Myatt, D. P. (2005): "Instant Exit from the Asymmetric War of Attrition," University of Oxford Department of Economics Working Paper no. 160.

Nalebuff, B., and J. Riley (1985): "Asymmetric Equilibria in the War of Attrition," *Journal of Theoretical Biology*, 113(3), 517–527.

Ordover, J. A., and A. Rubinstein (1986): "A Sequential Concession Game with Asymmetric Information," *Quarterly Journal of Economics*, 101(4), 879.

Ortner, J. (2013): "Optimism, Delay and (In)efficiency in a Stochastic Model of Bargaining," *Games and Economic Behavior*, 77(1), 352–366.

——— (2017): "A Theory of Political Gridlock," *Theoretical Economics*, 12(2), 555–586.

Osborne, M. J. (1985): "The Role of Risk Aversion in a Simple Bargaining model," in *Game-Theoretic Models of Bargaining*, ed. by A. E. Roth, pp. 181–214. Cambridge University Press, Cambridge.

Pitchik, C. (1981): "Equilibria of a Two-Person Non-Zerosum Noisy Game of Timing," *International Journal of Game Theory*, 10(3-4), 207–221.

Ponsati, C. (1995): "The Deadline Effect: A Theoretical Note," *Economics Letters*, 48(3-4), 281–285.

Ponsati, C., and J. Sákovics (1995): "The War of Attrition with Incomplete Information," *Mathematical Social Sciences*, 29(3), 239–254.

Powell, R. (2004): "Bargaining and Learning While Fighting," *American Journal of Political Science*, 48(2), 344–361.

——— (2017): "Taking Sides in Wars of Attrition," *American Political Science Review*, 111(2), 219–236.

Riley, J. G. (1979): "Evolutionary Equilibrium Strategies," *Journal of Theoretical Biology*, 76(2), 109–123.

——— (1980): "Strong Evolutionary Equilibrium and the War of Attrition," *Journal of Theoretical Biology*, 82(3), 383–400.

——— (1999): "Asymmetric Contests : A Resolution of the Tullock Paradox," in *Money, Markets and Method: Essays in Honour of Robert W. Clower*, ed. by P. Howitt, E. De Antoni, and A. Leijonhufvud. Edward Elgar.

Rothschild, M., and J. E. Stiglitz (1970): "Increasing Risk: I. A definition," *Journal of Economic Theory*, 2(3), 225–243.

Siegel, R. (2009): "All-Pay Contests," *Econometrica*, 77(1), 71–92.

——— (2010): "Asymmetric Contests with Conditional Investments," *American Economic Review*, 100(5), 2230–2260.

——— (2014a): "Asymmetric All-Pay Auctions with Interdependent Valuations," *Journal of Economic Theory*, 153(1), 684–702.

——— (2014b): "Asymmetric Contests with Head Starts and Nonmonotonic Costs," *American Economic Journal: Microeconomics*, 6(3), 59–105.

Slantchev, B. L. (2003a): "The Power to Hurt: Costly Conflict with Completely Informed States," *American Political Science Review*, 97(1), 123–133.

——— (2003b): "The Principle of Convergence in Wartime Negotiations," *American Political Science Review*, 97(4), 621–632.

Smith, A. (1998): "Fighting Battles, Winning Wars," *Journal of Conflict Resolution*, 42(3), 301–320.

Steg, J.-H. (2015): "Symmetric Equilibria in Stochastic Timing Games," Center for Mathematical Economics, Bielefeld University Working Paper.

Xiao, J. (2018): "All-Pay Contests with Performance Spillovers," *Mathematical Social Sciences*, 131(1), 35–39.